Chapter 2

Aluthge and Duggal transformations

2.1 Introduction

Let $\mathcal{H}$ be a separable Hilbert space with $2 \leq \text{dim}\mathcal{H} \leq \aleph_0$ and $\mathcal{L}(\mathcal{H})$ be the $C^*$-algebra of all bounded linear operators on $\mathcal{H}$. Let $T \in \mathcal{L}(\mathcal{H})$. Let $T = U|T|$ be the unique polar decomposition of $T$, where $U$ is a partial isometry such that $\ker U = \ker T = \ker |T|$ and $|T| = (T^*T)^{1/2}$. Obviously $|T|$ is a positive operator. Also $\|Tx\| = \|T|x\|$ for all $x \in \mathcal{H}$, and if $E = U^*U$ then $E$ is the initial projection of $U$ (ie., $E = P_X$ where $X = (\ker U)^\perp = (\ker T)^\perp$) and $E$ is the support of $T$ as well as the support of $|T|$. The following definition is due to Aluthge [1].

Definition 2.1.1 (Aluthge transformation [1]). If $T \in \mathcal{L}(\mathcal{H})$ and $T = U|T|$ is the polar decomposition of $T$, then

$$\tilde{T} = |T|^{1/2}U|T|^{1/2}$$

is called the Aluthge transformation of $T$.

Definition 2.1.2 (Duggal transformation [16]). If $T \in \mathcal{L}(\mathcal{H})$ and $T = U|T|$ is
the polar decomposition of $T$, then

$$\hat{T} = |T|U$$

is called the Duggal transformation of $T$.

**Definition 2.1.3** ($\lambda$–Aluthge transformation). If $T \in \mathcal{L}(\mathcal{H})$, $T = U|T|$ the polar decomposition of $T$, and $0 < \lambda < 1$, then $\Delta_\lambda(T) = |T|^\lambda U|T|^{1-\lambda}$ is called the $\lambda$–Aluthge transformation of $T$. When $\lambda = 1/2$, the $\lambda$–Aluthge transformation is the Aluthge transformation.

The notion of Aluthge transformation was first studied in [1] in relation with the $p$–hyponormal and log–hyponormal operators. Roughly speaking, the Aluthge transformation of an operator is closer to being normal. Aluthge transformation has received much attention in recent years. One reason is the connection of Aluthge transformation with the invariant subspace problem. Jung, Ko and Pearcy proved in [22] that $T$ has a nontrivial invariant subspace if and only if $\tilde{T}$ does. Another reason is related with the iterated Aluthge transformation.

In [16], Foias, Jung, Ko and Pearcy introduced the the concept of Duggal transformations, and proved several analogous results for Aluthge transformations and Duggal transformations. Yamazaki in [35] proved that for every $T \in \mathcal{L}(\mathcal{H})$, the sequence of the norms of the Aluthge iterates of $T$ converges to the spectral radius $r(T)$. Derming Wang in [33] gave another proof of this result. We started studying Aluthge and Duggal transformations hoping to prove, the analogue of the result of Yamazaki, that the sequence of the norms of the Duggal iterates of $T$ converges to the spectral radius $r(T)$, for every $T \in \mathcal{L}(\mathcal{H})$. Several finite dimensional examples suggested that the result is true for Duggal transformations. We succeeded in proving that the sequence of the norms of the Duggal iterates converges to the spectral radius, for certain classes of operators. We give this result as theorem 2.2.2. Further investigation led to an example of
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a finite dimensional operator showing that there exist operators such that the sequence of the norms of the Duggal iterates does not converge to the spectral radius. We exhibit this example in section 2.2.3.

**Definition 2.1.4.** Let $\mathcal{H}$ be a Hilbert space and let $T \in \mathcal{L}(\mathcal{H})$.

i. $T$ is called **hyponormal** if $T^*T \geq TT^*$.

ii. For $p > 0$, $T$ is $p$-**hyponormal** if $(T^*T)^p \geq (TT^*)^p$. (Thus $1$-hyponormal means simply hyponormal).

iii. If $T$ is invertible, $T$ is called **log-hyponormal** if $\log T^*T \geq \log TT^*$.

**Theorem 2.1.5.** [1] Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{L}(\mathcal{H})$.

i. For $0 < p < 1/2$, if $T$ is $p$-hyponormal, then $\tilde{T}$ is $p + 1/2$-hyponormal.

ii. For $1/2 \leq p \leq 1$, if $T$ is $p$-hyponormal, then $\tilde{T}$ is $1$-hyponormal.

**Theorem 2.1.6.** [31] Let $\mathcal{H}$ be a Hilbert space, $T \in \mathcal{L}(\mathcal{H})$, and $T$ be invertible. If $T$ is log-hyponormal, then $\tilde{T}$ is $1/2$-hyponormal.

It is well known that $\sigma(T) = \sigma(\tilde{T}) = \sigma(\hat{T})$ ([22], [16]). The following theorem shows some known results.

**Theorem 2.1.7.** Let $T \in \mathcal{L}(\mathcal{H})$.

i. $\| \tilde{T} \| \leq \| T \|$, $\| \hat{T} \| \leq \| T \|$.

ii. $T$ is quasinormal if and only if $T = \tilde{T}$ if and only if $T = \hat{T}$.

**Proof.** Let $T = U|T|$ be the polar decomposition of $T$. One can note that (if $U \neq 0$) $\| U \| = 1$ (see the remark 1.2.3 on page 12). Further, one can see that $\| T \| = \| |T|^2 \|^{1/2} = \| |T| \| = \| |T|^{1/2} \|^2$ and hence $\| |T|^{1/2} \| = \| T \|^{1/2}$ [24].
Now

\[
\| \tilde{T} \| = \| |T|^{1/2} U |T|^{1/2} \| \\
\leq \| |T|^{1/2} \| \cdot \| U \| \cdot \| |T|^{1/2} \| \\
= \| |T|^{1/2} \|^2 \\
= \| T \| \tag{2.1}
\]

\[
\| \hat{T} \| = \| |T| U \| \\
\leq \| |T| \| \cdot \| U \| \\
= \| |T| \| \\
= \| T \| \tag{2.2}
\]

Further, \( T = \hat{T} \implies T = |T| U \implies U |T| = |T| U \implies |T| \) commutes with \( U \implies |T|^{1/2} \) commutes with \( U \implies |T|^{1/2} U = U |T|^{1/2} \implies |T|^{1/2} U |T|^{1/2} = U |T| \implies \tilde{T} = T \). On the other hand, \( T = \tilde{T} \implies T = |T|^{1/2} U |T|^{1/2} \implies T |T|^{1/2} = |T|^{1/2} U |T| \implies T |T|^{1/2} = |T|^{1/2} T \implies |T|^{1/2} \) commutes with \( T \implies |T| \) commutes with \( T \implies T^* T \) commutes with \( T \implies T \) is quasinormal \( \implies U \) and \( |T| \) commute (see [18]) \( \implies |T| U = U |T| \implies \hat{T} = T \). Also, \( T \) is quasinormal \( \iff U \) and \( |T| \) commute \( \iff \hat{T} = T \). Thus \( T \) is quasinormal \( \iff T = \hat{T} \iff T = \tilde{T} \).

\[
\Box
\]

**Definition 2.1.8.** For \( T \in \mathcal{L}(\mathcal{H}) \), denote by \( Hol(\sigma(T)) \) the algebra of all complex-valued functions which are analytic on some neighborhood of \( \sigma(T) \), where linear combinations and products in \( Hol(\sigma(T)) \) are defined (with varying domains) in the obvious way. The (Riesz-Dunford) algebra \( \mathcal{A}_T \subseteq \mathcal{L}(\mathcal{H}) \) is defined as

\[
\mathcal{A}_T = \{ f(T) : f \in Hol(\sigma(T)) \},
\]

where the operator \( f(T) \in \mathcal{L}(\mathcal{H}) \) is defined by the Riesz-Dunford functional calculus as in 1.3.5.
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The following theorem in [16] gives useful information about \( \tilde{T} \) and \( \hat{T} \) by studying maps between the algebras \( A_T, A_{\tilde{T}} \) and \( A_{\hat{T}} \).

**Theorem 2.1.9.** [16] *For every* \( T \in \mathcal{L}(\mathcal{H}) \), with \( \tilde{T}, \hat{T} \), and \( \text{Hol}(\sigma(T)) \) *as defined above:*

(a) The maps \( \Phi : A_T \to A_{\tilde{T}} \) and \( \hat{\Phi} : A_T \to A_{\hat{T}} \) defined by

\[
\Phi(f(T)) = f(\tilde{T}), \quad \hat{\Phi}(f(T)) = f(\hat{T}), \quad f \in \text{Hol}(\sigma(T))
\]

are well defined contractive algebra homomorphisms. Thus

\[
\max\{\| f(\tilde{T}) \|, \| f(\hat{T}) \| \} \leq \| f(T) \|, \quad f \in \text{Hol}(\sigma(T)).
\]

(b) More generally, the maps \( \Phi \) and \( \hat{\Phi} \) are completely contractive, meaning that for every \( n \in \mathbb{N} \) and every \( n \times n \) matrix \( (f_{ij}) \) with entries from \( \text{Hol}(\sigma(T)) \),

\[
\max\{\| (f_{ij}(\tilde{T})) \|, \| (f_{ij}(\hat{T})) \| \} \leq \| (f_{ij}(T)) \|.
\]

(The norm here is the natural norm in the \( C^* \)-algebra \( \mathcal{M}_n(\mathcal{L}(\mathcal{H})) \)).

(c) Every spectral set for \( T \) is a spectral set for both \( \tilde{T} \) and \( \hat{T} \). For fixed \( K > 1 \), every \( K \)-spectral set for \( T \) is a \( K \)-spectral set for both \( \tilde{T} \) and \( \hat{T} \).

(d) If \( W(S) \) denotes the numerical range of an operator \( S \) in \( \mathcal{L}(\mathcal{H}) \), then

\[
\overline{W(f((\tilde{T}))) \cup W(f((\hat{T})))} \subset \overline{W(f((T)))}, \quad f \in \text{Hol}(\sigma(T))
\]

(the overbar denoting the closure).

**Remark 2.1.10.** Aluthge transformations and Duggal transformations enjoy several analogous properties. The following are some.
2.2. Aluthge and Duggal iterates

2.2.1 Aluthge and Duggal iterates

Definition 2.2.1 (Iterated Aluthge transformations). Denote \( \tilde{T}^{(0)} = T \), \( \tilde{T}^{(1)} = \tilde{T} \), \( \tilde{T}^{(2)} = \tilde{T}^{(1)} \), \ldots , \( \tilde{T}^{(n)} = \tilde{T}^{(n-1)} \), \ldots .

For every \( T \in \mathcal{L}(\mathcal{H}) \), the sequence \( \{ \| \tilde{T}^{(n)} \| \}_{n=0}^{\infty} \) is decreasing such that \( r(T) \leq \| \tilde{T}^{(n)} \| \leq \| T \| . \) (Proof: Since \( \sigma(T) = \sigma(\tilde{T}) = \sigma(\tilde{T}^{(n)}) \) for all \( n \in \mathbb{N} \), we have, \( r(T) = r(\tilde{T}^{(n)}) \leq \| \tilde{T}^{(n)} \| \) for all \( n \in \mathbb{N} \). The fact \( \| \tilde{T}^{(n)} \| \leq \| T \| \) follows easily from an application of the inequality (2.1) on page 30.) Hence

i. \( \| \tilde{T} \| \leq \| T \| , \| \hat{T} \| \leq \| T \| . \)

ii. \( \sigma(\tilde{T}) = \sigma(T) , \sigma(\hat{T}) = \sigma(T) \)

iii. \( T \) is quasinormal \( \iff T = \tilde{T} \iff T = \hat{T} . \)

iv. \( r(T) = r(\tilde{T}) = r(\hat{T}) . \)

v. \( \| f(\tilde{T}) \| \leq \| f(T) \| , \| f(\hat{T}) \| \leq \| f(T) \| , f \in \text{Hol}(\sigma(T)) . \)

vi. Every spectral set for \( T \) is a spectral set for \( \tilde{T} \), every spectral set for \( T \) is a spectral set for \( \hat{T} . \)

vii. Every K-spectral set for \( T \) is a K-spectral set for \( \tilde{T} \), every K-spectral set for \( T \) is a K-spectral set for \( \hat{T} . \)

viii. \( \| (f_{ij}(\tilde{T})) \| \leq \| (f_{ij}(T)) \| , \| (f_{ij}(\hat{T})) \| \leq \| (f_{ij}(T)) \| \) for every positive integer \( n \) and every \( n \times n \) matrix \( (f_{ij}) \) with entries in \( \text{Hol}(\sigma(T)) . \)

ix. \( \overline{W(\tilde{T})} \subset \overline{W(T)} , \overline{W(\tilde{T})} \subset \overline{W(T)} , \) where \( \overline{W(T)} \) denotes the closure of the numerical range \( W(T) \) of \( T . \)
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\begin{align*}
\{\| (\tilde{T}^{(n)}) \| \}_{n=0}^{\infty} \text{ is a convergent sequence. In 2002, Yamazaki in the excellent paper [35] proved that for every } T \in \mathcal{L}(\mathcal{H}), \text{ the sequence of the norms of the Aluthge iterates of } T \text{ converges to the spectral radius } r(T). \end{align*}

**Theorem 2.2.2.** [35] For every } T \in \mathcal{L}(\mathcal{H}), \text{ the sequence } \{\|\tilde{T}^{(n)}\|\} \text{ converges to } r(T).

In 2003, Derming Wang in [33] used McIntosh inequality and Heinz inequality to give another proof of the above theorem.

**Definition 2.2.3** (Iterated Duggal transformations). Denote \( \hat{T}^{(0)} = T, \hat{T}^{(1)} = \hat{T}, \hat{T}^{(2)} = (\hat{T}^{(1)}), \ldots, \hat{T}^{(n)} = (\hat{T}^{(n-1)}), \ldots. \)

In the coming sections, we investigate the convergence of the norms of the Duggal iterates of a bounded linear operator on a Hilbert space.

### 2.2.2 Convergence of the norms of Duggal iterates

We shall prove that \( \lim_{n \to \infty} \| \hat{T}^{(n)} \| = r(T) \) for operators \( T \) belonging to certain classes of operators in \( \mathcal{L}(\mathcal{H}) \). By the inequality (2.2), \( \| \hat{T}^{(n+1)} \| \leq \| \hat{T}^{(n)} \| \) for all \( n \in \mathbb{N} \). Moreover \( \sigma(\hat{T}^{(n)}) = \sigma(T) \), and hence \( r(\hat{T}^{(n)}) = r(T) \) for all \( n \geq 0 \). Thus \( \{\|\hat{T}^{(n)}\|\}_{n=0}^{\infty} \) is a decreasing sequence which is bounded below by \( r(T) \).

The following lemma is an easy consequence.

**Lemma 2.2.4.** There is an \( s \geq r(T) \) for which \( \lim_{n \to \infty} \| \hat{T}^{(n)} \| = s \).

**Remark 2.2.5.** We notice one more analogy between Aluthge and Duggal transformations.

The sequence \( \{\| (\tilde{T}^{(n)}) \| \}_{n=0}^{\infty} \) is decreasing such that \( r(T) \leq \| \tilde{T}^{(n)} \| \leq \| T \| , \text{ and } r(\tilde{T}^{(n)}) = r(T) \) for all \( n \). The sequence \( \{\| (\hat{T}^{(n)}) \| \}_{n=0}^{\infty} \) is decreasing such that \( r(T) \leq \| \hat{T}^{(n)} \| \leq \| T \| , \text{ and } r(\hat{T}^{(n)}) = r(T) \) for all \( n \).
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Theorem 2.2.6 (Mc Intosh inequality). For bounded linear operators $A, B$ and $X$,

$$\| A^* X B \| \leq \| A A^* X \|^{1/2} \| X B B^* \|^{1/2}. $$

Theorem 2.2.7 (Heinz inequality). For positive linear operators $A$ and $B$, and bounded linear operator $X$,

$$\| A^\alpha X B^\alpha \| \leq \| A X B \|^\alpha \| X \|^{1-\alpha}$$

for all $0 \leq \alpha \leq 1$.

Using these inequalities we prove the following results.

Lemma 2.2.8. For any positive integer $k$,

$$\| (\hat{T}^{(n+1)})^k \| \leq \| (\hat{T}^{(n)})^k \|$$

for all $n \geq 0$. Consequently, the decreasing sequence $\{ \| (\hat{T}^{(n)})^k \| \}_{n=0}^\infty$ is convergent.

Proof. Let $f(t) = t^k$, $t \in$ a neighborhood of $\sigma(T)$, and note that $\sigma(T) = \sigma(\hat{T}^{(n)})$. We have $f \in \text{Hol}(\sigma(T))$. Applying theorem 2.1.9 (a), the proof is complete. \qed

Lemma 2.2.9. If $\hat{T}^{(n)} = U_n |\hat{T}^{(n)}|$ is the polar decomposition of $\hat{T}^{(n)}$, then for any positive integer $k$,

$$\| (\hat{T}^{(n+1)})^k \| \leq \| \hat{T}^{(n)} \|^2 (\hat{T}^{(n)})^k \|^{1/2} \| (\hat{T}^{(n)})^k U_n U_n^* \|^{1/2}$$

Proof. We have $\hat{T}^{(n+1)} = |\hat{T}^{(n)}| U_n$ and therefore $(\hat{T}^{(n+1)})^k = |\hat{T}^{(n)}|^k (\hat{T}^{(n)})^{k-1} U_n$. Hence by theorem 2.2.6,

$$\| (\hat{T}^{(n+1)})^k \| \leq \| |\hat{T}^{(n)}|^k (\hat{T}^{(n)})^{k-1} U_n \| \leq \| |\hat{T}^{(n)}|^2 (\hat{T}^{(n)})^{k-1} \|^{1/2} \| (\hat{T}^{(n)})^{k-1} U_n U_n^* \|^{1/2}$$ \qed
Lemma 2.2.10. Let $n$ be a positive integer and $T \in \mathcal{L}(\mathcal{H})$ be an operator satisfying the condition $\| |\hat{T}^{(n)}| |^2 (\hat{T}^{(n)})^{k-1} \| \leq \| (\hat{T}^{(n)})^{k+1} \|$. Then

$$\| (\hat{T}^{(n+1)})^k \| \leq \| (\hat{T}^{(n)})^{k+1} \|^{1/2} \| (\hat{T}^{(n)})^{k-1} \|^{1/2}$$

Proof. By lemma 2.2.9,

$$\| (\hat{T}^{(n+1)})^k \| \leq \| |\hat{T}^{(n)}| |^2 (\hat{T}^{(n)})^{k-1} \|^{1/2} \| (\hat{T}^{(n)})^{k-1} U_n U_n^* \|^{1/2} \leq \| (\hat{T}^{(n)})^{k+1} \|^{1/2} \| (\hat{T}^{(n)})^{k-1} U_n U_n^* \|^{1/2} \leq \| (\hat{T}^{(n)})^{k+1} \|^{1/2} \| (\hat{T}^{(n)})^{k-1} \|^{1/2}$$

since $U_n U_n^*$ is a projection.

Lemma 2.2.11. Let $n$ be a positive integer and $T \in \mathcal{L}(\mathcal{H})$ be an operator satisfying the condition $|\hat{T}^{(n)}| \hat{T}^{(n)} = \hat{T}^{(n)} |\hat{T}^{(n)}|$. Then

$$\| (\hat{T}^{(n+1)})^k \| \leq \| (\hat{T}^{(n)})^{k+1} \|^{1/2} \| (\hat{T}^{(n)})^{k-1} \|^{1/2}$$

Proof.

$$\| |\hat{T}^{(n)}| \hat{T}^{(n)} |^{k-1} \| = \| |\hat{T}^{(n)}| \| |\hat{T}^{(n)}| (\hat{T}^{(n)})^{k-1} \| = \| \hat{T}^{(n)} |\hat{T}^{(n)}| (\hat{T}^{(n)})^{k-1} \| = \| |\hat{T}^{(n)}| (\hat{T}^{(n)})^{k} \| = \| (\hat{T}^{(n)})^{k+1} \|.$$ 

By lemma 2.2.10, the result follows.

Lemma 2.2.12. Let $T \in \mathcal{L}(\mathcal{H})$ be an operator satisfying the condition

$$\| |\hat{T}^{(n)}| \hat{T}^{(n)} |^{k-1} \| \leq \| (\hat{T}^{(n)})^{k+1} \|.$$
for all $k = 1, 2, \ldots$, and for all large positive integers $n$. Then $\lim_{n \to \infty} \| (\hat{T}^{(n)})^k \| = s^k$ for any positive integer $k$.

Proof. We prove the lemma by induction on $k$. By lemma 2.2.4, the result is true for $k = 1$. Suppose that the result is true for $1 \leq k \leq m$. By lemma 2.2.10, for large $n$,

$$\| (\hat{T}^{(n+1)})^m \| \leq \| (\hat{T}^{(n)})^{m+1} \|^{1/2} \| (\hat{T}^{(n)})^{m-1} \|^{1/2} \leq \| (\hat{T}^{(n)})^m \|^{1/2} \| \hat{T}^{(n)} \|^{1/2} \| (\hat{T}^{(n)})^{m-1} \|^{1/2}$$  \hspace{1cm} (2.3)

Put $\lim_{n \to \infty} \| (\hat{T}^{(n)})^{m+1} \| = t$ (the limit exists by lemma 2.2.8). Now, taking limits as $n \to \infty$ in (2.3), the induction hypothesis shows that

$$s^m \leq t^{1/2} s^{(m-1)/2} \leq s^{m/2} s^{1/2} s^{(m-1)/2} = s^m.$$

Therefore,

$$t^{1/2} s^{(m-1)/2} = s^m.$$

Hence

$$t = s^{m+1}.$$

The lemma follows by induction. \hfill \Box

Theorem 2.2.13. If $T \in \mathcal{L}(\mathcal{H})$ is such that $\| |\hat{T}^{(n)}|^2 (\hat{T}^{(n)})^{k-1} \| \leq \| (\hat{T}^{(n)})^{k+1} \|$ for all $k = 1, 2, \ldots$, and for all large positive integers $n$, then

$$\lim_{n \to \infty} \| \hat{T}^{(n)} \| = r(T).$$

Proof. By lemma 2.2.8, we see that for each fixed positive integer $k$, the sequence $\{\| (\hat{T}^{(n)})^k \|^{1/k}\}_{n=0}^{\infty}$ is convergent, and by lemma 2.2.12, it converges to $s$. 
Therefore,

\[ s \leq \left\| (\hat{T}^{(n)})^k \right\|^{1/k} \]

for all \( n \) and \( k \). By lemma 2.2.4,

\[ r(T) \leq s. \]

Suppose, if possible, \( r(T) < s \). For every fixed \( k \), the sequence \( \left\{ \left\| (\hat{T}^{(n)})^k \right\| \right\}_{n=0}^{\infty} \) is decreasing. Now fix an \( n \). We have

\[ \left\| (\hat{T}^{(n)})^k \right\| \leq \left\| (\hat{T}^{(0)})^k \right\| = \left\| T^k \right\| \]

for all \( k \). Therefore,

\[ \left\| (\hat{T}^{(n)})^k \right\|^{1/k} \leq \left\| T^k \right\|^{1/k} \]

for all \( k \). Since \( r(T) < s \), and \( \lim_{k \to \infty} \left\| T^k \right\|^{1/k} = r(T) \), we see that

\[ \left\| (\hat{T}^{(n)})^k \right\|^{1/k} < s \]

for sufficiently large \( k \). This is a contradiction. Hence

\[ s = r(T). \]

\[ \text{ie., } \lim_{n \to \infty} \left\| \hat{T}^{(n)} \right\| = r(T). \]

\[ \square \]

**Theorem 2.2.14.** Let \( T \in \mathcal{L}(\mathcal{H}) \) be an operator satisfying the condition that \( \hat{T}^{(n)} |\hat{T}^{(n)}| = |\hat{T}^{(n)}| \hat{T}^{(n)} \) for all large positive integers \( n \). Then

\[ \lim_{n \to \infty} \left\| \hat{T}^{(n)} \right\| = r(T). \]
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Proof. (See proof of lemma 2.2.11).}

\[ \| |\hat{T}^{(n)}|^2 \hat{T}^{(n)}|^{k-1} \| = \| \hat{T}^{(n)}|^{k+1} \| \]

for all \( k = 1, 2, \ldots \), and for all large \( n \). Hence by theorem 2.2.13, the proof is complete.

Remark 2.2.15. An operator \( S \) is quasinormal if and only if \( S|S| = |S|S \). (For \( S \) is quasinormal \( \iff \) \( S \) commutes with \( S^*S \) \( \iff \) \( S \) commutes with \( (S^*S)^{1/2} = |S| \)). Thus theorem 2.2.14 says that if \( \hat{T}^{(n)} \) is quasinormal for large positive integers \( n \), then \( \lim_{n \to \infty} \| \hat{T}^{(n)} \| = r(T) \). But this is obvious since if \( \hat{T}^{(n)} \) is quasinormal for some \( n \), then \( \hat{T}^{(n)} = \hat{T}^{(n)} \) i.e., \( \hat{T}^{(n+1)} = \hat{T}^{(n)} \) and hence \( \hat{T}^{(n)} = \hat{T}^{(n)} \) for all \( m \geq n \). Being a quasinormal operator, \( \hat{T}^{(n)} \) is normaloid, and therefore, \( \| \hat{T}^{(m)} \| = \| \hat{T}^{(n)} \| = r(\hat{T}^{(n)}) = r(T) \) for all \( m \geq n \). Thus \( \lim_{n \to \infty} \| \hat{T}^{(n)} \| = r(T) \).

Corollary 2.2.16. Let \( T \in \mathcal{L}(\mathcal{H}) \) be an operator satisfying \( |(\hat{T}^{(n)})^2| = |\hat{T}^{(n)}|^2 \) for all large positive integers \( n \). Then

\[ \lim_{n \to \infty} \| \hat{T}^{(n)} \| = r(T). \]

Proof. \( \| |\hat{T}^{(n)}|^2 \hat{T}^{(n)}|^{k-1} \| = \| |(\hat{T}^{(n)})^2| \hat{T}^{(n)}|^{k-1} \| = \| \hat{T}^{(n)}|^{k+1} \| \) for all large \( n \), and for all \( k = 1, 2, \ldots \). By theorem 2.2.13, the proof follows.

Remark 2.2.17. An operator \( S \) is quasinormal if and only if \( S \) is hyponormal and \( |S^2| = |S|^2 \) \[15\]. Thus theorem 2.2.14 can be deduced as a consequence of corollary 2.2.16.

If \( \hat{T}^{(n)} \) is quasinormal for some \( n \), then obviously, \( \| \hat{T}^{(n)} \| \to r(T) \) as \( n \to \infty \) (see remark 2.2.15). Similarly, if \( \hat{T}^{(n)} \) is normaloid for some \( n \), then for all \( m \geq n \), we have \( r(T) = r(\hat{T}^{(m)}) \leq \| \hat{T}^{(m)} \| \leq \| \hat{T}^{(n)} \| = r(\hat{T}^{(n)}) = r(T) \), hence \( \| \hat{T}^{(m)} \| = r(T) \), and therefore \( \lim_{n \to \infty} \| \hat{T}^{(n)} \| = r(T) \). As a special case, if \( T \) itself is normaloid, then \( \| \hat{T}^{(n)} \| = \| T \| = r(T) \) for all \( n \).
2.2. Aluthge and Duggal iterates

Remark 2.2.18. Now we pose the crucial question. Is it true that for every $T \in \mathcal{L}(\mathcal{H})$, the sequence $\{\|\hat{T}^{(n)}\|\}_{n=0}^{\infty}$ converge to the spectral radius $r(T)$?

2.2.3 The norms of the Duggal iterates of $T$ need not converge to $r(T)$

In 2002, Yamazaki [35], proved that for every $T \in \mathcal{L}(\mathcal{H})$, the Aluthge norm sequence $\{\|\hat{T}^{(n)}\|\}_{n=0}^{\infty}$ converges to the spectral radius $r(T)$. In 2003, T. Ando and T. Yamazaki [3], proved that in the case of a $2 \times 2$ matrix the sequence of the iterated Aluthge transformations itself converges. In 2006, Jorge Antezana, Enrique R. Pujals and Demetrio Stojanoff [4], proved the convergence of iterated Aluthge transformation sequence for diagonalizable matrices.

We construct below an example showing that the analogues of these three results fail in the case of Duggal transformations. Note that we thus answer the question in remark 2.2.18 in the negative.

Example 2.2.19. Let $\mathcal{H}$ be the Hilbert space $\mathbb{C}^2$ and consider $A \in \mathcal{L}(\mathcal{H})$ given by

$$A = \begin{bmatrix} -1 & -2 \\ 2 & 1 \end{bmatrix}.$$ 

Then the polar decomposition of $A$ is $A = U|A|$, where $U = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $|A| = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. We see that $\sigma(A) = \{-\sqrt{3}i, \sqrt{3}i\}$, $\|A\| = 3$, $r(A) = \sqrt{3}$. The Duggal transformation of $A$ is

$$\hat{A} = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}.$$
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So \( \| \hat{A} \| = 3 \). The polar decomposition of \( \hat{A} \) is \( \hat{A} = U_1 |\hat{A}| \), where \( U_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \) and \( |\hat{A}| = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \). Hence the second Duggal iterate \( \hat{A}^{(2)} \) of \( A \) is

\[
\hat{A}^{(2)} = \begin{bmatrix} -1 & -2 \\ 2 & 1 \end{bmatrix} = A.
\]

This shows that

\[
\hat{A}^{(n)} = \begin{cases} A & \text{if } n \text{ is even} \\ A_1 & \text{if } n \text{ is odd} \end{cases}
\]

where \( A_1 = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix} \). Hence \( \| \hat{A}^{(n)} \| = 3 \) for all \( n \). Thus \( \{ \| \hat{A}^{(n)} \| \}_{n=0}^{\infty} \) does not converge to \( r(A) \). More obviously, \( \{ \hat{A}^{(n)} \}_{n=0}^{\infty} \) does not converge. Also note that \( A \) is a \( 2 \times 2 \) diagonalizable matrix. ( \( A \) is diagonalizable because the eigenvalues of \( A \) are distinct ).

Remark 2.2.20. In 2007, Huajun Huang and Tin-Yau Tam, proved in [19] that the iterated \( \lambda \)-Aluthge sequence converges for an \( n \times n \) matrix if the nonzero eigenvalues of the matrix have distinct moduli. Earlier in [4], Jorge Antezana, Enrique R. Pujals and Demetrio Stojanoff proved the convergence of iterated Aluthge transformation sequence for diagonalizable matrices. If \( A \) is a normal \( n \times n \) matrix over \( \mathbb{C} \), then it is always possible to choose an orthonormal basis of \( \mathbb{C}^n \) such that the corresponding matrix is diagonal [13]. Conversely if \( A \) is an \( n \times n \) matrix over \( \mathbb{C} \) and if it is possible to choose an orthonormal basis of \( \mathbb{C}^n \) such that the corresponding matrix is diagonal, then obviously \( A \) is normal. If \( A \) is normal, then \( \hat{A}^{(n)} = A \) for all \( n \in \mathbb{N} \), and hence the sequence \( \{ \hat{A}^{(n)} \} \) converges trivially. Thus the result of Antezana, Pujals and Stojanoff is trivial in the case of matrices which are diagonalizable with respect to an orthonormal basis.

The question of whether for every \( T \in \mathcal{L}(\mathcal{H}) \) the sequence of iterated Aluthge transformation sequence converge remained unanswered for some time. Recently
2.3. More on Aluthge and Duggal transformations

M. Chô, I. B. Jung, and W. Y. Lee in [10] constructed a hyponormal bilateral weighted shift $T : \ell_2(\mathbb{Z}) \to \ell_2(\mathbb{Z})$ such that $\{\tilde{T}^{(n)}\}_{n=0}^\infty$ does not converge in the norm topology. However, the convergence of iterated Aluthge transformation sequence for $T \in L(H)$, where $H$ is a finite dimensional Hilbert space remains as an open problem.

Remark 2.2.21. Even though, in general, $\{\|\tilde{T}^{(n)}\|\}_{n=0}^\infty$ does not converge to $r(T)$ (as shown in the example), there are operators $T$ for which $\|\tilde{T}^{(n)}\| \to r(T)$. For instance, if $T$ is normaloid, (in particular, if $T$ is quasinormal, subnormal, or hyponormal), then $\|\tilde{T}^{(n)}\| \to r(T)$ (see remark 2.2.17). We proved in theorem 2.2.13, for certain class of operators $\|\tilde{T}^{(n)}\| \to r(T)$.

2.3 More on Aluthge and Duggal transformations

2.3.1 Invertible operators and Duggal transformations

In 2004, T. Ando in [2] proved the remarkable result that if $A$ is an $n \times n$ matrix over $\mathbb{C}$, then the convex hull of $\sigma(A)$ equals the numerical range $W(A)$ if and only if $A$ and the Aluthge transformation $\tilde{A}$ have the same numerical range. In this section we show that in the analogous case of Duggal transformations, the implication in one direction holds, and the converse fails. We give an example to show that the converse fails even for $2 \times 2$ matrices. Also, we prove that if $S$ and $T$ are unitarily equivalent, then so are $\hat{S}$ and $\hat{T}$.

Lemma 2.3.1. If $T \in L(H)$ is invertible, then $\hat{T}$ is invertible.

Proof. If $T$ is invertible, then $T$ has the polar decomposition $T = U|T|$, where $U$ is unitary [28]. Also $|T| = U^{-1}T$. Therefore, $|T|$ is invertible, and hence, $\hat{T} = |T|U$ is invertible. \qed
Lemma 2.3.2. Let $T \in \mathcal{L}(\mathcal{H})$ be any operator. If $T = U|T|$ is the polar decomposition of $T$, then $\hat{T} = U^*TU$.

Proof. Let $E = U^*U$. Then $E$ is a projection and $E|T| = |T|$. Therefore, $U^*TU = U^*U|T|U = E|T|U = |T|U = \hat{T}$. \hfill \Box

Theorem 2.3.3. Let $T \in \mathcal{L}(\mathcal{H})$. If $V$ is unitary and $S = V^*TV$, then $\hat{S} = V^*\hat{T}V$.

Proof. We have

$$S^*S = (V^*TV)^*(V^*TV) = V^*T^*VV^*TV = V^*T^*TV = V^*|T|^2V = (V^*|T|V)(V^*|T|V)$$

and $V^*|T|V$ is positive (note that $\langle V^*|T|Vx, x \rangle = \langle |T|Vx, Vx \rangle \geq 0 \ \forall x \in \mathcal{H}$). Therefore, $|S| = V^*|T|V$.

Let $T = U|T|$ be the polar decomposition of $T$. Then $\ker U = \ker T$. Let $U_1 = V^*UV$. Now $U_1^*U_1 = V^*U^*VV^*UV = V^*U^*UV$ and since $U^*U$ is a projection

$$V^*U^*UV = V^*(U^*U)^2V = (V^*U^*UV)^2$$

and hence $V^*U^*UV = U_1^*U_1$ is a projection. Thus $U_1$ is a partial isometry.

Let $x \in \mathcal{H}$. Then $x \in \ker U_1 \iff V^*UVx = 0 \iff UVx = 0 \iff Vx \in \ker U \iff Vx \in \ker T \iff TVx = 0 \iff V^*TVx = 0 \iff x \in \ker S$. Thus $\ker U_1 = \ker S$. \hfill \Box
2.3. More on Aluthge and Duggal transformations

Hence $S = U_1|S|$ is the polar decomposition of $S$. Therefore, $\hat{S} = |S|U_1 = V^*|T|VV^*UV = V^*|T|UV = V^*\hat{T}V$.

**Theorem 2.3.4.** Let $T \in \mathcal{L}(\mathcal{H})$ be invertible. If $T = U|T|$ is the polar decomposition of $T$, then for all $n \in \mathbb{N}$, the $n$th Duggal iterate $\hat{T}^{(n)} = (U^*)^nTU^n$.

**Proof.** We prove the result by induction. Since $T$ is invertible, $U$ is unitary. By lemma 2.3.2, $\hat{T} = U^*TU$. Thus the result is true for $n = 1$. (The case $n = 0$ is trivial).

Suppose that $n \geq 2$ and assume that the result is true for all $m \leq n - 1$. Then

$$
\begin{align*}
\hat{T}^{(n-1)} &= (U^*)^{n-1}TU^{n-1} \\
&= U^*[((U^*)^{n-2}TU^{n-2}]U \\
&= U^*\hat{T}^{(n-2)}U \
\end{align*}
$$

Therefore, by theorem 2.3.3,

$$
\begin{align*}
\hat{T}^{(n)} &= U^*\hat{T}^{(n-1)}U \\
&= U^*[(U^*)^{n-1}TU^{n-1}]U \\
&= (U^*)^nTU^n.
\end{align*}
$$

**Remark 2.3.5.** If $T$ is invertible, by theorem 2.3.4, every Duggal iterate of $T$ is unitarily equivalent to $T$. So if $T$ is invertible, every Duggal iterate of $T$ has the same numerical range as that of $T$.

**Remark 2.3.6.** In [2], T. Ando proved that if $A$ is an $n \times n$ matrix over $\mathbb{C}$, then the convex hull of $\sigma(A)$ equals $W(A)$ if and only if $A$ and $\tilde{A}$ have the same numerical range.
Consider the analogous case of Duggal transformations. Let $A$ be an $n \times n$ matrix over $\mathbb{C}$, and let $\mathcal{C}(\sigma(A))$ denote the convex hull of $\sigma(A)$. Suppose that $\mathcal{C}(\sigma(A)) = W(A)$. By theorem 2.1.9, $W(\hat{A}) \subset W(A)$. Since this is a finite dimensional case, numerical ranges are compact, and hence closed. Thus $W(\hat{A}) \subset W(A) = \mathcal{C}(\sigma(A)) = \mathcal{C}(\sigma(\hat{A})) \subset W(\hat{A})$. Therefore, $W(\hat{A}) = W(A)$. Thus if the convex hull of $\sigma(A)$ equals the numerical range $W(A)$, then $A$ and the Duggal transformation $\hat{A}$ have the same numerical range.

The converse fails in the case of Duggal transformations. For example if $A$ is invertible, then $A$ and $\hat{A}$ are unitarily equivalent, and therefore, $A$ and $\hat{A}$ have the same numerical range. But in this case, the convex hull of $\sigma(A)$ need not be equal to $W(A)$, as the following example shows.

**Example 2.3.7.** Let 

$$A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}.$$ 

Then $A$ is an invertible matrix and $\sigma(A) = \{-1, 1\}$.

It is fairly standard that if $A$ is a $2 \times 2$ matrix with distinct eigen values $\alpha$ and $\beta$, and corresponding eigen vectors $f$ and $g$, so normalized that $\|f\| = \|g\| = 1$, then $W(A)$ is a closed elliptical disc with foci at $\alpha$ and $\beta$; if $\gamma = |\langle f, g \rangle|$ and $\delta = \sqrt{1 - \gamma^2}$, then the minor axis is $\gamma|\alpha - \beta|/\delta$ and the major axis is $|\alpha - \beta|/\delta$. Also, if $A$ has only one eigen value $\alpha$, then $W(A)$ is the circular disc with center $\alpha$ and radius $\frac{1}{2} \| A - \alpha \|$. The results given in this paragraph can be seen in [18].

If $A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$, then $\alpha = 1$ and $\beta = -1$ are the distinct eigen values of $A$ with corresponding eigen vectors $f = (1, 0)$ and $g = (-1/\sqrt{5}, 2/\sqrt{5})$. We have $\|f\| = \|g\| = 1$. Let $\gamma = |\langle f, g \rangle|$ and $\delta = \sqrt{1 - \gamma^2}$. Then $\gamma|\alpha - \beta|/\delta = 1$ and $|\alpha - \beta|/\delta = \sqrt{5}$.

Therefore, the numerical range $W(A)$ is the closed elliptical disc with foci at
1 and $-1$; the minor axis is 1 and the major axis is $\sqrt{5}$.

The convex hull of $\sigma(A)$ is the straight line segment with end points $(-1, 0)$ and $(1, 0)$. Thus the convex hull of $\sigma(A)$ does not equal $W(A)$. Notice that since $A$ is invertible and by lemma 2.3.2, $\hat{A}$ is unitarily equivalent to $A$. So $A$ and $\hat{A}$ have the same numerical range.

By the remark 2.3.6 and the example 2.3.7, we have discussed the complete Duggal transformation analogue of Ando’s result.

### 2.3.2 When the partial isometry in the polar decomposition is a coisometry

Let $T \in \mathcal{L(H)}$ and let $T = U|T|$ be the polar decomposition of $T$. In this section we study the Aluthge and the Duggal transformations of $T$ when the partial isometry $U$ in the polar decomposition of $T$ happens to be a coisometry. A particular case is when $U$ is actually unitary. We know that when $T$ is invertible, the partial isometry $U$ in the polar decomposition of $T$ is a unitary operator.

We introduce in this section the concept of $n$–level spectral sets. We show that if the partial isometry $U$ in the polar decomposition of $T$ is a coisometry, then the obvious algebra homomorphism between the Riez Dunford algebras $\mathcal{A}_T$ and $\mathcal{A}_{\hat{T}}$ is a complete isometry. As a consequence, we prove that in such cases, the operators $T$ and $\hat{T}$ have the same collection of complete spectral sets. Also we show that for any invertible non-normaloid $T$, the sequence of the norms of Duggal iterates of $T$ cannot converge to the spectral radius of $T$; and this result is an improvement of the results of section 2.2.3.

**Lemma 2.3.8.** If $T = U|T|$ is the polar decomposition of $T$ and if $U$ is a coisometry, then $\| \hat{T} \| = \| T \|$. 
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Proof. We have $\hat{T} = U^*TU$. Therefore, $U\hat{T}U^* = UU^*TUU^* = T$, and therefore, $\| T \| = \| U\hat{T}U^* \| \leq \| U \| \cdot \| \hat{T} \| \cdot \| U^* \| \leq \| \hat{T} \|$. But by the inequality (2.2) on page 30, $\| \hat{T} \| \leq \| T \|$. Hence $\| \hat{T} \| = \| T \|$. 

If $T$ is invertible, we can prove the following stronger result.

Theorem 2.3.9. If $T \in \mathcal{L}(H)$ is invertible, then $\| \hat{T}^{(n)} \| = \| T \|$ for all $n \in \mathbb{N}$.

Proof. If $T$ is invertible, and $T = U|T|$ is the polar decomposition of $T$, then $U$ is unitary. Also $\hat{T} = U^*TU$. By theorem 2.3.4, $\hat{T}^{(n)} = (U^*)^nTU^n$ for all $n \in \mathbb{N}$. Therefore, $U^n\hat{T}^{(n)}(U^*)^n = T$. So, $\| T \| \leq \| \hat{T}^{(n)} \|$. But $\| \hat{T}^{(n)} \| \leq \| T \|$. 

If we apply the following lemma from [16], we can prove theorem 2.3.11 which is much more general than lemma 2.3.8.

Lemma 2.3.10 ([16]). If $T = U|T|$ is the polar decomposition of $T$, then for every $f \in \text{Hol}(\sigma(T))$, we have $f(T)U = UF(\hat{T})$.

Theorem 2.3.11. If $T = U|T|$ is the polar decomposition of $T$ and if $U$ is a coisometry, then for every $f \in \text{Hol}(\sigma(T))$, we have $\| f(\hat{T}) \| = \| f(T) \|$.

Proof. By theorem 2.1.9, $\| f(\hat{T}) \| \leq \| f(T) \|$. On the other hand, we have by lemma 2.3.10, $f(T)U = UF(\hat{T})$. Therefore, $UF(\hat{T})U^* = f(T)UU^* = f(T)$. So, $\| f(T) \| = \| UF(\hat{T})U^* \| \leq \| f(\hat{T}) \|$. 

Corollary 2.3.12. If $T \in \mathcal{L}(H)$ is invertible, then $\| f(\hat{T}) \| = \| f(T) \|$ for all $f \in \text{Hol}(\sigma(T))$.

Theorem 2.3.13. If $T = U|T|$ is the polar decomposition of $T$, and $U$ is coisometry, then the map $\hat{\Phi} : \mathcal{A}_T \to \mathcal{A}_T$ defined by $\hat{\Phi}(f(T)) = f(\hat{T})$, $f \in \text{Hol}(\sigma(T))$ is an isometry.
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Proof. By theorem 2.3.11,

\[ \| \hat{\Phi}(f(T)) \| = \| f(\hat{T}) \| = \| f(T) \|, \]

for all \( f \in Hol(\sigma(T)) \).

Corollary 2.3.14. If \( T \in \mathcal{L}(H) \) is invertible, then the map \( \hat{\Phi} : \mathcal{A}_T \to \mathcal{A}_{\hat{T}} \) defined by \( \hat{\Phi}(f(T)) = f(\hat{T}), \ f \in Hol(\sigma(T)) \) is an isometry.

Let \( T \in \mathcal{L}(H) \) be invertible. By an application of lemma 2.3.1, we see that \( \hat{T}^{(n)} \) is invertible for all \( n \in \mathbb{N} \). Also, \( \sigma(T) = \sigma(\hat{T}^{(n)}) \) for all \( n \in \mathbb{N} \). So by applying 2.3.12 inductively, we can prove the following result.

Theorem 2.3.15. If \( T \in \mathcal{L}(H) \) is invertible, then \( \| f(\hat{T}^{(n)}) \| = \| f(T) \| \) for all \( n \in \mathbb{N} \) and for all \( f \in Hol(\sigma(T)) \).

Remark 2.3.16. If \( T \) is invertible, then by theorem 2.3.9, \( \| \hat{T}^{(n)} \| = \| T \| \) for all \( n \in \mathbb{N} \), and hence \( \{ \| \hat{T}^{(n)} \| \} \) is a constant sequence converging to \( \| T \| \). Thus if \( T \) is any invertible non-normaloid, then the sequence \( \{ \| \hat{T}^{(n)} \| \} \) cannot converge to the spectral radius \( r(T) \). Referring back to the section 2.2.3, notice that the operator considered in example 2.2.19 was invertible and non-normaloid. For the operator in the example, we proved constructively that the norms of the Duggal iterates do not converge to the spectral radius. Now we realize that it was not accidental, and it is the case with every invertible non-normaloid.

Remark 2.3.17. Theorem 2.3.9 says that if \( T \) is invertible, then \( \| \hat{T}^{(n)} \| = \| T \| \) for all \( n \in \mathbb{N} \). But the condition that \( \| \hat{T}^{(n)} \| = \| T \| \) for all \( n \), does not imply \( T \) is invertible. It does not even imply that \( U \) is a coisometry. Consider the following example.

Example 2.3.18. Let \( A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \). Then the matrix \( A \) is not invertible. Since \( A \) is self-adjoint, \( \hat{A}^{(n)} = A \) for all \( n \), and therefore, \( \| \hat{A}^{(n)} \| = \| A \| \) for all \( n \).
The polar decomposition of $A$ is $A = U|A|$, where the partial isometry $U = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $|A| = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Notice that $U$ is not even a coisometry.

**Lemma 2.3.19.** Let $T = U|T|$ be the polar decomposition of $T$. If $U$ is a coisometry, then for every $n \times n$ matrix $(f_{ij})$ with $f_{ij} \in Hol(\sigma(T))$, we have $\| (f_{ij}(\hat{T})) \| = \| (f_{ij}(T)) \|$.

**Proof.** Let $(f_{ij})$ be an $n \times n$ matrix with $f_{ij} \in Hol(\sigma(T))$. By lemma 2.3.10, $f_{ij}(T)U = U f_{ij}(\hat{T})$ for all $i, j$. Therefore, $f_{ij}(T) = U f_{ij}(\hat{T})U^*$ for all $i, j$. Thus $(f_{ij}(T)) = (U f_{ij}(\hat{T})U^*)$. Therefore,

$$
\| (f_{ij}(T)) \| = \| (U f_{ij}(\hat{T})U^*) \|
\leq \| \begin{bmatrix} U & 0 & \cdots & 0 \\ 0 & U & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & U \end{bmatrix} (f_{ij}(\hat{T})) \begin{bmatrix} U^* & 0 & \cdots & 0 \\ 0 & U^* & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & U^* \end{bmatrix} \|.
$$

But the above diagonal matrices have norm less than or equal to 1. (For example, let $R = \begin{bmatrix} U & 0 & \cdots & 0 \\ 0 & U & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & U \end{bmatrix}$. Then $R \in M_n(\mathcal{L}(\mathcal{H}))$, $RR^* = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$.\)
Therefore, \( \| RR^* \| = 1 \). So, \( \| R^* \|^2 = \| R \|^2 = 1 \). Therefore, \( \| R \| = 1 \). Thus

\[
\| (f_{ij}(T)) \| \leq \| (f_{ij} (\widehat{T})) \|.
\]

On the other hand, by theorem 2.1.9,

\[
\| (f_{ij}(\widehat{T})) \| \leq \| (f_{ij}(T)) \|.
\]

Thus \( \| (f_{ij}(\widehat{T})) \| = \| (f_{ij}(T)) \| \).

**Theorem 2.3.20.** If \( T = U |T| \) is the polar decomposition of \( T \) and \( U \) a coisometry, then the map \( \widehat{\Phi} : \mathcal{A}_T \to \mathcal{A}_\widehat{T} \) defined by \( \widehat{\Phi}(f(T)) = f(\widehat{T}) \), \( f \in \text{Hol}(\sigma(T)) \) is a complete isometry.

**Proof.** By lemma 2.3.19,

\[
\| (f_{ij}(\widehat{T})) \| = \| (f_{ij}(T)) \| .
\]

for every \( n \times n \) matrix \((f_{ij})\) where \( f_{ij} \in \text{Hol}(\sigma(T)) \). In other words, the equality \( \| \widehat{\Phi}_n(f_{ij}) \| = \| (f_{ij}) \| \) holds for every positive integer \( n \) and for every \( n \times n \) matrix \((f_{ij})\) where \( f_{ij} \in \text{Hol}(\sigma(T)) \). Thus for all positive integers \( n, \| \widehat{\Phi}_n \| = 1 \), where \( \widehat{\Phi}_n \) denotes the \( n^{th} \) amplification of \( \widehat{\Phi} \). Hence, \( \widehat{\Phi} \) is a complete isometry.

**Corollary 2.3.21.** If \( T \) is invertible, then the map \( \widehat{\Phi} \) defined as above is a complete isometry.

Recall the definitions of spectral set and complete spectral set, given in section 1.5. Let \( X \) be a closed proper subset of \( \mathbb{C} \), and let \( \check{X} \) denote the closure of \( X \), when we regard \( X \) as a subset of the Riemann sphere \( \mathbb{S} \). We let \( \mathcal{R}(X) \) denote the quotients of polynomials with poles off \( \check{X} \), that is, the bounded, rational functions on \( X \) with a limit at \( \infty \). We regard \( \mathcal{R}(X) \) as a subalgebra of the
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$C^*$-algebra $C(\partial \hat{X})$, which defines norms on $\mathcal{R}(X)$ and each $\mathcal{M}_n(\mathcal{R}(X))$.

If $X$ is a closed, proper subset of $\mathbb{C}$, and $T \in \mathcal{L}(\mathcal{H})$, with $\sigma(T) \subset X$, then there is a functional calculus, i.e., a homomorphism $\rho : \mathcal{R}(X) \rightarrow \mathcal{L}(\mathcal{H})$, given by $\rho(f) = f(T)$, where $f(T) = p(T)q(T)^{-1}$ if $f = p/q$. If $\|\rho\| \leq 1$, then $X$ is called a spectral set for $T$. If $\|\rho\|_{cb} \leq 1$, then $X$ is called a complete spectral set for $T$.

Now let us introduce the concept of $n$-level spectral sets. They are discussed in more detail in section 4.3 on page 89.

**Definition 2.3.22 ($n$-level spectral set).** Let $n$ be a positive integer. Let $X$, $\rho$, and $T$ be as discussed above. If $\|\rho_n\| \leq 1$, where $\rho_n$ denotes the $n$th amplification of $\rho$, then we say that $X$ is an $n$-level spectral set for $T$.

**Lemma 2.3.23.** If $T = U|T|$ is the polar decomposition of $T$, and $U$ is a coisometry, then $T$ and $\hat{T}$ have the same collection of spectral sets.

**Proof.** By theorem 2.3.11,

$$\| f(\hat{T}) \| = \| f(T) \|,$$

for all $f \in Hol(\sigma(T))$. If $X$ is a closed set in the complex plane such that $\sigma(T) \supset X$, then $\mathcal{R}(X)$ is a subalgebra of $Hol(\sigma(T))$. Hence

$$\| f(\hat{T}) \| = \| f(T) \|,$$

for all $f \in \mathcal{R}(X)$. Therefore, $T$ and $\hat{T}$ have the same collection of spectral sets.

**Corollary 2.3.24.** If $T$ is invertible, then $T$ and $\hat{T}$ have the same collection of spectral sets.

Let $T = U|T|$ be the polar decomposition of $T$. If $U$ is a coisometry, then
Lemma 2.3.19 says that for every $n \times n$ matrix $(f_{ij})$ with $f_{ij} \in Hol(\sigma(T))$, we have $\| (f_{ij}(\hat{T})) \| = \| (f_{ij}(T)) \|$. Applying this we get the following result.

**Lemma 2.3.25.** If $T = U|T|$ is the polar decomposition of $T$, and $U$ is coisometry, then $T$ and $\hat{T}$ have the same collection of complete spectral sets. Also, for every fixed positive integer $n$, the operators $T$ and $\hat{T}$ have the same collection of $n$–level spectral sets.

**Theorem 2.3.26.** Let $T \in \mathcal{L}(H)$ be an invertible operator. If for some $n$, the $n^{th}$ Duggal iterate $\hat{T}^{(n)}$ is normal, then $T$ is normaloid. In fact, $f(T)$ is normaloid for every $f \in R(\sigma(T))$.

**Proof.** By theorem 2.3.15, $\| f(\hat{T}^{(n)}) \| = \| f(T) \|$ for all $f \in Hol(\sigma(T))$. If $X$ is a closed set in the complex plane such that $\sigma(T) \supset X$, then $R(X)$ is a subalgebra of $Hol(\sigma(T))$. It follows that $\hat{T}^{(n)}$ and $T$ have the same collection of spectral sets. Since $\hat{T}^{(n)}$ is normal, $\sigma(\hat{T}^{(n)})$ is a spectral set for $\hat{T}^{(n)}$. But $\sigma(\hat{T}^{(n)}) = \sigma(T)$. Thus $\sigma(T)$ is a spectral set for $T$. By a theorem in [7], $\sigma(T)$ is a spectral set for $T$ if and only if $f(T)$ is normaloid for every $f \in R(\sigma(T))$. In particular, $T$ is normaloid. \qed

2.3.3 Continuity of the maps $T \to \tilde{T}$ and $T \to \hat{T}$

Ken Dykema and Hanne Schultz proved in [14] that the Aluthge transformation map $T \to \tilde{T}$ is continuous on $\mathcal{L}(H)$. This result can be seen in [30] also. In this section we examine the continuity of the Duggal transformation map $T \to \hat{T}$. The method of proof in [14] to prove the Aluthge transformation map $T \to \tilde{T}$ is continuous (which uses continuous functional calculus), does not readily translate to the context of Duggal transformations. So we examine the continuity of the Duggal transformation map $T \to \hat{T}$ on the set of invertible operators in $\mathcal{L}(H)$ and prove that the map is continuous on the set of invertible operators. As a
consequence we show that the sequence of the Duggal iterates of an invertible operator $T$ converges to an invertible operator if and only if $T$ is quasinormal.

Further we obtain some results regarding the relation between the spectral sets of an operator and the spectral sets of the limit of the sequence of the Duggal iterates.

**Theorem 2.3.27.** [14]

i. Given $R \geq 1$ and $\epsilon > 0$, there are real polynomials $p$ and $q$ such that for every $T \in \mathcal{L}(\mathcal{H})$ with $\| T \| \geq R$, we have $\| \tilde{T} - p(T^*T)Tq(T^*T) \| < \epsilon$.

ii. For every $T \in \mathcal{L}(\mathcal{H})$, the Aluthge transformation $\tilde{T}$ of $T$ belongs to the $C^*$-algebra generated by $T$ and the identity.

**Theorem 2.3.28.** [14] The Aluthge transformation map $T \mapsto \tilde{T}$ is $(\| . \|, \| . \|)$ continuous on $\mathcal{L}(\mathcal{H})$.

**Remark 2.3.29.** Let $T \in \mathcal{L}(\mathcal{H})$. Suppose that the Aluthge transformation sequence $\{ \tilde{T}^{(n)} \}$ is convergent and that $\tilde{T}^{(n)} \rightarrow S$ in $\mathcal{L}(\mathcal{H})$ as $n \rightarrow \infty$. Then $S$ is quasinormal. (The fact that $S$ is quasinormal can be proved as follows. Define $\Delta(T) = \tilde{T}$ for all $T \in \mathcal{L}(\mathcal{H})$. By theorem 2.3.28, the map $\Delta : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H})$ is continuous. Therefore, $\Delta(\tilde{T}^{(n)}) \rightarrow \Delta(S)$ as $n \rightarrow \infty$. But $\Delta(\tilde{T}^{(n)}) = \tilde{T}^{(n+1)}$, and $\tilde{T}^{(n+1)} \rightarrow S$ as $n \rightarrow \infty$. Hence $\Delta(S) = S$. i.e., $\tilde{S} = S$. Thus $S$ is quasinormal).

Let $\mathcal{H}$ be a finite dimensional Hilbert space. Every quasinormal operator on $\mathcal{H}$ is normal (see remarks after 1.2.11 ). Hence if $T \in \mathcal{L}(\mathcal{H})$, and $\tilde{T}^{(n)} \rightarrow S$ in $\mathcal{L}(\mathcal{H})$ as $n \rightarrow \infty$, then $S$ is normal.

**Definition 2.3.30.** Let $\mathcal{H}$ be a Hilbert space. Let $\Delta : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H})$ and $\Gamma : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H})$ be defined by

$$\Delta(T) = \tilde{T}, \quad \Gamma(T) = \hat{T} \text{ for } T \in \mathcal{L}(\mathcal{H}).$$
2.3. More on Aluthge and Duggal transformations

Let $\mathcal{D}$ be a subset of $\mathcal{L}(\mathcal{H})$. We say that $\mathcal{D}$ is a Duggal continuity family if the map $\Gamma|_{\mathcal{D}} : \mathcal{D} \to \mathcal{L}(\mathcal{H})$ is continuous.

Notice that by theorem 2.3.28, the Aluthge transformation map $\Delta$ is continuous on all of $\mathcal{L}(\mathcal{H})$.

Our next aim is to prove that the set of all invertible operators in $\mathcal{L}(\mathcal{H})$ is a Duggal continuity family, or in other words, our aim is to prove that the map $T \to \hat{T}$ is continuous on the set of invertible operators.

**Theorem 2.3.31.** [11] Let $\mathcal{A}$ be a unital complex $C^*$-algebra. Let $A, B$ be subsets of $\mathbb{C}$ and $f : A \to B$ a homeomorphism. Put $A_0 = \{ x \in N_A : \sigma(x) \subset A \}$, $B_0 = \{ x \in N_A : \sigma(x) \subset B \}$ where $N_A$ denotes the set of all normal elements of the $C^*$-algebra $\mathcal{A}$. Then $\{ f(x) : x \in A_0 \} = B_0$, and the map $x \to f(x) : A_0 \to B_0$ is a homeomorphism.

Notice that $f \in C(\sigma(x))$ for every $x \in A_0$, and hence by the continuous functional calculus on $C^*$-algebras, $f(x)$ is defined as an element in $\mathcal{A}$.

If $A = B = \mathbb{R}_+$ and $\alpha \in \mathbb{R}_+ \setminus \{0\}$, then $f : A \to B$ defined by $f(t) = t^\alpha$ is a homeomorphism. Here, $A_0 = B_0 = \{ x \in N_A : \sigma(x) \subset \mathbb{R}_+ \} = \mathcal{A}_+$, the set of all positive elements in $\mathcal{A}$. Hence by the above theorem, the map $x \to x^\alpha : \mathcal{A}_+ \to \mathcal{A}_+$ is a homeomorphism.

If $x \in \mathcal{A}$, then $|x| = (x^*x)^{1/2}$. Since the maps $x \to x^*x : \mathcal{A} \to \mathcal{A}_+$ and $x \to x^{1/2} : \mathcal{A}_+ \to \mathcal{A}_+$ are continuous, the theorem below follows.

**Theorem 2.3.32.** [11] Let $\mathcal{A}$ be a unital complex $C^*$-algebra and $\mathcal{A}_+$ be the set of all positive elements in $\mathcal{A}$. The map

$$x \to |x| : \mathcal{A} \to \mathcal{A}_+$$

is continuous.
2.3. More on Aluthge and Duggal transformations

**Theorem 2.3.33.** Let \( \mathcal{H} \) be a Hilbert space and \( S \) be the set of all invertible operators in \( \mathcal{L}(\mathcal{H}) \). Then \( S \) is a Duggal continuity family. In other words, the map \( T \rightarrow \hat{T} \) is continuous on \( S \).

**Proof.** If \( T \in \mathcal{L}(\mathcal{H}) \), let \( T = \theta(T)\mu(T) \) be the polar decomposition of \( T \). We have, \( \mu(T) = |T| = (T^*T)^{1/2} \). If \( T \in S \), then \( \mu(T) \in S \), and in this case, \( \theta(T) = T\mu(T)^{-1} \).

By theorem 2.3.32, the map \( T \rightarrow \mu(T) \) is continuous on \( \mathcal{L}(\mathcal{H}) \). Also, the inversion map \( S \rightarrow S^{-1} : S \rightarrow \mathcal{L}(\mathcal{H}) \) is continuous. Hence the map \( \theta : S \rightarrow \mathcal{L}(\mathcal{H}) \) is continuous. If \( T \in \mathcal{L}(\mathcal{H}) \), then \( \hat{T} = \mu(T)\theta(T) \).

Suppose that \( \{S_n\} \) is a sequence in \( S \) such that \( S_n \rightarrow S \) in \( S \) as \( n \rightarrow \infty \). Since \( \theta : S \rightarrow \mathcal{L}(\mathcal{H}) \) is continuous, and the map \( T \rightarrow \mu(T) \) is continuous on \( \mathcal{L}(\mathcal{H}) \), we see that \( \theta(S_n) \rightarrow \theta(S) \) and \( \mu(S_n) \rightarrow \mu(S) \) in \( \mathcal{L}(\mathcal{H}) \) as \( n \rightarrow \infty \). Therefore, \( \hat{S}_n \rightarrow \hat{S} \) in \( \mathcal{L}(\mathcal{H}) \) as \( n \rightarrow \infty \). Thus the map \( T \rightarrow \hat{T} \) is continuous on \( S \). In other words, \( \Gamma_{|S} : S \rightarrow \mathcal{L}(\mathcal{H}) \) is continuous. Hence \( S \) is a Duggal continuity family. \( \square \)

**Theorem 2.3.34.** Let \( \mathcal{H} \) be a Hilbert space. Suppose that \( D \) is a Duggal continuity family in \( \mathcal{L}(\mathcal{H}) \). Let \( T \in D \) be such that

i. \( \hat{T}^{(n)} \in D \) for all \( n \in \mathbb{N} \).

ii. \( \hat{T}^{(n)} \rightarrow S \) in \( \mathcal{L}(\mathcal{H}) \) as \( n \rightarrow \infty \).

iii. \( S \in D \).

Then \( S \) is quasinormal.

**Proof.** Since \( D \) is a Duggal continuity family, the Duggal transformation map
2.3. More on Aluthge and Duggal transformations

\( \Gamma_D : D \to \mathcal{L}(\mathcal{H}) \) is continuous. Therefore,

\[
\Gamma(S) = \Gamma \left( \lim_{n \to \infty} \hat{T}^{(n)} \right) \\
= \lim_{n \to \infty} \Gamma \left( \hat{T}^{(n)} \right) \\
= \lim_{n \to \infty} \hat{T}^{(n+1)} \\
= S.
\]

Hence, \( S \) is quasinormal. \( \square \)

**Corollary 2.3.35.** Let \( \mathcal{H} \) be a finite dimensional Hilbert space. Suppose that \( D \) is a Duggal continuity family in \( \mathcal{L}(\mathcal{H}) \). Let \( T \in D \) be such that

1. \( \hat{T}^{(n)} \in D \) for all \( n \in \mathbb{N} \).
2. \( \hat{T}^{(n)} \to S \) in \( \mathcal{L}(\mathcal{H}) \) as \( n \to \infty \).
3. \( S \in D \).

Then \( S \) is normal.

**Proof.** On finite dimensional Hilbert spaces every quasinormal operator is normal (see remarks after 1.2.11 on page 15). Therefore, by theorem 2.3.34, the proof follows. \( \square \)

**Theorem 2.3.36.** Let \( T \in \mathcal{L}(\mathcal{H}) \) be such that

1. \( T \) is invertible
2. \( \{ \hat{T}^{(n)} \} \) converges to \( S \in \mathcal{L}(\mathcal{H}) \) such that \( S \) is invertible.

Then \( S \) is quasinormal.
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Proof. Since $T$ is invertible, $\hat{T}^{(n)}$ is invertible for every $n$. By theorem 2.3.33, the set $S$ of invertible operators in $\mathcal{L}(\mathcal{H})$ is a Duggal continuity family. By theorem 2.3.34, $S$ is quasinormal. \hfill \Box

Corollary 2.3.37. Let $\mathcal{H}$ be a finite dimensional Hilbert space. Let $T \in \mathcal{L}(\mathcal{H})$ be such that

(i) $T$ is invertible

(ii) $\{\hat{T}^{(n)}\}$ converges to $S \in \mathcal{L}(\mathcal{H})$ such that $S$ is invertible.

Then $S$ is normal.

Lemma 2.3.38. Let $U, S \in \mathcal{L}(\mathcal{H})$ and assume that $U$ is unitary. Then

i. $S$ is normal if and only if $U^*SU$ is normal.

ii. $S$ is quasinormal if and only if $U^*SU$ is quasinormal.

Proof. $S$ normal $\Rightarrow$ $S^*S = SS^* \Rightarrow (U^*SU)^*(U^*SU) = U^*S^*UU^*SU = U^*S^*SU = U^*SU^*SU = (U^*SU)(U^*SU)^* \Rightarrow U^*SU$ normal.

$S$ quasinormal $\Rightarrow$ $S(S^*S) = (S^*S)S \Rightarrow (U^*SU)[(U^*SU)^*(U^*SU)] = U^*SUU^*SU^*SU = U^*S(S^*S)U = U^*(S^*S)SU = U^*S^*UU^*SU = [(U^*SU)^*(U^*SU)](U^*SU) \Rightarrow U^*SU$ quasinormal.

On the other hand, since $U^*$ is unitary, $U^*SU$ normal (quasinormal) $\Rightarrow$ $(U^*)(U^*SU)U^*$ normal (quasinormal) $\Rightarrow$ $S$ normal (quasinormal). \hfill \Box

Theorem 2.3.39. Let $\mathcal{N}$ be the set of all normal operators and $\mathcal{Q}$ be the set of all quasinormal operators in $\mathcal{L}(\mathcal{H})$. Let $T \in \mathcal{L}(\mathcal{H})$ be invertible. Then

i. $T$ and $\hat{T}$ are at the same distance from $\mathcal{N}$.

ii. $T$ and $\hat{T}$ are at the same distance from $\mathcal{Q}$. 


Proof. Let $T = U|T|$ be the polar decomposition of $T$. By lemma 2.3.2, $\hat{T} = U^*TU$. Since $T$ is invertible, $U$ is unitary. Therefore, $\| U^*RU \| = \| R \|$ for every $R \in \mathcal{L}(\mathcal{H})$. By lemma 2.3.38, $\mathcal{N} = \{ U^*SU : S \in \mathcal{N} \}$ and $\mathcal{Q} = \{ U^*SU : S \in \mathcal{Q} \}$. Therefore,

$$
\text{dist}(\hat{T}, \mathcal{N}) = \inf \{ \| \hat{T} - S \| : S \in \mathcal{N} \} = \inf \{ \| \hat{T} - U^*SU \| : S \in \mathcal{N} \} = \inf \{ \| U^*TU - U^*SU \| : S \in \mathcal{N} \} = \inf \{ \| U^*(T - S)U \| : S \in \mathcal{N} \} = \inf \{ \| T - S \| : S \in \mathcal{N} \} = \text{dist}(T, \mathcal{N}).
$$

Similarly, $\text{dist}(\hat{T}, \mathcal{Q}) = \text{dist}(T, \mathcal{Q}).$ □

Let $T \in \mathcal{L}(\mathcal{H})$ be invertible. Then $\{ \hat{T}^{(n)} \}_{n=0}^{\infty}$ is a sequence of invertible operators. The following theorem shows that this sequence can converge to an invertible operator in $\mathcal{L}(\mathcal{H})$ only when $T$ is quasinormal. Notice that if $T$ is quasinormal then $\hat{T}^{(n)} = T$ for all $n$.

**Theorem 2.3.40.** Let $T \in \mathcal{L}(\mathcal{H})$ be invertible. Then $\{ \hat{T}^{(n)} \}_{n=0}^{\infty}$ converges to an invertible operator in $\mathcal{L}(\mathcal{H})$ if and only if $T$ is quasinormal.

**Proof.** If $T$ is quasinormal then $\hat{T}^{(n)} = T$ for all $n$, and therefore one part of the proof is trivial.

Conversely, suppose that $\{ \hat{T}^{(n)} \}_{n=0}^{\infty}$ converges to $S$ in $\mathcal{L}(\mathcal{H})$, and assume that $S$ is invertible. Let $\mathcal{Q}$ be the set of all quasinormal operators in $\mathcal{L}(\mathcal{H})$. Then $\mathcal{Q}$ is a closed subset of $\mathcal{L}(\mathcal{H})$. By theorem 2.3.36, $S \in \mathcal{Q}$. By repeated application
of theorem 2.3.39, we see that \( \text{dist} \left( T, Q \right) = \text{dist} \left( \hat{T}^{(n)}, Q \right) \) for all \( n \). Therefore,

\[
\begin{align*}
\text{dist} \left( T, Q \right) & = \text{dist} \left( \hat{T}^{(n)}, Q \right) \\
& \leq \| \hat{T}^{(n)} - S \|
\end{align*}
\]

for every \( n \). Since \( \hat{T}^{(n)} \rightarrow S \) as \( n \rightarrow \infty \), it follows that \( \text{dist} \left( T, Q \right) = 0 \). Since \( Q \) is closed, \( T \in Q \).

\[ \square \]

**Corollary 2.3.41.** Let \( \mathcal{H} \) be a finite dimensional Hilbert space. Let \( T \in \mathcal{L}(\mathcal{H}) \) be invertible. Then \( \{ \hat{T}^{(n)} \}_{n=0}^{\infty} \) converges to an invertible operator in \( \mathcal{L}(\mathcal{H}) \) if and only if \( T \) is normal.

**Remark 2.3.42.** The following is another form of the corollary 2.3.41 and it appeared in [5]. If \( \mathcal{G}l_r (\mathbb{C}) \) is the general linear group of \( r \times r \) invertible complex matrices and \( T \in \mathcal{G}l_r (\mathbb{C}) \), then the sequence \( \{ \hat{T}^{(n)} \} \) can not converge (in \( \mathcal{G}l_r (\mathbb{C}) \)), unless \( T \) is normal.

**Definition 2.3.43.** Let \( T \in \mathcal{L}(\mathcal{H}) \). Let \( X \) be a closed proper subset of \( \mathbb{C} \) with \( \sigma(T) \subset X \). Let \( f \) be a rational function with poles off \( \hat{X} \). We shall say that \( f \in R(X,T) \) if it satisfies the condition that \( f(\hat{T}^{(n)}) \rightarrow f(S) \) in \( \mathcal{L}(\mathcal{H}) \) whenever \( \hat{T}^{(n)} \rightarrow S \) in \( \mathcal{L}(\mathcal{H}) \) as \( n \rightarrow \infty \).

If \( p \) is any polynomial, then the map \( A \rightarrow p(A) \) is continuous on \( \mathcal{L}(\mathcal{H}) \). Therefore, if \( T \in \mathcal{L}(\mathcal{H}) \) is any operator, and if \( X \) is any closed proper subset of \( \mathbb{C} \) with \( \sigma(T) \subset X \), then \( p \in R(X,T) \).

**Theorem 2.3.44.** Let \( T \in \mathcal{L}(\mathcal{H}) \) be invertible, and \( X \) be any closed proper subset of \( \mathbb{C} \) such that \( \sigma(T) \subset X \) and \( R(X,T) = R(X) \). Suppose that \( \hat{T}^{(n)} \rightarrow S \) in \( \mathcal{L}(\mathcal{H}) \) as \( n \rightarrow \infty \), and that \( \sigma(S) \subset X \). If \( X \) is a spectral set for \( T \), then \( X \) is a spectral set for \( S \).

**Proof.** Let \( f \in R(X) \). Then \( f \in Hol(\sigma(T)) \). Since \( T \) is invertible, by theorem 2.3.15, \( \| f(\hat{T}^{(n)}) \| = \| f(T) \| \) for all \( n \in \mathbb{N} \). Since \( f \in R(X,T) \), we have
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\[ f(\hat{T}^{(n)}) \to f(S) \text{ in } \mathcal{L}(\mathcal{H}) \text{ as } n \to \infty. \] Therefore, \( \| f(\hat{T}^{(n)}) \| \to \| f(S) \| \) as \( n \to \infty. \) Hence \( \| f(S) \| = \| f(T) \|. \) This proves the theorem. \( \square \)

The following theorem talks about the upper semi-continuity of the spectrum. We use this theorem to prove the useful result in theorem 2.3.46.

**Theorem 2.3.45.** [28] Suppose \( A \) is a Banach algebra, \( x \in A, \) \( \Omega \) is an open set in \( \mathbb{C}, \) and \( \sigma(x) \subset \Omega. \) Then there exists \( \delta > 0 \) such that \( \sigma(x+y) \subset \Omega \) for every \( y \in A \) with \( \| y \| < \delta. \)

**Theorem 2.3.46.** Let \( \mathcal{H} \) be a Hilbert space and \( T \in \mathcal{L}(\mathcal{H}). \) If \( \{\hat{T}^{(n)}\} \) converges to \( S \in \mathcal{L}(\mathcal{H}), \) then \( \text{Hol}(\sigma(S)) \subset \text{Hol}(\sigma(T)). \)

**Proof.** Let \( \Omega \) be an open set in \( \mathbb{C} \) with \( \Omega \supset \sigma(S). \) By theorem 2.3.45, there exists \( \delta > 0 \) such that \( \sigma(S+R) \subset \Omega \) for every \( R \in \mathcal{L}(\mathcal{H}) \) with \( \| R \| < \delta. \) Since \( \hat{T}^{(n)} \to S \) as \( n \to \infty, \) there exists a positive integer \( n_0 \) such that \( \| \hat{T}^{(n)} - S \| < \delta \) for all \( n \geq n_0. \) Hence \( \sigma(S + \hat{T}^{(n_0)} - S) \subset \Omega, \) i.e., \( \sigma(\hat{T}^{(n_0)}) \subset \Omega. \) But \( \sigma(\hat{T}^{(n_0)}) = \sigma(T). \) So \( \sigma(T) \subset \Omega. \) Thus if \( \Omega \) is an open set in \( \mathbb{C} \) with \( \Omega \supset \sigma(S), \) then \( \Omega \supset \sigma(T). \)

Now, \( f \in \text{Hol}(\sigma(S)) \Rightarrow f \) is holomorphic on an open set \( \Omega \) that contains \( \sigma(S) \Rightarrow f \) is holomorphic on an open set \( \Omega \) that contains \( \sigma(T) \Rightarrow f \in \text{Hol}(\sigma(T)). \)

**Remark 2.3.47.** The analogue of theorem 2.3.46 is true in the case of Aluthge transformations. The proof uses the fact that \( \sigma(\hat{T}^{(n)}) = \sigma(T) \) for all \( n. \) We state the result in the following theorem.

**Theorem 2.3.48.** Let \( \mathcal{H} \) be a Hilbert space and \( T \in \mathcal{L}(\mathcal{H}). \) If \( \{\hat{T}^{(n)}\} \) converges to \( S \in \mathcal{L}(\mathcal{H}), \) then \( \text{Hol}(\sigma(S)) \subset \text{Hol}(\sigma(T)). \)

**Theorem 2.3.49.** Let \( \mathcal{H} \) be a Hilbert space and \( T \in \mathcal{L}(\mathcal{H}) \) be an invertible operator. Suppose that \( \hat{T}^{(n)} \to S \) in \( \mathcal{L}(\mathcal{H}), \) and \( X \) is a closed proper subset of \( \mathbb{C} \) such that \( X \) contains a neighborhood of the spectrum \( \sigma(S). \) Further, assume that
2.4. Contractivity of the maps $f(T) \to f(\tilde{T})$ and $f(T) \to f(\hat{T})$

$f(\hat{T}^{(n)}) \to f(S)$ for all $f \in \mathcal{R}(X)$. Then $X$ is a spectral set for $T$ if and only if $X$ is a spectral set for $S$.

Proof. As in the first paragraph of the proof of theorem 2.3.46, we see that $X$ contains a neighborhood of $\sigma(T)$. Now, let $f \in \mathcal{R}(X)$. Then $f \in Hol(\sigma(S)) \subset Hol(\sigma(T))$. Since $T$ is invertible, by theorem 2.3.15, $\| f(\hat{T}^{(n)}) \| = \| f(T) \|$ for all $n \in \mathbb{N}$. It follows that $\| f(T) \| = \| f(S) \|$. Hence, $X$ is a spectral set for $T$ if and only if $X$ is a spectral set for $S$. \qed

2.4 Contractivity and positivity of the maps

$f(T) \to f(\tilde{T})$ and $f(T) \to f(\hat{T})$

Let $\mathcal{H}$ be an arbitrary Hilbert space whose dimension satisfies $2 \leq \text{dim} \mathcal{H} \leq \aleph_0$. If $T \in \mathcal{L}(\mathcal{H})$, let

$$\mathcal{A}_T = \{ f(T) : f \in Hol(\sigma(T)) \}.$$

Then $\mathcal{A}_T$ is a subalgebra of the $C^*$-algebra $\mathcal{L}(\mathcal{H})$. In [16], Foias, Jung, Ko, and Pearcy proved that the maps $f(T) \to f(\tilde{T})$ and $f(T) \to f(\hat{T})$ are completely contractive algebra homomorphisms from $\mathcal{A}_T$ onto $\mathcal{A}_{\tilde{T}}$ and from $\mathcal{A}_T$ onto $\mathcal{A}_{\hat{T}}$ respectively. Also, these maps are unital.

Let $T \in \mathcal{L}(\mathcal{H})$ be such that $\mathcal{A}_T$ is closed in $\mathcal{L}(\mathcal{H})$. In this case $\mathcal{A}_T$ is a closed subalgebra of $\mathcal{L}(\mathcal{H})$ and therefore is a subspace, that is, a closed linear manifold. In such cases the set $\mathcal{A}_T + (\mathcal{A}_T)^*$ is an operator system in the $C^*$-algebra $\mathcal{L}(\mathcal{H})$.

Notice that there are operators in $\mathcal{L}(\mathcal{H})$ such that $\mathcal{A}_T$ is closed in $\mathcal{L}(\mathcal{H})$. For example, if $\mathcal{H}$ is finite dimensional, then $\mathcal{A}_T$ is a closed subalgebra of $\mathcal{L}(\mathcal{H})$ for every $T \in \mathcal{L}(\mathcal{H})$.

If $T = I$, the identity operator on any Hilbert space $\mathcal{H}$, then for every function
2.4. Contractivity of the maps $f(T) \rightarrow f(\tilde{T})$ and $f(T) \rightarrow f(\hat{T})$

\[ f \in Hol(\sigma(T)), \]

\[ f(T) = f(I) = \frac{1}{2\pi i} \int_C f(z)(zI - I)^{-1}dz \quad \text{where } C \text{ is a smooth closed curve whose} \]
\[ \text{interior contains } \sigma(I) \]

\[ = \frac{1}{2\pi i} \int_C f(z)(z - 1)^{-1}I dz \]

\[ = I \frac{1}{2\pi i} \int_C \frac{f(z)}{z - 1}dz \]

\[ = I \frac{1}{2\pi i} f(1) 2\pi i \quad \text{since } 1 \in \sigma(I) \]

\[ = f(1)I, \]

which shows that $\mathcal{A}_T = \mathcal{C}_\mathcal{H}$, where $\mathcal{C}_\mathcal{H}$ denotes the set of all scalar operators in $\mathcal{L}(\mathcal{H})$. Obviously, $\mathcal{C}_\mathcal{H}$ is a closed subalgebra of $\mathcal{L}(\mathcal{H})$.

We use the following two results from [26] to prove some consequences.

**Theorem 2.4.1** ([26]). Let $A$ be a unital $C^*$-algebra and let $M$ be a subspace of $A$ containing $1$. If $B$ is a unital $C^*$-algebra and $\phi : M \rightarrow B$ is a unital contraction, then $\phi$ extends uniquely to a positive map $\tilde{\phi} : M + M^* \rightarrow B$ with $\tilde{\phi}$ given by $\tilde{\phi}(a + b^*) = \phi(a) + \phi(b)^*$.

**Theorem 2.4.2** ([26]). If $S$ is an operator system in a unital $C^*$-algebra $A$, $B$ a unital $C^*$-algebra and if $\phi : S \rightarrow B$ is a unital positive map, then $\phi$ is self-adjoint.

**Theorem 2.4.3.** Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{L}(\mathcal{H})$ be such that $\mathcal{A}_T$ is closed in $\mathcal{L}(\mathcal{H})$. If $f, g \in Hol(\sigma(T))$ such that $(f(T))^* = g(T)$, then $(f(\tilde{T}))^* = g(\tilde{T})$, $(f(\hat{T}))^* = g(\hat{T})$.

**Proof.** Being a closed subalgebra of $\mathcal{L}(\mathcal{H})$, the set $\mathcal{A}_T$ is a subspace of the unital $C^*$-algebra $\mathcal{L}(\mathcal{H})$ and $I \in \mathcal{A}_T$. By theorem 2.1.9, the maps $\tilde{\Phi} : \mathcal{A}_T \rightarrow \mathcal{A}_{\tilde{T}}$ and
2.4. Contractivity of the maps $f(T) \to f(\tilde{T})$ and $f(T) \to f(\hat{T})$

\[ \Phi : \mathcal{A}_T \to \mathcal{A}_{\tilde{T}} \text{ defined by} \]
\[ \Phi(h(T)) = h(\tilde{T}), \tilde{\Phi}(h(T)) = h(\hat{T}), \ h \in Hol(\sigma(T)) \]

are well-defined and contractive. Also $\hat{\Phi}$ and $\tilde{\Phi}$ are unital (Let $h$ be the constant polynomial $h(z) = 1$. Then $h \in Hol(\sigma(T)) = Hol(\sigma(\tilde{T})) = Hol(\sigma(\hat{T}))$ and $h(T) = h(\tilde{T}) = h(\hat{T}) = I$). Since $\hat{\Phi}$ is a unital contraction of the algebra $\mathcal{A}_T$ into the algebra $\mathcal{A}_{\tilde{T}} \subset \mathcal{L}(\mathcal{H})$, by theorem 2.4.1, $\hat{\Phi}$ extends uniquely to a positive map $\hat{\Psi} : \mathcal{A}_T + (\mathcal{A}_T)^* \to \mathcal{L}(\mathcal{H})$ defined by $\hat{\Psi}(f(T) + g(T)^*) = \hat{\Phi}(f(T)) + (\hat{\Phi}(g(T)))^*$. Since $\mathcal{A}_T + (\mathcal{A}_T)^*$ is an operator system, by theorem 2.4.2, $\hat{\Psi}$ is self-adjoint. Therefore,

\[ (f(\tilde{T}))^* = (\hat{\Phi}(f(T)))^* \]
\[ = (\hat{\Psi}(f(T)))^* \]
\[ = \hat{\Psi}((f(T))^*) \]
\[ = \hat{\Psi}(g(T)) \]
\[ = \hat{\Phi}(g(T)) \]
\[ = g(\tilde{T}). \]

The proof goes similar in the case of Aluthge transformations. \qed

**Corollary 2.4.4.** Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{L}(\mathcal{H})$ be such that $\mathcal{A}_T$ is closed in $\mathcal{L}(\mathcal{H})$. If $T^* = g(T)$ for some $g \in Hol(\sigma(T))$, then $(\tilde{T})^* = g(\tilde{T})$ and $(\hat{T})^* = g(\hat{T})$.

**Proof.** Apply theorem 2.4.3 taking $f \in Hol(\sigma(T))$ defined by $f(z) = z$. \qed

**Remark 2.4.5.** Note that $\mathcal{A}_T$ is a commutative algebra. The fact that $\mathcal{A}_T$ is a commutative algebra can be proved as follows: By the definition of the holomorphic functional calculus in 1.3.6, the mapping $f \to f(T) : Hol(\sigma(T)) \to \mathcal{L}(\mathcal{H})$
is an algebra homomorphism, and the range of this homomorphism is $A_T$. The function algebra $Hol(\sigma(T))$ is commutative. Therefore, the algebra $A_T$ is commutative. Thus if $T^* = g(T)$ for some $g \in Hol(\sigma(T))$, then since both $T$ and $T^*$ belong to $A_T$, we see that $T$ and $T^*$ commute, or in other words, $T$ is normal. So in this case $\hat{T} = \tilde{T} = T$. Thus the corollary 2.4.4, which we proved using theorem 2.4.3, is true even otherwise.