Chapter 3

Polar decomposition of Aluthge and Duggal transformations

3.1 Introduction

Let $T = U|T|$ be the polar decomposition of an operator $T \in \mathcal{L}(\mathcal{H})$. Then one can think about the polar decomposition of Aluthge transformation $\tilde{T}$, and of Duggal transformation $\hat{T}$. In [20], Masatoshi Ito, Takeaki Yamazaki, and Masahiro Yanagida obtained results on the polar decomposition of Aluthge transformation. In [21], Ito, Yamazaki, and Yanagida showed results on the polar decomposition of the product of two operators and of Aluthge transformation. They also showed properties and characterizations of binormal and centered operators from the viewpoint of the polar decomposition and Aluthge transformation.

In [20], Ito, Yamazaki, Yanagida gave an example of a binormal, invertible operator $T$ such that the Aluthge transformation $\tilde{T}$ is not binormal. In this chapter, first we show that if $T$ is a binormal, invertible operator, then the Duggal transformation $\hat{T}$ is binormal. We discuss some consequences of applying Aluthge transformation and Duggal transformation successively on an invertible
3.1. Introduction

Let $T = U|T|$ be the polar decomposition of an operator $T$. A theorem in [21] says that $T$ is binormal if and only if $\tilde{T} = \tilde{U}|\tilde{T}$ is the polar decomposition of the Aluthge transformation $\tilde{T}$. We discuss the similar situation of Duggal transformations. We give necessary and sufficient condition for $\hat{T}$ to have the polar decomposition $\hat{T} = \hat{U}|\hat{T}$. As a consequence, we prove that if $T$ is binormal, then $\tilde{T} = \tilde{U}|\tilde{T}$ is the polar decomposition of $\tilde{T}$. Characterization of operators $T = U|T|$, having the property that $\tilde{T}^{(n)} = \tilde{U}^{(n)}|\tilde{T}^{(n)}$ is the polar decomposition of $\tilde{T}^{(n)}$ for all $n = 1, 2, \ldots$, exists. In theorem 3.2.27, we prove that the final space of $U$ is invariant under every $|\hat{T}^{(n)}|$, if $\hat{T}^{(n)} = \hat{U}^{(n)}|\hat{T}^{(n)}$ is the polar decomposition of $\hat{T}^{(n)}$ for all $n = 1, 2, \ldots$.

In [9], Ximena Catepillan and Waclaw Szymanski proved the semigroup properties of factors in the polar decomposition of operators. In section 3.2.3, we use them to give a modification of the proof of a theorem in [21].

In [29], M. Schreiber discussed operators for which the closure of the numerical range is a spectral set. He characterized such operators by means of normal dilations. He obtained relations between the spectrality of the numerical range and the equality of the convex hull of the spectrum with the closure of the numerical range. In section 3.3, we discuss some consequences of these results on Aluthge and Duggal transformations. In theorem 3.3.6, we prove a general version of one part of Ando’s theorem. We proceed to prove that if $A$ is an $n \times n$ matrix over $\mathbb{C}$, and if the convex hull of the spectrum of $A$ is a spectral set for the matrix $A$, then the matrices $A$, $\tilde{A}$, and $\hat{A}$ have the same numerical range.
3.2. Aluthge and Duggal transformations of binormal operators

3.2 Aluthge and Duggal transformations of binormal and centered operators

3.2.1 Aluthge and Duggal transformations of binormal or centered invertible operators

The following lemma shows results about Aluthge and Duggal transformation of invertible operators.

**Lemma 3.2.1.** If \( T \in \mathcal{L}(\mathcal{H}) \) is invertible, then \( \tilde{T} \) and \( \hat{T} \) are invertible. In this case,

\[
\tilde{T} = |T|^{1/2} T |T|^{-1/2}
\]
\[
\hat{T} = |T| T |T|^{-1}
\]
(see lemma 2.3.1).

**Definition 3.2.2.** An operator \( T \) is said to be **binormal** if \([ |T|, |T^*| ] = 0\), where \([ A, B \] = AB − BA\). The operator \( T \) is said to be **centered** if the following sequence

\[
\ldots, T^3(T^3)^*, T^2(T^2)^*, TT^*, T^*T, (T^2)^*T^2, (T^3)^*T^3, \ldots
\]

is commutative.

Binormal and centered operators were defined by S. L. Campbell in [8] and B. B. Morrel and P. S. Muhly in [25], respectively. Relations among these classes and that of quasinormal operators are easily obtained as follows.

\[
\text{quasinormal} \subset \text{centered} \subset \text{binormal}.
\]
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Theorem 3.2.3. [20] Let \( T \in \mathcal{L}(H) \), and suppose that \( T = U|T| \) is the polar decomposition of \( T \). Then \( \tilde{T} = U|\tilde{T}| \) if and only if \( T \) is binormal. (Note that, this assertion does not mean that \( \tilde{T} = U|\tilde{T}| \) is the polar decomposition of \( \tilde{T} \), when \( T \) is binormal).

Theorem 3.2.4. [20] Let \( T \in \mathcal{L}(H) \), and suppose that \( T = U|T| \) be the polar decomposition of \( T \). If \( T \) is binormal, then \( \tilde{T} = U^*UU|\tilde{T}| \) is the polar decomposition of \( \tilde{T} \).

Theorem 3.2.5. Let \( T \in \mathcal{L}(H) \) be invertible. Suppose that \( T = U|T| \) is the polar decomposition of \( T \). If \( T \) is binormal, then \( \tilde{T} = U|\tilde{T}| \) is the polar decomposition of \( \tilde{T} \).

Proof. Since \( T \) is invertible, \( U \) is unitary. By theorem 3.2.4, the proof follows.

If \( S, T \) and \( V \) are operators with \( S = V^*TV \), and \( V \) unitary, then it can be easily verified that \( |S| = V^*|T|V \) and \( |S|^{1/2} = V^*|T|^{1/2}V \). Further, if the polar decomposition of \( T \) is \( T = U|T| \), then the polar decomposition of \( S \) is \( S = (V^*UV)|S| \). Hence \( \tilde{S} = V^*\tilde{T}V \).

Remark 3.2.6. The binormality of \( T \) does not imply the binormality of the Aluthge transformation \( \tilde{T} \). The following example is from the paper of Ito, Yamazaki, and Yanagida [20]. It gives a binormal operator \( T \) such that the Aluthge transformation \( \tilde{T} \) is not binormal. We give the example here not only for the sake of completeness but also for the reason that it gives a simple example of an invertible operator which is not binormal. Notice that the operator in the example is invertible. We shall show that the analogous case of the Duggal transformations is different. We prove in theorem 3.2.9 that, if \( T \) is an invertible operator, then the binormality of the operator \( T \) implies the binormality of the Duggal transformation \( \hat{T} \).
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Example 3.2.7. [20] There exists a binormal operator $T$ such that the Aluthge transformation $\tilde{T}$ is not binormal. Let

$$T = \begin{bmatrix} 0 & 0 & 5 \\ 1/2 & \sqrt{3}/2 & 0 \\ \sqrt{3}/2 & -1/2 & 0 \end{bmatrix}$$

and $T = U|T|$ be the polar decomposition of $T$. Then $T$ is binormal since

$$T^*TT^* = TT^*T^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 25 \end{bmatrix}$$

and also

$$|T| = (T^*T)^{1/2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix},$$

so that

$$U = T|T|^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & \sqrt{3}/2 & 0 \\ \sqrt{3}/2 & -1/2 & 0 \end{bmatrix}.$$ 

Therefore,

$$\tilde{T} = |T|^{1/2}U|T|^{1/2} = \begin{bmatrix} 0 & 0 & \sqrt{5} \\ 1/2 & \sqrt{3}/2 & 0 \\ \sqrt{15}/2 & -\sqrt{5}/2 & 0 \end{bmatrix}.$$ 

We get that

$$(\tilde{T})^*\tilde{T}\tilde{T}^* = \begin{bmatrix} 20 & -\sqrt{3} & 0 \\ -5\sqrt{3} & 2 & 0 \\ 0 & 0 & 25 \end{bmatrix}.$$
and
\[
\tilde{T}(T)^* \tilde{T} = \begin{bmatrix}
20 & -5\sqrt{3} & 0 \\
-\sqrt{3} & 2 & 0 \\
0 & 0 & 25
\end{bmatrix}.
\]
Hence \( \tilde{T} \) is not binormal.

**Theorem 3.2.8.** Let \( T = U|T| \) is the polar decomposition of the operator \( T \), and \( U \) a coisometry. If \( T \) is binormal, then \( \hat{T} \) is binormal.

**Proof.** We have \( \hat{T} = U^*TU \). Therefore, \( (\hat{T})^* \hat{T} = U^*|T|^2 U \geq 0 \), and \( |\hat{T}| = U^*|T| U \). On the other hand, \( \hat{T}(\hat{T})^* = U^*|T^*|^2 U \geq 0 \), and \( |(\hat{T})^*| = U^*|T^*| U \). Hence, \( |\hat{T}| |(\hat{T})^*| = |(\hat{T})^*| |\hat{T}| \), and \( \hat{T} \) is binormal.

If \( S \) and \( T \) are unitarily equivalent operators, then \( S \) is binormal if and only if \( T \) is binormal.

**Theorem 3.2.9.** Let \( T \) be invertible. Then \( T \) is binormal if and only if \( \hat{T} \) is binormal.

**Proof.** Since \( T \) is invertible, the polar decomposition of \( T \) has the form \( T = U|T| \) where \( U \) is unitary. Also, \( \hat{T} = U^*TU \). Thus \( T \) and \( \hat{T} \) are unitarily equivalent.

**Theorem 3.2.10.** Let \( T \in \mathcal{L}(H) \) be invertible. If \( T \) is binormal, then \( \tilde{T} = \tilde{T} \).

**Proof.** Let \( T = U|T| \) be the polar decomposition of \( T \). Since \( T \) is invertible, \( U \) is unitary. By lemma 2.3.2, \( \hat{T} = U^*TU \). i.e., \( \hat{T} \) is unitarily equivalent to \( T \). Therefore, \( \tilde{T} = U^*\tilde{T}U \).

By theorem 3.2.5, \( \tilde{T} = U|\tilde{T}| \) is the polar decomposition of \( \tilde{T} \), and therefore, again by lemma 2.3.2, \( \tilde{T} = U^*\tilde{T}U \).

For convenience of notation we make the following definitions. We use these notations till the end of the present section.
Definition 3.2.11. Let $T$ be an operator. Define $\Delta(T) = \tilde{T}$, $\Gamma(T) = \hat{T}$. For every non-negative integer $n$, define $\Delta^n(T) = \tilde{T}^{(n)}$, $\Gamma^n(T) = \hat{T}^{(n)}$.

Theorem 3.2.10 says that if $T$ is invertible and binormal, then $\Gamma(\Delta(T)) = \Delta(\Gamma(T))$.

The following characterization of centered operators from the viewpoint of the polar decomposition and the Aluthge transformation can be seen in [20].

Theorem 3.2.12. [20] Let $T$ be an operator. Then $\Delta^n(T)$ is binormal for all $n \geq 0$ if and only if $T$ is a centered operator.

Theorem 3.2.13. Let $T$ be invertible and centered. Then $\Gamma(\Delta^n(T)) = \Delta^n(\Gamma(T))$ for all $n \geq 0$.

Proof. Since $T$ is centered, $T$ is binormal. Therefore, the result is true for the case $n = 1$, by theorem 3.2.10. Suppose that the result is true for $n = m - 1$. Then $\Gamma(\Delta^{m-1}(T)) = \Delta^{m-1}(\Gamma(T))$.

Now, $\Delta^{m-1}(T)$ is invertible since $T$ is invertible. By theorem 3.2.12, $\Delta^{m-1}(T)$ is binormal. Therefore, by theorem 3.2.10, $\Gamma[\Delta(\Delta^{m-1}(T))] = \Delta[\Gamma(\Delta^{m-1}(T))]$. Hence,

$$
\Gamma(\Delta^m(T)) = \Gamma[\Delta(\Delta^{m-1}(T))] = \Delta[\Gamma(\Delta^{m-1}(T))] = \Delta[\Delta^{m-1}(\Gamma(T))] = \Delta^m(\Gamma(T))
$$

The theorem follows by induction.

Theorem 3.2.14. If $T$ is invertible and binormal, then $\Delta(\Gamma^n(T)) = \Gamma^n(\Delta(T))$ for all $n \geq 0$. 

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Proof. The result is true for $n = 1$, by theorem 3.2.10.

Suppose that the result is true for $n = m - 1$, i.e., $\Delta(\Gamma^{m-1}(T)) = \Gamma^{m-1}(\Delta(T))$.

Since $T$ is invertible and binormal, by lemma 3.2.1 and theorem 3.2.9, $\Gamma^n(T)$ is invertible and binormal for all $n \geq 0$. Since $\Gamma^{m-1}(T)$ is binormal and invertible, by theorem 3.2.10, $\Delta[\Gamma(\Gamma^{m-1}(T))] = \Gamma[\Delta(\Gamma^{m-1}(T))]$. Hence,

$$
\Delta(\Gamma^m(T)) = \Delta[\Gamma(\Gamma^{m-1}(T))]
$$

$$
= \Gamma[\Delta(\Gamma^{m-1}(T))]
$$

$$
= \Gamma[\Gamma^{m-1}(\Delta(T))]
$$

$$
= \Gamma^m(\Delta(T))
$$

The theorem follows by induction. \qed

Every centered operator is binormal. So combining theorem 3.2.13 and theorem 3.2.14 we have the following result.

**Theorem 3.2.15.** Let $T$ be invertible and centered. Then

$$
\Gamma(\Delta^n(T)) = \Delta^n(\Gamma(T)), \quad \Delta(\Gamma^n(T)) = \Gamma^n(\Delta(T))
$$

for all $n \geq 0$.

**Corollary 3.2.16.** Let $T$ be invertible and centered. Then

$$
\Gamma^m(\Delta^n(T)) = \Delta^n(\Gamma^m(T))
$$

for all $m, n \geq 0$.

Proof. The proof follows by theorem 3.2.12, theorem 3.2.14, and theorem 3.2.15. \qed
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3.2.2 Polar decomposition of Aluthge and Duggal transformations

The following result gives the polar decomposition of the product of two operators.

**Theorem 3.2.17.** [17] Let $T = U|T|$ and $S = V|S|$ be the polar decompositions. If $T$ and $S$ are doubly commutative (i.e., $[T, S] = [T, S^*] = 0$), then

$$TS = UV|TS|$$

is the polar decomposition of $TS$.

The following is a generalization of this result.

**Theorem 3.2.18.** [21] Let $T = U|T|$, $S = V|S|$ and $|T| = W|T| |S^*| = W$ be the polar decompositions. Then

$$TS = UWV|TS|$$

is also the polar decomposition.

This theorem can be used to obtain the polar decomposition of the Duggal transformation of an invertible operator as follows.

**Theorem 3.2.19.** Let $T \in \mathcal{L}(\mathcal{H})$ be invertible. If $T = U|T|$ is the polar decomposition of $T$, then

$$\hat{T} = U|\hat{T}|$$

is the polar decomposition of $\hat{T}$.

**Proof.** Since $T$ is invertible, $U$ is unitary and $|T|$ is invertible. We have, $|U| = |U^*| = I$ the identity operator, because $UU^* = U^*U = I = I^2$ and $I \geq 0$. Since
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\[ \ker I = \ker U = \ker U^* = \{0\}, \]
we see that \( U = UI \) is the polar decomposition of \( U \). We have \( |T|^* |T| = |T|^2 \) and \( |T| \geq 0 \). Therefore, \( |T| \) is invertible, \( \ker |T| = \ker I = \{0\} \), and hence \( |T| = I \ | |T| \) is the polar decomposition of \( |T| \). Also \( |T| |U^*| = |T| \). Replacing \( T \) and \( S \) by \( |T| \) and \( |U| \) respectively in 3.2.18, we obtain \( |T|U = I|U|U \ | |T|U | \) is the polar decomposition of \( |T|U \), or in other words, \( \hat{T} = U|\hat{T}| \) is the polar decomposition of \( \hat{T} \). \( \square \)

Remark 3.2.20. Theorem 3.2.19 shows that the Duggal analogue of theorem 3.2.3 is false. By theorem 3.2.19, whenever \( T = U |T| \) is invertible, \( \hat{T} = U|\hat{T}| \). An invertible operator need not always be binormal. For an example of invertible non-binormal operator, consider the matrix \( \tilde{T} \) in example 3.2.7.

Lemma 3.2.21. If \( U \) is a partial isometry, then \( |U| = U^*U \) and \( U = U |U| \) is the polar decomposition of \( U \). Also, \( \hat{U} = \tilde{U} = U^*UU \).

Proof. Since \( U^*U \) is a projection, \( (U^*U)^2 = U^*U \) and \( U^*U \geq 0 \). Therefore, \( |U| = (U^*U)^{1/2} = U^*U \). Since \( U^*U \) is the support of \( U \), we have \( UU^*U = U \).

i.e., \( U |U| = U \). The kernel condition for the polar decomposition is satisfied automatically. Hence \( U = U |U| \) is the polar decomposition of \( U \).

Since \( |U| = U^*U = (U^*U)^2 \) and \( U^*U \geq 0 \), we have \( |U|^{1/2} = U^*U \). Therefore, \( \hat{U} = |U|U = U^*UU \) and \( \tilde{U} = |U|^{1/2}U \ |U|^{1/2} = (U^*U)(U^*U) = U^*UU \). \( \square \)

Let \( T = U|T| \) and \( S = V|S| \) be the polar decompositions. The following theorem gives an equivalent condition so that \( TS = UV |TS| \) becomes the polar decomposition.

Theorem 3.2.22. [21] Let \( T = U|T| \) and \( S = V|S| \) be the polar decompositions. Then \( |T| |S^*| = |S^*| |T| \) if and only if \( TS = UV|TS| \)
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is the polar decomposition.

We use this theorem to prove the following result on the polar decomposition of the Duggal transformation of an operator.

**Theorem 3.2.23.** Let \( T = U|T| \) be the polar decomposition of \( T \). Then \( \hat{T} = \hat{U} |\hat{T}| \) is the polar decomposition of \( \hat{T} \) if and only if \( |T| \, |U^*| = |U^*| \, |T| \).

**Proof.** Since \( U \) is a partial isometry, \( (\ker U)^\perp \) is the initial space of \( U \). Since \( U^*U \) is the support of both \( T \) and \( U \), we see that \( \text{ran} \,(U^*U) = (\ker U)^\perp = (\ker T)^\perp \).

Also, \( U^*U \) is self-adjoint. Therefore, \( \ker |T| = \ker T = [\text{ran} \,(U^*U)]^\perp = \ker (U^*U) \). Every projection is a partial isometry, in particular, \( U^*U \) is a partial isometry. Further, \( (U^*U) \, |T| = |T| \, (U^*U) = |T| \). Hence \( |T| = (U^*U) \, |T| \) is the polar decomposition of \( |T| \), recalling \( |T| = |T| \).

By lemma 3.2.21, \( U = U \, |U| \) is the polar decomposition of \( U \) and \( \hat{U} = U^*UU \).

Replacing \( T \) and \( S \) in theorem 3.2.22 by \( |T| \) and \( U \) respectively, we see that

\[
| \, |T| \, | \cdot |U^*| = |U^*| \cdot | \, |T| \, |
\]

if and only if

\[
|T|U = U^*UU \mid |T|U \mid
\]

is the polar decomposition of \( |T|U \).

ie.,

\[
|T| \cdot |U^*| = |U^*| \cdot |T|
\]

if and only if

\[
\hat{T} = U^*UU \mid \hat{T} \mid
\]

is the polar decomposition of \( \hat{T} \).
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\[ |T| \cdot |U^*| = |U^*| \cdot |T| \]

if and only if

\[ \hat{T} = \hat{U} |\hat{T}| \]

is the polar decomposition of \( \hat{T} \).

Let \( T = U|T| \) be the polar decomposition of \( T \). A theorem in [21] says that \( T \) is binormal if and only if \( \tilde{T} = \tilde{U} |\tilde{T}| \) is the polar decomposition of the Aluthge transformation \( \tilde{T} \). In the following two theorems we discuss the similar situation of Duggal transformations.

**Theorem 3.2.24.** Let \( T = U|T| \) be the polar decomposition of \( T \). If \( T \) is binormal, then \( \hat{T} = \hat{U} |\hat{T}| \) is the polar decomposition of \( \hat{T} \).

**Proof.** Let \( F =UU^* \). Then \( F \) is the support of \( T^* \). If \( T \) is binormal, then \( \ker |T^*| \) is invariant under \( |T| \). Therefore, \( (\ker |T^*|)^\perp \) is invariant under \( |T| \). But \( (\ker |T^*|)^\perp = (\ker T^*)^\perp = \text{ran } F \). Therefore, \( F |T| F = |T| F \), and hence \( |T| F = F |T| \). It follows that \( |T| |U^*| = |U^*| |T| \). By theorem 3.2.23, \( \hat{T} = \hat{U} |\hat{T}| \) is the polar decomposition of \( \hat{T} \). \( \square \)

**Theorem 3.2.25.** Let \( T = U|T| \) be the polar decomposition of \( T \), and \( E, F \) the initial and final projections, respectively, of the partial isometry \( U \). If \( \hat{T} = \hat{U} |\hat{T}| \) is the polar decomposition of \( \hat{T} \), then \( EF = FE \), or equivalently, \( U \) is binormal.

**Proof.** We have, \( E = U^*U \) and \( F = UU^* \). If \( \hat{T} = \hat{U} |\hat{T}| \) is the polar decomposition of \( \hat{T} \), then by theorem 3.2.23, \( |T| |U^*| = |U^*| |T| \). Thus, \( |T| F = F |T| \), and therefore, \( \text{ran } |T| \) is invariant under \( F \). Hence \( (\text{ran } |T|)^\perp \) is invariant under \( F \). But \( (\text{ran } |T|)^\perp = (\ker T)^\perp = \text{ran } E \). Hence \( EF = FE \).

Next, \( U \) is binormal, if and only if, \( |U| |U^*| = |U^*| |U| \), if and only if, \( EF = FE \). \( \square \)
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Remark 3.2.26. Let $T$ be an operator, and $T = U|T|$ the polar decomposition of $T$. Let $E, F$ be the initial and final projections, respectively, of the partial isometry $U$. By lemma 3.2.21, $\hat{U} = \tilde{U} = U^*UU$. Theorem 3.1 in [21] says that if $U$ is a partial isometry, then $U$ is binormal, if and only if, $\tilde{U}$ is a partial isometry, if and only if, $U^2$ is a partial isometry. Thus if $\hat{T} = \hat{U} |\hat{T}|$ is the polar decomposition of $\hat{T}$, then the following hold.

i. $|T| |U^*| = |U^*| |T|.$

ii. $EF = FE.$

iii. $U$ is binormal.

iv. $\hat{U}$ is a partial isometry.

v. $\tilde{U}$ is a partial isometry.

vi. $U^2$ is a partial isometry.

On the other hand, if $T$ is binormal, then $\hat{T} = \hat{U} |\hat{T}|$ is the polar decomposition of $\hat{T}$, and this in turn implies each of the above statements.

Theorem 3.2.27. Let $T = U |T|$ be the polar decomposition of $T$. If $\hat{T}^{(n)} = \hat{U}^{(n)} |\hat{T}^{(n)}|$ is the polar decomposition of $\hat{T}^{(n)}$ for all $n = 1, 2, \ldots$, then $|U^*|$ commutes with every $|\hat{T}^{(n)}|$, or in other words, $\text{ran} U$ is invariant under every $|\hat{T}^{(n)}|.$

Proof. Let $E = U^*U$ and $F = UU^*$. First note that $\text{ran} U = \text{ran} F$.

By theorem 3.2.23 and theorem 3.2.25, we have

$$|T| |U^*| = |U^*| |T|$$

(3.1)

and $EF = FE.$
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Since \((\hat{U})^*\) is a partial isometry,

\[
|(\hat{U})^*| = \hat{U}(\hat{U})^* = U^*UU^*U = EFE = EF.
\]

Also,

\[
(\hat{T})^*\hat{T} = (|T|U)^*|T|U = U^*|T|^2U \\
= U^*|T|^2UU^*U \\
= U^*|T|UU^*|T|U \text{ by (3.1)} \\
= (U^*|T|U)^2
\]

and \(U^*|T|U \geq 0\). Therefore, \(|\hat{T}| = U^*|T|U\).

Since \(\hat{T}^{(2)} = \hat{U}^{(2)}|\hat{T}^{(2)}|\) and \(\hat{T} = \hat{U}|\hat{T}|\) are polar decompositions, by theorem 3.2.23,

\[
|\hat{T}| |(\hat{U})^*| = |(\hat{U})^*| |\hat{T}|.
\]

But

\[
|\hat{T}| |(\hat{U})^*| = U^*|T|UEF = U^*|T|UF = |\hat{T}|F
\]

and

\[
|(\hat{U})^*| |\hat{T}| = EFU^*|T|U = FEU^*|T|U = FU^*|T|U = F|\hat{T}|.
\]

Thus \(|\hat{T}|F = F|\hat{T}|\). Therefore, \(|\hat{T}| |U^*| = |U^*| |\hat{T}|\).

Next, since \(\hat{T}^{(n)} = \hat{U}^{(n)}|\hat{T}^{(n)}|\) is the polar decomposition of \(\hat{T}^{(n)}\) for \(n \leq 3\), by the same argument as above, we see that

\[
|\hat{T}^{(2)}| = (\hat{U})^*|\hat{T}|(\hat{U}).
\]

Also,

\[
|\hat{T}^{(2)}||{(\hat{U})^*}||\hat{U}| = |(\hat{U})^*||\hat{T}^{(2)}|.
\]
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ie.,

\[ |\hat{T}^{(2)}| EF = EF |\hat{T}^{(2)}|. \]

Now, \( \hat{U}E = U^*UU^*U = U^*UU = \hat{U} \) and \( E(\hat{U})^* = (\hat{U})^* \). Therefore,

\[ |\hat{T}^{(2)}| EF = |\hat{T}^{(2)}| F \]

and

\[ EF |\hat{T}^{(2)}| = FE |\hat{T}^{(2)}| = F |\hat{T}^{(2)}|. \]

Thus

\[ |\hat{T}^{(2)}| F = F |\hat{T}^{(2)}|. \]

Therefore, \( |\hat{T}^{(2)}| |U^*| = |U^*| |\hat{T}^{(2)}|. \)

Proceeding like this, we obtain \( |\hat{T}^{(n)}| |U^*| = |U^*| |\hat{T}^{(n)}| \) for all \( n \). \( \square \)

3.2.3 Semigroup properties of factors in the polar decomposition and some applications

Masatoshi Ito, Takeaki Yamazaki and Masahiro Yanagida in [21] proved the following theorem 3.2.28 using properties of the polar decomposition of the product of operators. In this section we give a modification of the proof of this theorem using semigroup properties of factors in the polar decomposition.

**Theorem 3.2.28.** [21] Let \( T \in \mathcal{L}(\mathcal{H}) \) and \( T = U|T| \) be the polar decomposition of \( T \). Then the following are equivalent.

i. \( T \) is centered.

ii. \( \hat{T}^{(n)} = \hat{U}^{(n)} |\hat{T}^{(n)}| \) is the polar decomposition for all nonnegative integer \( n \).

iii. \( T^n = U^n |T^n| \) is the polar decomposition for all natural number \( n \).
3.2. Aluthge and Duggal transformations of binormal operators

Theorem 3.2.29. [21] Let $T \in \mathcal{L}(\mathcal{H})$ and $T = U|T|$ be the polar decomposition of $T$. Then $T$ is binormal if and only if $\tilde{T} = \tilde{U} \tilde{|T|}$ is the polar decomposition of $\tilde{T}$.

Definition 3.2.30. Let $S$ be a commutative semigroup with unit. A mapping $\pi : S \rightarrow \mathcal{L}(\mathcal{H})$ is called a

i. **semigroup homomorphism** if $\pi(s + t) = \pi(s)\pi(t)$, $s, t \in S$ and $\pi(0) = I$.

ii. **normal homomorphism** if the set $\{\pi(s), \pi(t)^*, s, t \in S\}$ is commutative.

iii. **quasinormal homomorphism** if the set $\{\pi(s)^*\pi(s), \pi(t), s, t \in S\}$ is commutative.

iv. **subnormal homomorphism** if there exists a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and a normal homomorphism $\tau : S \rightarrow \mathcal{L}(\mathcal{H})$ such that $\mathcal{H}$ is invariant for each $\tau(s)$ and $\tau(s)|_{\mathcal{H}} = \pi(s)$, $s \in S$.

v. **centered homomorphism** if the set $\{\pi(s)^*\pi(s), \pi(t)^*\pi(t), s, t \in S\}$ is commutative.

All these special kinds of homomorphisms are, clearly, assumed to be semigroup homomorphisms. Notice that $\mathcal{L}(\mathcal{H})$ is considered here as a semigroup under the operation of multiplication of operators.

Let $\pi : S \rightarrow \mathcal{L}(\mathcal{H})$ be a semigroup homomorphism. Let

$$\pi(s) = \theta(s)\mu(s)$$

be the polar decomposition of the operator $\pi(s)$, $s \in S$.

The following two theorems in [9] discuss the semigroup properties of the factors $\theta$ and $\mu$ in the polar decomposition of $\pi$. 
Theorem 3.2.31. [9]

i. If $\pi$ is centered, then $\theta$ is a semigroup homomorphism.

ii. Assume additionally that for each $s, t \in S$ there exists $r \in S$ such that $s = t + r$ or $t = s + r$. If $\theta$ is a semigroup homomorphism, then $\pi$ is centered.

The semigroup $\mathbb{N}$ satisfies the additional condition in (ii) of theorem 3.2.31, but the semigroup $\mathbb{N} \times \mathbb{N}$ does not.

Theorem 3.2.32. [9] Let $S$ be a commutative semigroup with unit and let $\pi : S \to \mathcal{L}(\mathcal{H})$ be a semigroup homomorphism. $\pi$ is a quasinormal homomorphism if and only if $\mu$ is a semigroup homomorphism.

Proof of theorem 3.2.28. $\mathbb{N}$ is a semigroup that satisfies the additional condition in (ii) of theorem 3.2.31. Let $\pi : \mathbb{N} \to \mathcal{L}(\mathcal{H})$ be defined by

$$\pi(n) = T^n, \ n \in \mathbb{N}.$$ 

Then $\pi$ is a semigroup homomorphism. Let

$$\pi(n) = \theta(n)\mu(n)$$ 

be the polar decomposition of the operator $\pi(n), \ n \in \mathbb{N}$. Now, $\theta(1) = U$ and $\mu(1) = |T|$. 

Assume that $T$ is centered. Then the set

$$\{\pi(n)^*\pi(n), \pi(m)^*\pi(m), \ n, m \in \mathbb{N}\} = \{(T^n)^*T^n, T^n(T^m)^*, \ n, m \in \mathbb{N}\}$$ 

is commutative, and therefore, we see that $\pi$ is a centered homomorphism. By
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Theorem 3.2.31, \( \theta : \mathbb{N} \to \mathcal{L}(\mathcal{H}) \) is a semigroup homomorphism. Therefore, for every \( n \in \mathbb{N} \), \( U^n = (\theta(1))^n = \theta(n) \).

Hence the polar decomposition of \( T^n = \pi(n) \) is

\[
T^n = \theta(n)\mu(n) = \theta(n)\vert\pi(n)\vert = U^n |T^n|
\]

for every \( n \in \mathbb{N} \). This proves (i) \( \Rightarrow \) (iii).

Assume that \( T^n = U^n |T^n| \) is the polar decomposition for all natural number \( n \). Then \( \theta(n) = U^n \) for all \( n \in \mathbb{N} \). Therefore, for \( n, m \in \mathbb{N} \),

\[
\theta(n + m) = U^{n+m} = U^n U^m = \theta(n)\theta(m),
\]

and hence \( \theta \) is a semigroup homomorphism. By theorem 3.2.31, \( \pi \) is a centered homomorphism. It follows that \( T \) is centered. This proves (iii) \( \Rightarrow \) (i).

The rest of the proof is essentially the same as that in [21]. We give it here for completeness.

To prove (i) \( \Rightarrow \) (ii). Assume that \( T \) is centered. By theorem 3.2.12, \( \tilde{T}^{(n)} \) is binormal for all \( n \in \mathbb{N} \). By theorem 3.2.29,

\[
\tilde{T} = \tilde{U} \vert \tilde{T} \vert \text{ is the polar decomposition of } \tilde{T}.
\]

Next since \( \tilde{T} \) is binormal and since (3.2) holds, we have by theorem 3.2.29,

\[
\tilde{T}^{(2)} = \tilde{U}^{(2)} \vert \tilde{T}^{(2)} \vert \text{ is the polar decomposition of } \tilde{T}^{(2)}.
\]

Repeating this method, we have \( \tilde{T}^{(n)} = \tilde{U}^{(n)} \vert \tilde{T}^{(n)} \vert \) is the polar decomposition for all nonnegative integer \( n \).

To prove (ii) \( \Rightarrow \) (i). Assume that \( \tilde{T}^{(n)} = \tilde{U}^{(n)} \vert \tilde{T}^{(n)} \vert \) is the polar decomposition
for all nonnegative integer \( n \). By theorem 3.2.29, \( \tilde{T}^{(n)} \) is binormal for all \( n \in \mathbb{N} \). Hence \( T \) is centered by theorem 3.2.12.

\[ 3.3 \] Spectral sets and numerical range of Aluthge and Duggal transformations

In [29], M. Schreiber characterized by means of normal dilations those operators the closure of whose numerical range is a spectral set. He obtained results on the equality of the convex hull of the spectrum with the closure of the numerical range, in relation to the spectrality of the numerical range. In this section we discuss Aluthge and Duggal transformations in the context of these results.

The Toeplitz - Hausdorff theorem states that the numerical range \( W(T) \) of an operator \( T \in \mathcal{L}(\mathcal{H}) \) is always convex. If \( T \) is normal, then \( W(T) \) is the closed convex hull \( \mathcal{C}(\sigma(T)) \) of the spectrum \( \sigma(T) \) of \( T \). These facts can be seen in [18].

We defined dilations in 1.2.10 as follows. Let \( T \in \mathcal{L}(\mathcal{H}) \). If there exists a normal operator \( N \) on a larger Hilbert space \( \mathcal{K} \supset \mathcal{H} \) such that \( Tx = PNx \) for all \( x \in \mathcal{H} \), where \( P \) is the orthogonal projection of \( \mathcal{K} \) onto \( \mathcal{H} \), then \( N \) is called a normal dilation of \( T \). If, in addition, \( T^n x = PN^n x \) for all \( x \in \mathcal{H} \) and all \( n = 0, 1, 2, \ldots \), then \( N \) is called a strong normal dilation of \( T \).

**Theorem 3.3.1.** [16] Every spectral set for \( T \) is a spectral set for \( \tilde{T} \) and a spectral set for \( \hat{T} \). Also \( \overline{W(T)} \subset \overline{W(T)} \) and \( \overline{W(T)} \subset \overline{W(T)} \).

**Theorem 3.3.2.** [29] Let \( \mathcal{C}(X) \) denote the convex hull of \( X \). If \( \mathcal{C}(\sigma(T)) \) is a spectral set for \( T \), then there exists a strong normal dilation \( N \) of \( T \) such that \( \mathcal{C}(\sigma(T)) = \overline{W(T)} = \overline{W(N)} \).

**Theorem 3.3.3.** [29] If there exists a strong normal dilation \( N \) of \( T \) such that \( \overline{W(N)} = \overline{W(T)} \), then \( \overline{W(T)} \) is a spectral set for \( T \).
3.3. Spectral sets and numerical range

**Theorem 3.3.4.** [2] Let $A$ be an $n \times n$ matrix over $C$. The convex hull of the eigen values of $A$ equals $W(A)$ if and only if $A$ and $\tilde{A}$ have the same numerical range.

**Theorem 3.3.5.** [18] If $\mathcal{H}$ is a finite dimensional Hilbert space and $T \in \mathcal{L}(H)$, then the numerical range $W(T)$ is compact.

**Theorem 3.3.6.** Let $T \in \mathcal{L}(H)$ be such that $C(\sigma(T)) = W(T)$. Then $\overline{W(\tilde{T})} = W(T)$. Also $\overline{W(\hat{T})} = W(T)$. (In other words, if $T$ is convexoid, then so are $\tilde{T}$ and $\hat{T}$).

*Proof.* By theorem 3.3.1, $\overline{W(\tilde{T})} \subset \overline{W(T)} = C(\sigma(T)) = \overline{W(\tilde{T})} \subset \overline{C(\sigma(T))}$. The proof of $\overline{W(\hat{T})} = W(T)$ is similar. \qed

**Remark 3.3.7.** This theorem generalizes one part of Ando’s theorem 3.3.4. The finite dimensional case was discussed in remark 2.3.6.

**Theorem 3.3.8.** Let $T \in \mathcal{L}(H)$ be such that $C(\sigma(T)) = \overline{W(T)}$. Suppose there exists a strong normal dilation $N$ of $T$ such that $\overline{W(N)} = \overline{W(T)}$. Then

(a) there exists a strong normal dilation $N_1$ of $\tilde{T}$ such that $\overline{W(N_1)} = \overline{W(T)}$, 

(b) there exists a strong normal dilation $N_2$ of $\hat{T}$ such that $\overline{W(N_2)} = \overline{W(T)}$.

( Remembe $\overline{W(T)} = \overline{W(\tilde{T})} = \overline{W(\hat{T})}$, in this case by theorem 3.3.6 ).

*Proof.* By theorem 3.3.3, $\overline{W(T)}$ is a spectral set for $T$. By theorem 3.3.1, $\overline{W(T)}$ is a spectral set for $\tilde{T}$. But $C(\sigma(\tilde{T})) = C(\sigma(T)) = \overline{W(T)}$. Thus $C(\sigma(\tilde{T}))$ is a spectral set for $\tilde{T}$. By theorem 3.3.2, there exists a strong normal dilation $N_1$ of $\tilde{T}$ such that $\overline{W(N_1)} = \overline{W(T)}$, proving (a).

The proof of (b) is similar. \qed
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Theorem 3.3.9. Let $T \in \mathcal{L}(H)$. If $\mathcal{C}(\sigma(T))$ is a spectral set for $T$, then $\overline{W(T)} = \overline{W(\tilde{T})} = \overline{W(\hat{T})}$.

Proof. If $\mathcal{C}(\sigma(T))$ is a spectral set for $T$, by theorem 3.3.2, $\mathcal{C}(\sigma(T)) = \overline{W(T)}$. Therefore, by theorem 3.3.6, $\overline{W(T)} = \overline{W(\tilde{T})} = \overline{W(\hat{T})}$. \hfill $\Box$

Corollary 3.3.10. Let $A$ be an $n \times n$ matrix over $\mathbb{C}$. If $\mathcal{C}(\sigma(A))$ is a spectral set for $A$, then $A, \tilde{A},$ and $\hat{A}$ have the same numerical range.

Proof. Obvious by theorem 3.3.5 and theorem 3.3.9. \hfill $\Box$