CHAPTER – VI

WIENER STRENGTH OF A GRAPH

In this chapter we introduce a new concept, namely the Wiener strength of a connected graph and initiate a study on it.

If \( e \) is an edge, but not a bridge of a connected graph \( G \), then \( W(G - e) \) is defined and \( W(G) < W(G - e) \). Thus removable of an edge \( e \), lying in a cycle, increases the Wiener number. The least increase in \( W(G) \) caused by a non-bridge edge removal can be considered as a measure of strength of \( G \). We call this least increase as the Wiener strength of \( G \).

**Definition :** 6.1

If \( G \) is a connected graph, which is not a tree, the Wiener strength of \( G \) is defined as

\[
WS(G) = \min \{ W(G - e) - W(G) / e \text{ lies in a cycle} \}.
\]

In this chapter we initiate a study on \( WS(G) \).

**Examples :**

1. \( WS(K_n) = 1 \), for all \( n \geq 3 \)
2. \( WS(K_{m,n}) = 2 \), for all \( n \geq m \geq 2 \)
3. \( WS(C_n \times K_2) = 2 \), for all \( n > 3 \) and \( WS(C_3 \times K_2) = 1 \)
4. \( WS(M_n) = 2 \)
5. \( WS(P) = 7 \) where \( P \) is a Peterson graph.

For the following graph \( G, WS(G) = 1 \).

(a) \( K_n, \ n \geq 3 \)
(b) Wheel \( W_n = C_n + K_1 \)
(c) Fan \( F_n = P_n + K_1 \)
(d) \( \Delta_n = K_2 + \overline{K}_n \), \( \Delta_5 \) is shown in Fig. 6.1 (a)
(e) The graph $G$ obtained from $P_n$ by a cycle $C_3$ at a pendant vertex of $P_n$ (fig. 6.1 (b))

(f) The graph $G$ obtained from $C_3$ by attaching some pendant vertices at a vertex $v$ of $C_3$ (fig. 6.1 (c))

Fig. 6.1 : Examples for graphs $G$ for which $WS(G) = 1$

Notation : If $u \in V(G), N_i(u) = \{u \in V(G) \mid d(u,v) = i\}$, where $i$ is a positive integer.

The following theorem characterizes the graphs $G$ for which $WS(G) = 1$.

**Theorem : 6.2**

Let $e = u_1u_2$ be an edge of $G$, which is not a bridge. Then $W(G-e) = W(G) + 1$ if and only if

(i) $e$ lies in a cycle $C_3$ of length 3, and

(ii) $w \in N_i(u_j) \cap N_j(u_i) \Rightarrow |N_i(w) \cap N_j(u_i)| \geq 2$, where $\{i, j\} = \{1, 2\}$.

**Proof:**

Let $H = G - e$. Then $d_G(u_i,u_j) = 1, d_H(u_i,u_j) \geq 2$, and for all $u, v \in V(G), \ d_G(u,v) \leq d_H(u,v)$. 

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Thus $W(G - e) = 1 + W(G)$

\[ d_H(u_1, u_2) = 2 \text{ and } d_H(u, v) = d_G(u, v) \text{ for all } \{u, v\} \neq \{u_1, u_2\} \]

\[ N(u_1) \cap N(u_2) \neq 0, \text{ and for all } \{w_1, w_2\} \neq \{u_1, u_2\}, \text{ there is a } w_1 - w_2 \]

shortest path $P$ in $G$ such that $e \not\in P$.

\[ e \text{ lies in a cycle } C_3 \text{ of length } 3; \text{ and if } \{i, j\} = \{1, 2\} \text{ and } w u i u j \text{ is a } \]

shortest $w - u_j$ path in $G$, then $w v u_j$ is a path in $G$ for some $v \neq u_i$.

\[ e \text{ lies in a cycle } C_3 \text{ of length } 3 \text{ and for } \{i, j\} = \{1, 2\}, \text{ if } \]

\[ w \in N_1(u_i) \cap N_2(u_j) \] then $|N_1(w) \cap N_1(u_j)| \geq 2$.

**Remark : 1**

Let $WS(G) = 1$ and $W(G - e) = W(G) + 1$ where $e = u_i u_2$. If $G$ is an induced subgraph of $H$ such that $\deg_H(u_i) = \deg_H(u_i)$, for $i = 1, 2$, then $WS(H) = 1$.

**Remark : 2**

$WS(G) = 1$ if and only if there exists an edge $e = u_i u_2$ such that $A = N_1(u_1) \cap N_1(u_2) \neq \emptyset$, and if $B = N_1(u_1) - N_1(u_2)$ and $C = N_1(u_2) - N_1(u_1)$ then $B$ is dominated by $A \cup C$ and $C$ is dominated by $A \cup B$.

**Theorem : 6.3**

Let $e = u_i u_2$ be an edge, which is not a bridge, of a connected graph $G$. Then $W(G - e) = 2 + W(G)$ if and only if the following holds.

(a) If $e$ lies in a cycle $C_3$ of length 3 then there exists a unique vertex $w$ and a unique $i \in \{1, 2\}$ such that

(i) $d(w, u_i) = 1$ and $d(w, u_j) = 2$, where $j \neq i \in \{1, 2\}$

(ii) $N_1(u_j) \cap N_1(w) = \{u_i\}$

(iii) $x \in N_1(w) \cap N_1(u_j) \Rightarrow |N_2(x) \cap N_1(u_j)| \geq 2$

(iv) $x \in N_1(u_i) \cap N_2(u_j)$ and $x \neq w \Rightarrow N_1(u_j) \cap N_1(x) \neq \{u_i\}$

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(v) \( x \in N_1(u_j) \cap N_2(u_i) \Rightarrow N_1(u_i) \cap N_1(x) \neq \{u_j\} \)

(or)

(b) (i) \( N(u_i) \cap N(u_2) = \emptyset \)

(ii) \( e \) lies in a cycle \( C_4 \) of length 4

(iii) For \( \{i, j\} = \{1, 2\}, w \in N_1(u_i) \cap N_2(u_j) \) then \( |N_1(w) \cap N_1(u_j)| \geq 2 \)

Proof:

Let \( H = G - e \).

Case (i)

Assume that \( e \) lies on a cycle \( C_3 \) of length 3. Then \( d_H(u_1, u_2) = 2 = 1 + d_G(u_1, u_2) \). So in this case \( W(G-e) = 2 + W(G) \) if and only if there is exactly one pair \( \{w_1, w_2\} \neq \{u_1, u_2\} \) such that \( d_H(u,v) = d_G(u,v) \) for all \( \{u,v\} \neq \{w_1, w_2\}, \{u_1, u_2\} \) and \( d_H(w_1, w_2) = 1 + d_G(w_1, w_2) \). Every \( w_1 - w_2 \) path in \( G \) contains \( e \). As \( \{w_1, w_2\} \neq \{u_1, u_2\} \) it follows that \( d_G(w_1, w_2) \geq 2 \). First we claim that \( d_G(w_1, w_2) = 2 \). If \( d_G(w_1, w_2) \geq 3 \), consider a \( w_1 - w_2 \) shortest path \( P \) in \( G \). Either \( w_1 \) or \( w_2 \) is not incident with the edge \( e \). Say \( w_2 \) is not incident with \( e \). Let \( x \) be the vertex in \( P \) which is adjacent to \( w_2 \). (so that \( w_2 x \neq e \)).

Now \( d_G(w_1, x) = d_G(w_2, w_1) - 1 \) and \( d_H(w_1, x) \neq d_G(w_1, x) \).

Otherwise \( d_H(w_1, x) = d_G(w_2, w_1) \) and hence \( d_H(w_1, w_2) \leq d_H(w_1, x) + 1 = d_G(w_1, x) + 1 = d_G(w_2, w_1) \) which is a contradiction. As \( \{w_1, x\} \neq \{u, v\} \{d_G(w_1, x) \geq 2 \} \) \( d_H(w_1, x) \neq d_G(w_1, x) \) is a contradiction to the uniqueness property of the pair \( \{w_1, w_2\} \). Thus \( d_G(w_1, w_2) = 2 \).

As any \( w_1 - w_2 \) shortest path contains \( e \), from \( d_G(w_1, w_2) = 2 \), we conclude that \( \{w_1, w_2\} \cap \{u_1, u_2\} = \emptyset \).
Let \( w_2 = u_1 \). (Then \( w_1 \neq u_1, u_2 \)). As \( d_G(u_1, w_1) \neq d_H(u_1, w_1) \), it follows that \( N(u_1) \cap N(w_1) = \{ u_2 \} \). For any \( w \neq u_1, w_1, d_G(w, u_1) = d_H(w, u_1) \) and hence

(i) \( w \in N_1(u_2) \cap N_2(u_1) \Rightarrow |N_1(u_1) \cap N_1(w)| \geq 2 \)

(ii) \( w \in N_1(u_1) \cap N_2(u_2) \Rightarrow |N_1(w) \cap N_1(u_2)| \geq 2 \)

(iii) \( w \in N_2(u_2) \cap N_1(u_1) \Rightarrow |N_1(u_1) \cap N_2(w)| \geq 2 \).

**Case ii:**

Assume that \( e \) does not lie on any cycle \( C_3 \) of length 3. Then \( d_H(u_1, u_2) \geq 3 \) and for all pair \( \{ w_1, w_2 \} \neq \{ u_1, u_2 \} \), \( d_H(w_1, w_2) = d_G(w_1, w_2) \).

Hence if \( w \in N_1(u_1) \cap N_2(u_2) \) then there is a \( w_1 - u_2 \) path of length two in \( G - e \). It follows that \( \{ u_1 \} \subset N_1(w) \cap N_1(u_2) \), and hence \( |N_1(w) \cap N_1(u_2)| \geq 2 \).

Similarly if \( w \in N_1(u_2) \cap N_2(u_1) \) then \( |N_1(w) \cap N_1(u_1)| \geq 2 \).

**Characterization of graphs \( G \) for which \( WS(G) = 1 \) or \( 2 \).**

The Theorem 6.2 characterizes edges \( e \) for which \( W(G - e) = W(G) + 1 \). In fact the theorem characterizes graphs \( G \) for which \( WS(G) = 1 \). For a connected graph \( G \) with \( m \geq n \), \( WS(G) = 1 \) if and only if \( G \) contains an edge satisfying the sufficient condition given in the Theorem 6.2.

**Theorem : 6.4**

Let \( G \) be a connected graph with \( m \geq n \). Then \( WS(G) = 1 \) if and only if there is an edge \( e = u_1u_2 \) in \( G \) such that

(i) \( e \) lies in a cycle \( C_3 \) of length 3, and

(ii) \( w \in N_1(u_i) \cap N_1(u_j) \Rightarrow |N_1(w) \cap N_1(u_j)| \geq 2 \), where \( \{ i, j \} = \{ 1, 2 \} \).

The remark 2 also characterizes the graphs \( G \) for which \( WS(G) = 1 \).
The Theorem 6.3 characterizes the edge $e$ of a connected graph $G$ for which $w(G-e) = W(G) + 2$. Thus $WS(G) = 2$ if and only if no edge of $G$ satisfies the sufficient condition given in theorem 6.2 and there is an edge of $G$ satisfying the sufficient condition given in the theorem 6.3. Thus we have the following theorem.

**Theorem : 6.5**

For a connected graph $G, WS(G) = 2$ if and only if

(i) for any edge $e = u_iu_j$ on a cycle $C_3$ of length 3 there is a vertex $w \in N_1(u_i) \cap N_2(u_j)$ such that $N_1(w) \cap N_1(u_j) = \{u_j\}$, for some $i \in \{1,2\}$, where $j \neq i \in \{1,2\}$ and

(ii) there exists an edge $e = u_iu_j$ satisfying one of the following two conditions ;

(C 1) $e$ lies in a cycle $C_3$ of length 3 and there exists a unique vertex $w$ and a unique $i \in \{1,2\}$ such that

1. $d(w,u_i) = 1$ and $d_{N_1}(u_j) = 2$, where $\{i,j\} = \{1,2\}$
2. $N_1(u_j) \cap N_1(w) = \{u_j\}$
3. $x \in N_1(w) \cap N_3(u_j) \Rightarrow \left| N_2(x) \cap N_1(u_j) \right| \geq 2$
4. $x \in N_1(u_j) \cap N_2(u_j)$ and $x \neq w \Rightarrow N_1(u_j) \cap N_1(x) = \{u_j\}$.
5. $x \in N_1(u_j) \cap N_2(u_i) \Rightarrow N_1(u_i) \cap N_1(x) \neq \{u_j\}$.

(C 2) (1) $N_1(u_i) \cap N_1(u_j) = \emptyset$

2. $e$ lies in a cycle $C_4$ of length 4, and

3. for $\{i,j\} = \{1,2\}$, $w \in N_1(u_i) \cap N_2(u_j)$ \Rightarrow $\left| N_1(w) \cap N_1(u_j) \right| \geq 2$. 

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Wiener Strength of the Cycle $C_n$:

If we remove any edge $e$ from $C_n$ then it leaves the graph $P_n$. Therefore,

$W(C_n - e) = W(P_n)$ for any $e \in C_n$ \quad \therefore W(C_n - e) - W(C_n) = W(P_n) - W(C_n)$

But $W(P_n) = \left(\frac{n+1}{3}\right)$ and $W(C_n) = \begin{cases} \frac{n^3}{8} & \text{if } n \text{ is even} \\ \frac{n^3 - n}{8} & \text{if } n \text{ is odd} \end{cases}$

\therefore W(C_n - e) - W(C_n) = W(P_n) - W(C_n)

\[= \begin{cases} \frac{n(n^2 - 1)}{6} - \frac{n^3}{8} & \text{if } n \text{ is even} \\ \frac{n(n^2 - 1)}{6} - \frac{(n^3 - n)}{8} & \text{if } n \text{ is odd} \end{cases} \]

\[= \begin{cases} \frac{n(n^2 - 4)}{24} & \text{if } n \text{ is even} \\ \frac{n(n^2 - 1)}{24} & \text{if } n \text{ is odd} \end{cases} \]

Remark :3

$\frac{n(n^2 - 4)}{24}$ cannot be 1 for any even $n$

But $\frac{n(n^2 - 1)}{24} = 1$ only when $n = 3$.

\therefore WS($C_n$) = 1 if and only if $n = 3$.

Remark :4

If $n$ is odd, WS($C_{n+2}$) = $\frac{\left(n + 2\right)(n^2 + 4n + 3)}{24} = \frac{(n + 2)(n^2 - 1 + 4n + 4)}{24}$
\[
\begin{align*}
&= \frac{n(n^2 - 1)}{24} + \frac{1}{24} \left[ n(4n + 4) + 2(n^2 - 1 + 4n + 4) \right] \\
&= \frac{n(n^2 - 1)}{24} + \frac{1}{24} \left[ 4n(n + 1) + 2(n^2 + 4n + 3) \right] \\
&= \frac{n(n^2 - 1)}{24} + \frac{2}{24} \left[ 2n^2 + 2n + n^2 + 4n + 3 \right] \\
&= \frac{n(n^2 - 1)}{24} + \frac{(n + 1)^2}{4} = \text{WS}(C_n) + \frac{(n + 1)^2}{4}
\end{align*}
\]

If \( n \) is even, \( \text{WS}(C_{n+2}) = \frac{(n + 2)}{24} \left( (n + 2)^2 - 4 \right) \)

\[
= \frac{(n + 2)}{24} (n^2 + 4n + 4 - 4) \\
= \frac{n}{24} (n^2 - 4) + \frac{1}{24} \left( n(4n + 4) + 2(n^2 + 4n) \right) \\
= \text{WS}(C_n) + \frac{n}{4} (n + 2)
\]

\[
\therefore \text{For odd } n, \text{WS}(C_{n+2}) = \text{WS}(C_n) + \frac{(n + 1)^2}{4}
\]

For even \( n \), \( \text{WS}(C_{n+2}) = \text{WS}(C_n) + \frac{n(n + 2)}{4} \)

<table>
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<th>( n )</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
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<td>( \text{WS}(C_n) )</td>
<td>1</td>
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<td>14</td>
<td>20</td>
<td>30</td>
<td>40</td>
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**Wiener strength of unicyclic graphs:**

Let \( G \) be a unicyclic graph. Let \( v_0, v_1, v_2, ..., v_{n-1}, v_0 \) be the unique cycle of \( G \).

Let \( e_i \) be the edge \( v_i v_{i+1} \) (addition \( i + 1 \) is taken under modulo \( n \)). Let \( T_i \) be the (tree) component of the graph \( G - \{e_{i-1}, e_i\} \) that contains the vertex \( v_i \). Let \( l_i \) be the order of \( T_i \) (i.e., \( l_i \) be the number of vertices of the tree \( T_i \)). Note that \( l_i \) may be equal to 1.
Fix the edge $e_i = v_iv_{i+1}$. Let $H = G - e_i$. Take $w \in T_j$ and $w' \in T_k$ for some $j$ and $k \in \{0, 1, 2, ..., n-1\}$. Then $d_H(w, w') > d_G(w, w')$ if and only if $e_i$ lies in every shortest $(w-w')$-path in $G$. This is possible if and only if $T_j = T_{r+s}$ and $T_k = T_{r-s}$ for some $r \geq 1, s \geq 0$ and $r + s \leq \left\lfloor \frac{n-1}{2} \right\rfloor$. In fact if $w \in T_{r+s}, w' \in T_{r-s}$ where $1 \leq r, o \leq s$ and $r + s \leq \left\lfloor \frac{n-1}{2} \right\rfloor$, we observe the following.

Case : 1 when $n$ is odd:

<table>
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<tr>
<th>$r + s$</th>
<th>$\left\lfloor \frac{n-1}{2} \right\rfloor$</th>
<th>$\left\lfloor \frac{n}{2} \right\rfloor - 1$</th>
<th>$\left\lfloor \frac{n}{2} \right\rfloor - 2$</th>
<th>....</th>
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<tbody>
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<td>$d_H(w, w') - d_G(w, w')$</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td>...</td>
<td>$n-2$</td>
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</table>

Case : 2 when $n$ is even:

<table>
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<tr>
<th>$r + s$</th>
<th>$\left\lfloor \frac{n}{2} \right\rfloor - 1$</th>
<th>$\left\lfloor \frac{n}{2} \right\rfloor - 2$</th>
<th>$\left\lfloor \frac{n}{2} \right\rfloor - 3$</th>
<th>....</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_H(w, w') - d_G(w, w')$</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>...</td>
<td>$n-2$</td>
</tr>
</tbody>
</table>

Hence $W(G - e_i) - W(G) = \sum_{k=1}^{\left\lfloor \frac{n-1}{2} \right\rfloor} (n-2k) \left( \sum_{l=r-s}^{l=r+s} l_{-l} l_{+l} \right)$

Thus $WS(G) = \min_i \left\{ W(G - e_i) - W(G) \right\}$

$$= \min_{0 \leq (5n-1)} \sum_{k=1}^{\left\lfloor \frac{n-1}{2} \right\rfloor} (n-2k) \left( \sum_{l=r-s}^{l=r+s} l_{-l} l_{+l} \right)$$
Corollary : 1

If \( l_i = m \), for all \( i = 0, 1, 2, ..., n - 1 \) then

\[
WS(G) = \sum_{k=1}^{\frac{n-1}{2}} (n - 2k) \sum_{1 \leq r < s \leq k} l_r l_s
\]

\[
= \sum_{k=1}^{\frac{n-1}{2}} (n - 2k) km^2
\]

\[
= m^2 n \sum_{k=1}^{\frac{n-1}{2}} k - 2m^2 \sum_{k=1}^{\frac{n-1}{2}} k^2
\]

\[
= \begin{cases} 
\frac{m^2 n (n-1)}{2} \left( \frac{n+1}{2} \right) - \frac{2m^2}{6} \left( \frac{n-1}{2} \right) \left( \frac{n+1}{2} \right) (n) & \text{if } n \text{ is odd} \\
\frac{m^2 n (n-2)}{2} \left( \frac{n}{2} \right) - \frac{2m^2}{6} \left( \frac{n-2}{2} \right) \left( \frac{n}{2} \right) (n-1) & \text{if } n \text{ is even}
\end{cases}
\]

\[
= \begin{cases} 
\frac{m^2 n (n^2 - 1)}{24} & \text{if } n \text{ is odd} \\
\frac{m^2 n (n^2 - 4)}{24} & \text{if } n \text{ is even}.
\end{cases}
\]

Note :

(i) If \( m = 1, \quad G = C_n \)

\[
WS(G) = \begin{cases} 
\frac{n(n^2 - 1)}{24} & \text{if } n \text{ is odd} \\
\frac{n(n^2 - 4)}{24} & \text{if } n \text{ is even}.
\end{cases}
\]

(ii) If \( m = 2, \quad G \) is a crown graph (ie \( G = C_n oK_1 \))

\[
WS(G) = \begin{cases} 
\frac{n(n^2 - 1)}{6} & \text{if } n \text{ is odd} \\
\frac{n(n^2 - 4)}{6} & \text{if } n \text{ is even}.
\end{cases}
\]
Characterization of connected graphs \( G \) with \( W(G) = W(T) - (m - n + 1) \) for some spanning tree \( T \) of \( G \).

First we note that for each \( n \geq 3 \), the complete graph \( K_n \) is such a graph. Let \( V(K_n) = \{u_1, u_2, ..., u_n\} \) and \( A = \{\text{edge } u_i u_j / 1 \leq i \leq j \leq n - 1\} \) Then

(i) \( K_n - A = (K_{l,n-1}) \) is a spanning tree of \( K_n \) and \( W(K_n) = W(K_n - A) - |A| \)

(ii) for any subset \( B \) of \( A, K_n - A \) is a spanning tree of \( K_n - B \) and

\[ W(K_n - B) = W(K_n - A) - |A - B| \]

The following lemma is obvious.

**Lemma : 6.5**

If \( e = uv \) is an edge of a connected graph such that \( W(G - e) = W(G) + 1 \) then \( d(u, v) = 2 \) in \( G - e \) and \( d_G(x, y) = d_{G-e}(x, y) \) for all pairs \( \{x, y\} \neq \{u, v\} \) of vertices of \( G \). \([d_G(x, y) \text{ denotes the distance between the vertices } x \text{ and } y \text{ in the graph } G] \)

Now we obtain the following characterization theorem :

**Theorem : 6.6**

Let \( G \) be a connected graph with order \( n \) and size \( m \) \((m \geq n)\) Then \( W(G) = W(T) - (m - n + 1) \) for some spanning tree \( T \) of \( G \) if and only if \( G \) contains a spanning tree \( T \) with atleast \((m - n + 1)\) pairs \( \{u_i, v_i\} \) of pendant vertices (of \( T \)) with \( d_T(u_i, v_i) = 2 \) and \( G \) is obtained from \( T \) by adding exactly \((m - n + 1)\) such edges \( u_i v_i \).
Proof:
Assume that $T$ is a spanning tree of $G$ such that $W(G) = W(T) - (m-n+1)$. Let $E(G) \setminus E(T) = \{e_1, ..., e_{m-n+1}\}$ and for each $i$, $1 \leq i \leq m-n+1$, let $e_i = u_i, v_i$. Let 

$G_i = T + \{e_1, ..., e_i\}$, for all $1 \leq i \leq m-n+1$.

As $W(R) = W(G_{m-n+1}) < ... < W(G_i) < W(G_{i-1}) < ... < W(G_1) < W(T)$ and $W(T) - W(G) = m-n+1$, we have $W(G_{m-n+1}) = W(G_i) + 1$ for all $i$.

Then by the lemma 6.5, it follows that for all $x, y \in V(G)$

$$d_T(x, y) = \begin{cases} 2 & \text{if } (x, y) = \{u_i, v_i\} \text{ for some } i \\ d_G(x, y) & \text{otherwise} \end{cases}$$

Let $w_i$ be the vertex of $T$ such that $u_i, w_i, v_i$ is a path of length 2 in $T$. We claim that $\deg_T(u_i) = \deg_T(v_i) = 1$, for all $i$ (i.e., $\deg_T(u)$ denotes the degree of the vertex $u$ in the tree $T$).

If possible, assume that $\deg_T(u_i) \neq 1$ for some $i$. Select a vertex $v \neq w_i$ such that $u_i v$ is an edge in $T$. Then $v, w_i, u_i v$ is the $v_i - v$ path in $T$ and hence $d_T(v, v_i) = 3$. But as $v, u_i v$ is a path in $G$, $d_G(v, v_i) \leq 2$.

As $d_T(v, v_i) \neq 2$, $\{v, v_i\} \neq \{u_j, v_j\}$ for any $j$. As $\{v, v_i\} \neq \{u_i, v_i\}$ for all $j$ and as $d_T(v, v_i) \neq d_G(v, v_i)$ we get a contradiction. Then $\deg_T(u_i) = \deg(v_i) = 1$ for all $i$. It follows that $T$ contains (at least) $m-n+1$ pairs $\{u_i, v_i\}$ of pendant vertices (of $T$) such that $d_T(u_i, v_i) = 2$, and $T$ is obtained from $G$ by omitting $(m-n+1)$ edges $u_i v_i$.

Converse Part:

Let $T$ be a tree on $n$ vertices having $(m-n+1)$ pairs $\{u_i, v_i\}$ of pendant vertices such that $d_T(u_i, v_i) = 2$. Let $\{w_i\} = N_i(u_i) = N_i(v_i)$ in $T$. Let $G$ be obtained from $T$ be adding $(m-n+1)$ edges $u_i v_i$. In other words, $G = T + \{e_i = u_i, v_i / 1 \leq i \leq m-n+1\}$

Let $G_i = T + \{e_1, ..., e_i\}$ for all $1 \leq i \leq m-n+1$. Note that $G = G_{m-n+1}$. We claim that for each $i$, $W(G_i - e) = 1 + W(G_i)$. In $G_i$, the edge $e_i$ lies on the cycle $u_i w_i v_i u_i$, 

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Assume that in $G_i, w \in N_n(u_i) \cap N_2(v_i)$. Then clearly $w \neq w_i$, as $w_i \in N_1(v_i)$. So the edge $u_iw$ of $G_i$ is not an edge of $T$, (In $T$, $\deg_T(u_i)=1$ and $u_iw_i$ is an edge) and hence $\{u_i, w\}=\{u_j, v_j\}$ for some $j < i$. In the tree $T, N(v_i) = N(u_i) = N(u_j) = N(v_j) = \{w_j\}$ and so $w_j = w_i$. Now $v_iw_iw$ is a path of length two in $T$ (also in $G$). Thus in $G_i, \{u_i, w_i\} \subseteq N_1(w) \cap N_1(v_i)$ and hence the edge $e_i$ of $G_i$ satisfies the sufficient condition stated in the Theorem 6.2.

Thus we have $W(G_i-e) = W(G_i)+1$, for all $i$, and as $W(T) > W(G_1) > W(G_2) > ... > W(G_{m-n+1}) = W(G)$, it follows that $W(G) = W(T) - (m - n + 1)$.

**Bounds for WS$(G)$:**

Let us find the lower bound for $WS(G)$. As $W(G-e) > W(G)$, it is clear that $1 \leq WS(G)$, for all connected graphs with $n \geq m$.

**Theorem : 6.7**

For any connected graph $G$ with order $n$ and size $m, (m \geq n), 1 \leq WS(G)$.

Further for given $n$ and $m$, with $n \leq m \leq \frac{1}{2} n(n-1)$, there is a graph $G$ with $n$ vertices and $m$ edges such that $WS(G)=1$.

**Proof :**

It is enough to construct a graph $G$ with $WS(G)=1$. 

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Construction : 1

Consider $K_n$. Let $V(K_n) = \{v_i/1 \leq i \leq n\}$

Let $A = \{edges u_i v_j/1 \leq i < j \leq n - 1\}$. Select a subset $B$ of $A$ such that $|B| = m - n + 1$, and $D = A - B$. If $G = K_n - D$, then $G$ is a graph with order $n$, size $m$ and with $WS(G) = 1$.

Construction : 2

Given $n$ and $m$, with $3 \leq n \leq m \leq \frac{1}{2}n(n-1)$ Let $H$ be any graph with $n - 2$ vertices and $k -$ edges, where $k = \min\left\{m - 3, \frac{1}{2}(n^2 - 5n + 6)\right\}$. Obtain $G$ from $H$ as follows: Let $V(H) = \{v_i/1 \leq i \leq n - 2\}$. Then $V(G) = V(H) \cup \{w_1, w_2\}$ and $E(G) = E(H) \cup \{w_1w_2, w_1v_i, w_2v_i/1 \leq i \leq l\}$ where $l = \frac{1}{2}(m - 1 - k)$

Then $WS(G) = 1$.

Remark : 5

If $WS(G) = 1$, then the girth of $G$ is three. But the converse is not true. There are graphs with girth $(G) = 3$ and $WS(G) \neq 1$.

Let $P = v_1v_2...v_{n-1}$ be the path of length $n - 2$.

Let $i \leq \left\lfloor \frac{n-2}{2} \right\rfloor$ Obtain $G$ from $P$ as follows:

$V(G) = V(P) \cup \{w\}$ and $E(G) = E(P) \cup \{v_iw, v_{i+1}w\}$

Then $WS(G) = i$
Remark : 6

Let girth \((G) = k\) for a connected graph \(G\). If \(u, v \in V(G)\) and if \(d(u, v) < \left\lfloor \frac{k}{2} \right\rfloor\), then there is a unique \(u - v\) path of length \(d(u, v)\) and the lengths of all other \(u - v\) paths will be greater than or equal to \(k - d(u, v)\). Hence we have the following theorem.

Theorem 6.8 :

If \(G\) is a graph with girth \(k\), then \(WS(C_k) \leq WS(G)\) and the lower bound is attained (i) for all \(n \geq k\) and (ii) for all \(d \geq \left\lfloor \frac{k}{2} \right\rfloor\) (where \(n\) is the order of \(G\) and \(d\) is the diameter of \(G\)).

We find upper bounds for \(WS(G)\) for certain special cases.

1) For all unicyclic graphs with \(n\) vertices

\[
WS(G) \leq \begin{cases} \frac{n(n^2 - 1)}{24} & \text{if } n \text{ is odd} \\ \frac{n(n^2 - 4)}{24} & \text{if } n \text{ is even} \end{cases}
\]

2) If \(\text{diam}(G) = 1\) then \(G = K_n\) and \(WS(G) = 1\).

3) If \(\text{diam}(G) = n - 2\) then \(1 \leq WS(G) \leq 2 \left\lfloor \frac{n-2}{2} \right\rfloor\). Further the upperbound is attained.

Proof :

Let \(P = v_1, ..., v_{n-1}\) be a path in \(G\) such that \(d(v_1, v_{n-1}) = n - 2\). Let \(V(G) \setminus V(P) = \{v_n\}\).

Then \(2 \leq |N(v_n)| \leq 3\) and \(d(u, w) \leq 2\) whenever \(u, w \in N(v_n)\).
If \( N(v_n) = \{u_i, u_{i+1}, u_{i+1}\} \) for some \( i \), then \( WS(G) = 1 \). If \( N(v_n) = \{u_i, u_{i+2}\} \) for some \( i \), then \( WS(G) = 2(\min \{i,n-i-2\}) \leq 2 \left\lfloor \frac{n-2}{2} \right\rfloor \).

If \( N(v_n) = \{u_i, u_{i+1}\} \) for some \( i \), then \( WS(G) = \min \{i,n-i-2\} \leq 2 \left\lfloor \frac{n-2}{2} \right\rfloor \).

Thus \( 1 \leq WS(G) \leq 2 \left\lfloor \frac{n-2}{2} \right\rfloor \).