CHAPTER V

PSEUDO-PARA-SASAKIAN MANIFOLDS

In this chapter, we study Para-Sasakian manifolds with pseudo Riemannian metric. We have also considered the isometric immersions of Pseudo Para-Sasakian manifolds in Euclidean spaces, as well as their isometric immersions in pseudo Riemannian manifolds of constant curvature.

5.1. PSEUDO-PARA-SASAKIAN MANIFOLDS.

Let $M(\phi, \xi, \gamma)$ be an almost paracontact manifold and let $g$ be a pseudo Riemannian metric (i.e., a non-degenerate bilinear form) defined on $M$, such that

\begin{equation}
(5.1) \quad g(\xi, \xi) = \varepsilon \gamma(x) = \varepsilon g(\xi, x),
\end{equation}

\begin{equation}
\varepsilon g(\phi x, \phi y) = g(x,y) - \varepsilon \gamma(x) \gamma(y), \quad \varepsilon = \pm 1.
\end{equation}

Then we shall call $M$ an almost paracontact pseudo metric manifold or manifold with $(\phi, \xi, \gamma, \varepsilon, g)$-structure.
Further if \( M \) is almost paracontact pseudo metric manifold and it also satisfies

\[
(5.1.2) \quad d\eta(X,Y) = 0, \quad (D_X\eta)(Y) = -g(\phi X, Y),
\]

where \( D \) is Riemannian connection of \( g \), then \( M \) is called pseudo paracontact manifold.

An almost paracontact pseudo metric manifold \( M(\phi, \xi, \eta, g, \varepsilon) \) is said to be pseudo para-Sasakian manifold if

\[
(5.1.3) \quad (D_X\phi)(Y) = \varepsilon \eta(Y)X - 2\varepsilon \eta(X)\eta(Y)\xi + g(X,Y)\xi, \tag{5.1.3}
\]

where \( D \) is Riemannian connection of \( g \).

Now we shall show that we can take \( \varepsilon = 1 \), in the above definitions of pseudo-structures without any loss of generality.

**Lemma 5.1.1.** Let \( M(\phi, \xi, \eta, g, \varepsilon) \) be a pseudo paracontact (resp. pseudo Para-Sasakian) manifold. If we put

\[
\bar{g} = -g, \quad \bar{\xi} = -\xi, \quad \bar{\eta} = -\eta, \quad \bar{\phi} = \phi \quad \text{and} \quad \bar{\varepsilon} = -\varepsilon,
\]
then $M(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g}, \tilde{e})$ is also a pseudo paracontact (resp. pseudo Para-Sasakian) manifold.

**Proof.** It is easy to check that $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g}, \tilde{e})$ is an almost paracontact pseudo-metric structure on $M$. Suppose that $(\phi, \xi, \eta, g, e)$ is a pseudo paracontact structure on $M$. Since the parallelism with respect to $g$ and $\tilde{g}$ are same, we get

$$(D_X \tilde{\eta})(Y) = -(\tilde{D}_X \eta)(Y) = g(\phi X, Y) = -\tilde{g}(\phi X, Y)$$

where $D$ and $\tilde{D}$ are Riemannian connection of $g$ and $\tilde{g}$ respectively. We also have

$$d \tilde{\eta}(X, Y) = -d \eta(X, Y) = 0.$$ 

Thus $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g}, \tilde{e})$ is also a pseudo paracontact structure on $M$.

Now let $(\phi, \xi, \eta, g, e)$ be a pseudo para-Sasakian structure on $M$. Then we have

$$(\tilde{D}_X \tilde{\phi})(Y) = (D_X \phi)(Y) = e \tilde{\eta}(Y) X - 2e \eta(\tilde{Y}) \tilde{\eta}(X) \tilde{\xi} + \tilde{g}(X, Y) \tilde{\xi}$$

$$= e \tilde{\eta}(Y) X - 2e \tilde{\eta}(X) \tilde{\eta}(Y) \tilde{\xi} + \tilde{g}(X, Y) \tilde{\xi}$$
Hence \((\phi, \overline{\xi}, \overline{\eta}, \overline{\xi}, \overline{\xi})\) is also a pseudo para-Sasakian structure on \(M\).

On account of the above lemma, henceforth we shall assume \(\xi = 1\), and hence drop it.

**Lemma 5.1.2.** In a pseudo para-Sasakian manifold \(M(\phi, \overline{\xi}, \overline{\eta}, \overline{\xi})\) we have

\[
\begin{align*}
(i) & \quad D_X \overline{\xi} = -\phi X \\
(ii) & \quad d \overline{\eta}(X, Y) = 0 \\
(iii) & \quad (D_X \overline{\eta})(Y) + (D_Y \overline{\eta})(X) = -2g(\phi X, Y).
\end{align*}
\]

**Proof.** Putting \(Y = \overline{\xi}\) in (5.1.3) we get

\[
(D_X \phi)(\overline{\xi}) = X - \overline{\eta}(X) \overline{\xi}
\]

i.e.,

\[-\phi(D_X \overline{\xi}) = \phi(\phi X)\]

which gives

\[
D_X \overline{\xi} = -\phi X + s \overline{\xi}
\]

for some real valued function \(s\). Operating \(\overline{\eta}\) on above
equation we get  
\[ s = \gamma(D_X \xi) = g(D_X \xi, \xi) = \frac{1}{2} x.g(\xi, \xi) = 0. \]

Hence we have (i). Equations (ii) and (iii) follow directly from (i).

**Theorem 5.1.1.** Given two pseudo para-Sasakian manifolds \( M(\phi, \xi, \eta, g) \) and \( \tilde{M}(\phi, \tilde{\xi}, \tilde{\eta}, \tilde{g}) \), each of dimension \( n \), with \( g \) and \( \tilde{g} \) having the same signatures. If \( M \) and \( \tilde{M} \) are both simply connected, complete and of constant curvature \(-1\), then there exists an isometry \( f : M \to \tilde{M} \) such that

\[ f_\ast \xi = \tilde{\xi}, \quad f_\ast \eta = \tilde{\eta}, \quad f_\ast \phi = \tilde{\phi} f_\ast. \]

**Proof.** For \( x \in M \) and \( \tilde{x} \in \tilde{M} \), let \( T_x(M) \) and \( T_{\tilde{x}}(\tilde{M}) \) denote the tangent spaces to \( M \) and \( \tilde{M} \) at the points \( x \) and \( \tilde{x} \) respectively. Since \( g \) and \( \tilde{g} \) have the same signature we can find an isometry \( F : T_x(M) \to T_{\tilde{x}}(\tilde{M}) \) such that \([8]\]

\[ F \xi_x = \tilde{\xi}_{\tilde{x}}, \quad \tilde{\eta}(Fx) = \eta(x) \text{ and } F\phi = \tilde{\phi} \circ F, x \in T_x(M). \]

Further since \( M \) and \( \tilde{M} \) are simply connected, complete and are of the same constant curvature, we have a unique isometry \([36]\).
such that
\[ f(x) = x \text{ and } f_\ast |_{T_x(M)} = F. \]

For any tangent vector \( X \) of \( M \), we have
\[
(5.1.4) \quad \bar{D}_X(f_\ast \bar{\omega}) = f_\ast(D_{f_\ast X} \bar{\omega}) = -f_\ast(f^{-1}_{\ast} \bar{\phi} X) = -\bar{\phi} X,
\]
where \( D \) and \( \bar{D} \) are Riemannian connections of \( g \) and \( \bar{g} \) respectively. Since \( M \) is a pseudo para-Sasakian manifold we have
\[
(5.1.5) \quad \bar{D}_X(\bar{\omega}) = -\bar{\phi} X.
\]
From (5.1.4) and (5.1.5) we get
\[
(5.1.6) \quad f_\ast \bar{\omega} = \bar{\omega} \quad \text{and} \quad f_\ast \bar{\eta} = \eta
\]
Finally for any \( X \in \mathfrak{X}(M) \) and \( Y \in \mathfrak{X}(\bar{M}) \) we have
\[
g(f_\ast \phi X, \bar{Y})f_\ast f_\ast g(\phi X, f_\ast Y) = g(\phi X, f_\ast Y) = -(D_X \bar{\eta})(f_\ast Y)
\]
\[ = - (D_{\Phi X} \gamma)(Y) \circ f \]
\[ = g(\Phi f^* X, Y) \circ f, \]

showing that \( f^* \circ \phi = \Phi \circ f^* \), completing the proof of the theorem.

Let \( R \) and \( R_\perp \) denote the curvature tensor and the Ricci tensor of a pseudo para-Sasakian manifold \( M(\phi, \xi, \eta, g) \). Now we prove the following

**Lemma 5.1.3.** In a pseudo para-Sasakian manifold \( M(\phi, \xi, \eta, g) \) we have

\[ R(X, Y) \xi = \gamma(X) Y - \gamma(Y) X \]  
\[ R(X, \xi) Y = g(X, Y) \xi - \gamma(Y) X \]  
\[ R_\perp(\xi, X) = (1 - m) \gamma(X). \]

**Proof.** We have

\[ g(R(X, Y) \xi, Z) = g(D_X D_Y \xi - D_Y D_X \xi - D_{[X, Y]} \xi, Z). \]

Using (1) of Lemma (5.1.2), we get
\[ g(R(X,Y) \xi, Z) = -g((D_X \phi)(Y) - (D_Y \phi)(X), Z), \]

which in consequence of (5.1.3) gives (5.1.7). The equations (5.1.8) and (5.1.9) follow directly from (5.1.7).

**Theorem 5.1.2.** If in a pseudo para-Sasakian manifold \( M(\phi, \xi, \eta, g), R(X,Y) \xi = 0 \), then the sectional curvature of the plane section orthogonal to \( \xi \) is \(-1\).

**Proof.** Let \( X \) and \( Y \) be vectors tangent to \( M \), which span a plane orthogonal to \( \xi \). Then

\[ (5.1.10) \quad \eta(X) = \eta(Y) = 0. \]

We have

\[ (R(X, \xi), R)(X,Y) \]

\[ = R(X, \xi)R(X,Y)Y - R(R(X, \xi)X, Y)Y - R(X, R(X, \xi))Y \]

\[ - R(X, Y)R(X, \xi)Y. \]

Using (5.1.7), (5.1.8) and (5.1.10) we get

\[ (R(X, \xi)R)(X,Y)Y = -\eta(R(X,Y)X) + g(X, R(X,Y)Y)\xi \]

\[ + g(X,X)g(Y,Y)\xi - g(X,Y)\xi^2. \]
Taking the left hand side zero and equating the components of \( \Xi \) we get

\[
g(X, R(X,Y)Y) = -\left[g(X,X)g(Y,Y) - g(X,Y)^2\right]
\]

i.e.

\[
K(X, Y) = -1,
\]

where \( K(X,Y) \) is the sectional curvature of the plane section determined by \( X \) and \( Y \).

**THEOREM 5.1.3.** If in a pseudo para-Sasakian manifold \( M(\Phi, \Xi, \gamma, g) \), \( R(X,Y)R_1 = 0 \), then it is an Einstein manifold with Ricci scalar \(-(n-1)\).

**PROOF.** We have

\[
(R(X, \Xi)R_1)(\Xi, Y) = R(X, \Xi)R_1(\Xi, Y) - R_1(R(X, \Xi)\Xi, Y) - R_1(\Xi, R(X, \Xi)Y).
\]

Using (5.1.7), (5.1.8) and (5.1.9) we get

\[
(R(X, \Xi)R_1)(\Xi, Y) = R_1(X,Y) - (1-n)g(X,Y),
\]

which by virtue of given condition yields
$R^c(X,Y) = -(n-1) g(X,Y),$

showing that $M$ is an Einstein manifold with Ricci scalar $-(n-1)$.

5.2. PSEUDO-PARA-SASAKIAN MANIFOLDS ISOMETRICALLY IMMERSED IN $R^{n+1}$

Let $\mathbb{R}^n$ be endowed with a pseudo Riemannian metric $g_s$ which is defined by the parallel displacement of the inner product

$$\langle X, Y \rangle = -\sum_{i=1}^{s} x_i y_i + \sum_{j=s+1}^{n} x_j y_j,$$

and denote this space $(\mathbb{R}^n, g_s)$ by $E^n_s$. Then the signature of $g_s$ is $n$ and $E^n_s$ is complete manifold with constant curvature zero [36].

Let $M(\phi, \xi, \tau, g)$ be a pseudo para-Sasakian manifold and suppose we have an isometry

$$f : M \to E^n_s.$$

For each $x \in M$, we can choose a unit vector field $\xi$ normal
to $M$ such that $g_{B}(C, 0) = \epsilon$, $\epsilon = \pm 1$. Since $E_{B}$ has zero curvature, the Gauss equation (1.5.3) expressing the curvature $R$ of $M$ has the form

$$R(X, Y)Z = g \left\{ g(Z, AX)AX - g(Z, AX)AY \right\},$$

where $A$ is the field of symmetric endomorphisms, which corresponds to the second fundamental form of $M$.

**Theorem 5.2.1.** Suppose a pseudo para-Sasakian manifold $M(\phi, \xi, \gamma, g)$ is properly and isometrically immersed in $E_{B}$, such that $\xi$ is not an eigen vector of $A$. Then $M$ is of constant curvature $-1$.

**Proof.** We have from (5.2.2)

$$R(X, \xi)Y = g \left\{ \gamma(AY)AX - g(AX, Y)AY \right\}$$

Since the immersion is proper, $A$ can be expressed as a real diagonal matrix with respect to a suitable orthogonal frame at each point of $M$. Let $(e_{1}, e_{2}, \ldots, e_{n})$ be such a frame. Then

$$Ae_{i} = \lambda_{i}e_{i}, \quad 1 \leq i \leq n, \quad \sum_{i} \lambda_{i} = R.$$

We have from (5.2.3) and (5.1.8) with $X = e_{1}$ and
\[ Y = e_j \]

\[ g(e_i, e_j) \Xi - \gamma(e_j) e_1 = \Theta \sum_{i \neq j} \gamma(e_j) e_1 - \gamma_i g(e_i, e_j) \Xi \]

If \( i \neq j \), we get

\[ \gamma(e_j) e_1 = - \Theta \gamma_i g(e_i, e_j) \Xi \]

The above equation informs that either \( \gamma(e_j) = 0 \) for some \( j \) or \( \Theta \gamma_i g_{ij} = -1 \) \((i \neq j)\).

Now suppose \( \gamma(e_{j_0}) = 0 \) for some \( j_0 \), then (5.2.5) implies

\[ \Xi = - \Theta \gamma_{j_0} A(\Xi) \]

Hence \( \gamma_{j_0} \neq 0 \) and \( A(\Xi) = \left( \frac{-1}{\gamma_{j_0}} \right) \Xi \). This shows that \( \Xi \) is an eigen vector of \( A \), which is not possible by the statement.

Thus we are left with \( \Theta \gamma_i g_{ij} = -1 \) for all \( i \neq j \). Then \( \gamma_i \neq 0 \) for all \( i \) and therefore

\[ \gamma_1 = \gamma_2 = \ldots = \gamma_n = \gamma \text{ say,} \]

which shows that \( A = \gamma I \) and \( \Theta \gamma^2 = -1 \), hence by (5.2.2)
we get

\[ R(X,Y)Z = - \left\{ g(Y,Z)X - g(X,Z)Y \right\} . \]

This completes the proof of the theorem.

5.3. PSEUDO PARA-SASAKIAN MANIFOLDS ISOMETRICALLY IMMERSED IN PSEUDO RIEMANNIAN MANIFOLDS OF CONSTANT CURVATURE.

Let \((M, g)\) be a pseudo Riemannian manifold of constant curvature \(C\) and let \(M(\phi, \xi, \eta, g)\) be a pseudo para-Sasakian manifold which is isometrically immersed in \((M, g)\). The Gauss equation is

\[ (5.3.1) \quad R(X,Y)Z = C \left[ g(Y,Z)X - g(X,Z)Y \right] + g \left[ g(AX,Z)AX - g(AX,Y)AY \right] \]

where \(R\) is the curvature tensor of \(M\) and \(A\) is the field of the second fundamental form of operators which corresponds to the field of normal vectors \(N\) to \(M\) and \(g(N,N) = \xi\), \(\xi = \pm 1\).

From (5.3.1) we have

\[ (5.3.2) \quad R(X, \xi)Y = C \left[ \gamma(Y)X - g(X,Y)\xi \right] + g \left[ \gamma(AX)AX - g(AX,Y)AY \right] \]
Now we prove the following interesting theorem.

**THEOREM 5.3.1.** Let \( M(\phi, \Xi, \gamma, g) \) be a pseudo para-Sasakian manifold which is properly and isometrically immersed in a pseudo Riemannian manifold \((\widetilde{M}, \tilde{g})\) of constant curvature \( C \neq -1 \). Then either \( M \) is totally umbilic in \( \widetilde{M} \) and of constant curvature \(-1\), or \( M \) is locally isometric to the product of two manifolds of constant curvature.

**PROOF.** Since the immersion is proper, \( A \) has its representation in terms of a real diagonal matrix, subject to a suitable orthogonal frame \((\mathbf{e}_1, \ldots, \mathbf{e}_n)\) at each point of \( M \). Then we have

\[
A\mathbf{e}_i = \rho_i \mathbf{e}_i, \quad 1 \leq i \leq n, \quad \rho_i \in \mathbb{R}.
\]

Using this in (5.3.2) and (5.1.7) we get

\[
(5.3.3) \quad 0 \left[ \gamma(\mathbf{e}_j)\mathbf{e}_1 - g(\mathbf{e}_1, \mathbf{e}_j) \Xi \right] + \epsilon \left[ \rho_i \rho_j \gamma(\mathbf{e}_j)\mathbf{e}_1 - \rho_j g(\mathbf{e}_1, \mathbf{e}_j) \Xi \right] = g(\mathbf{e}_1, \mathbf{e}_j) \Xi - \gamma(\mathbf{e}_j)\mathbf{e}_1.
\]

In particular we have for \( i \neq j \)

\[
(5.3.4) \quad \left[ 0 + \epsilon \rho_i \rho_j + \frac{1}{n} \gamma(\mathbf{e}_j)\mathbf{e}_1 \right] = 0.
\]
Put \( K = 1 + C \), then as \( C \neq -1, K \neq 0 \) and

\[
[g \gamma_i \gamma_j + K] \gamma(e_j)e_i = 0.
\]

Then we have either

(a) \( e \gamma_i \gamma_j = -K \) for all \( i \neq j \)

or

(b) \( \gamma(e_j) = 0 \) for some \( j \).

(a): In this case since \( K \neq 0 \),

\[
\gamma_1 = \gamma_2 = \ldots = \gamma_n = \hat{\gamma} \text{ say}
\]

and thus

(5.3.5)

\[
g \gamma^2 = -K.
\]

This shows that \( \lambda = \gamma^2 \), i.e., \( M \) is totally umbilic and in light of (5.3.5), (5.3.1) becomes

\[
R(X,Y)Z = g \left\{ g(Y,Z)X - g(X,Z)Y \right\} + g \left[ g(Y,Z)X - g(X,Z)Y \right] \gamma^2
\]

\[
= (-1) \left[ g(Y,Z)X - g(X,Z)Y \right]
\]

i.e., \( M \) is umbilic and of constant curvature \(-1\).
(b) : In this case, (5.3.3) with \( i = j \) gives

\[- e \rho j A(\xi) = \xi + c \xi = K \xi\]

Hence \( \rho j \neq 0 \) and

\[
(5.3.6) \quad A(\xi) = \frac{-e K}{\rho j} \xi
\]

This shows that \( \xi \) is an eigen vector of \( A \), let us choose \( e_1 = \xi \), and hence

\[\gamma_j(e_j) = g(\xi, e_j) = 0 \quad \text{for} \quad j = 2, 3, \ldots, n.\]

Thus (5.3.6) still hold for \( j = 2, \ldots, n \), which shows that

\[\rho_2 = \ldots = \rho_n = \rho \neq 0\]

and

\[\rho_1 \rho = -e K.\]

This shows that there are atmost two distinct eigen values \( \rho_i \) and \( \rho \) of \( A \) at each point \( x \in M \).

Let \( T_1 \) and \( T \) be the distributions defined by

\[T_1 : \text{Spanned by } \xi, \quad \text{and } T_x = \{ x \in T_x(M) : A(x) = \rho x \}\]

obviously \( T_1 \) is involutive. Now for \( T \) let \( Z \in T_x - \gamma(z) = 0 \),
and since \( \mathcal{M} \) is pseudo para-Sasakian manifold, we have by Lemma (5.1.2)

\[
\gamma([X,Y]) = d\gamma(X,Y) = 0, \quad X,Y \in T_x
\]

showing that \([X,Y] \in T_x\). Hence \( T \) is also involutive. Thus for this case (cf. [15], Theo. 2.5) \( \mathcal{M} \) is locally isometric to the product of two manifolds each of constant curvature. Completing the proof of theorem.

**Theorem 5.3.2.** Let \( \mathcal{M}(\phi, \Xi, \gamma, g) \) be a pseudo para-Sasakian manifold which is properly and isometrically immersed in a pseudo-Riemannian manifold \((\mathcal{M},g)\) of constant curvature \(-1\), and let \( \mathcal{M} \) satisfies the condition

\[
\text{R}(X,Y) \cdot R_q = 0.
\]

Then \( \mathcal{M} \) is totally geodesic or developable hypersurface of \( \mathcal{M} \), in particular it is of constant curvature \(-1\).

**Proof.** We know from Theorem (5.1.3) that if \( \mathcal{M}(\phi, \Xi, \gamma, g) \) satisfies \( \text{R}(X,Y) \cdot R_q = 0 \), then it is an Einstein space, with Ricci scalar \( r = -(n-1) \). Now we state the following lemma [8].

**Lemma.** Let \( (N, f) \), \( m \geq 3 \) be a pseudo-Einstein manifold with Ricci curvature \( r \), which is properly and isometrically \( -m+1 \)-immersed in a pseudo Riemannian manifold \((\mathcal{M}, f)\) of constant curvature \(-1\).
curvature \( C \), then if \( r = (m-1) C \), \((M, f)\) is either totally geodesic or a developable hypersurface in \( M^{m+1} \), in particular it is of constant curvature \( C \).

We observe that in our case \( C = -1 \) and \( r = -(n-1) = (n-1) C \). Hence applying this lemma to \( M \) we get the result.

**Theorem 5.3.3.** Suppose a pseudo para-Sasakian manifold \( M(\phi, \xi, \eta, g) \) is improperly and isometrically immersed in a pseudo Riemannian manifold \((M, g)\) of constant curvature \(-1\). If rank \( A \geq 2 \) at \( x \in M \), then \( A(\xi_{\lambda}) = 0 \).

**Proof.** Since the immersion is proper, we have

\[
A e_i = \sum_{\rho} \xi_{\rho} e_i, \quad 1 \leq i \leq \eta, \quad \rho \in \mathbb{R}.
\]

Now (5.3.3) with \( C = -1 \), implies

\[
(5.3.7) \quad \sum_{\rho} \xi_{\rho} \eta(e_j) e_i = \sum_{\rho} g(e_i, e_j) A_{\xi_{\rho}}, \quad 1 \leq i, j \leq \eta.
\]

Since \( \text{rank } A \geq 2 \), we may suppose \( \rho_1 \rho_2 \neq 0 \). Then (5.3.7) with \( i = 1 \) and \( j = 2 \) gives
\[ \gamma(e_2) = 0. \]

Hence putting \( i = j = 1 \) in (5.3.7) we get

\[ A_{\mathcal{E}_2} = 0. \]