CHAPTER 3
HERMITE-BASED SHEFFER POLYNOMIALS:
PROPERTIES AND APPLICATIONS

3.1. INTRODUCTION

Sequences of polynomials play an important role in various branches of science. One of the important classes of polynomial sequences is the class of Sheffer sequences. We have discussed Sheffer sequences and the related concepts in Section 1.5 of Chapter 1. This class may be defined in many ways, most commonly by a generating function and by a differential recurrence relation. A polynomial sequence \( \{s_n(x)\}_{n=0}^{\infty} \) (\( s_n(x) \) being a polynomial of degree \( n \)) is called Sheffer A-type zero [99] (which we shall hereafter call Sheffer-type) if \( s_n(x) \) possesses the exponential generating function of the form

\[
A(t) \exp (xH(t)) = \sum_{n=0}^{\infty} s_n(x) \frac{t^n}{n!},
\]

where \( A(t) \) and \( H(t) \) have (at least the formal) expansions

\[
A(t) = \sum_{n=0}^{\infty} A_n \frac{t^n}{n!}, \quad A_0 \neq 0
\]

(3.1.2a)

and

\[
H(t) = \sum_{n=1}^{\infty} H_n \frac{t^n}{n!}, \quad H_1 \neq 0
\]

(3.1.2b)

respectively.

The Sheffer class contains important sequences such as the Hermite, Laguerre, Bessel, Bernoulli, Poisson-Charlier, factorial polynomials et cetera. (A detailed study of these polynomials and more will be discussed in the next section). These polynomials are important from the viewpoint of applications in physics and number theory. Properties of Sheffer sequences are naturally handled within the framework of modern classical umbral calculus by Roman [102].

The sequences of Sheffer A-type zero are called poweroids by Steffensen [121], from which the idea of monomiality came. The monomiality principle is presented in detail in Section 1.5 of Chapter 1.
It has been shown in [92] that if \( s_n(x) \) are of Sheffer-type then it is possible to give explicit representations of the multiplicative and derivative operators \( \hat{M} \) and \( \hat{P} \) of Eq. (1.5.22). Conversely if \( \hat{M} = \hat{M}(X, D) \) and \( \hat{P} = \hat{P}(D) \) then \( s_n(x) \) of equation (1.5.4) are necessarily of Sheffer-type. A general theorem [19,102] states that a polynomial sequence \( s_n(x) \) satisfying the monomiality principle equation (1.5.22) with an operator \( \hat{P} \) given as a function of the derivative operator only \( \hat{P} = \hat{P}(D) \) is uniquely determined by two (formal) power series

\[
f(t) = \sum_{n=0}^{\infty} f_n \frac{t^n}{n!}, \quad f_0 = 0, \quad f_1 \neq 0 \tag{3.1.3a}
\]

and

\[
g(t) = \sum_{n=0}^{\infty} g_n \frac{t^n}{n!}, \quad g_0 \neq 0. \tag{3.1.3b}
\]

The exponential generating function of \( s_n(x) \) is then given by

\[
G(x,t) = \sum_{n=0}^{\infty} s_n(x) \frac{t^n}{n!} = \frac{1}{g(f^{-1}(t))} \exp \left( x f^{-1}(t) \right), \tag{3.1.4}
\]

where \( f^{-1}(t) \) is the compositional inverse of \( f(t) \). The associated raising and lowering operators of equation (1.5.22) are given by

\[
\hat{M} = \left[ X - \frac{g'(D)}{g(D)} \right] \frac{1}{f'(D)} \tag{3.1.5a}
\]

and

\[
\hat{P} = f(D), \tag{3.1.5b}
\]

respectively.

We note that \( X \) enters \( \hat{M} \) only linearly and the order of \( X \) and \( D \) in \( \hat{M}(X, D) \) matters. By direct calculation one may check that any pair \( \hat{M}, \hat{P} \) from Eqs. (3.1.5a), (3.1.5b) satisfies the commutation relation (1.5.23), see for details [19,102].

In view of Eqs. (3.1.1) and (3.1.4), we have

\[
A(t) = \frac{1}{g(f^{-1}(t))} \tag{3.1.6a}
\]

and

\[
H(t) = f^{-1}(t). \tag{3.1.6b}
\]

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Several types of polynomial sequences are studied using the monomiality principle, see for example [21,22,28,36,38-41,45,46,92]. We recall that the 2-variable Hermite Kampe de Feriet polynomials $2VHKdP H_n(x,y)$ [6], defined by

$$H_n(x,y) = n! \sum_{r=0}^{\left[\frac{n}{2}\right]} \frac{x^{n-2r} y^r}{r! (n-2r)!}, \quad (3.1.7)$$

have shown to be quasi-monomials under the action of the operators [21, p. 148(1.9)]

$$\hat{M} = x + 2y \frac{\partial}{\partial x}, \quad (3.1.8a)$$

$$\hat{P} = \frac{\partial}{\partial x}. \quad (3.1.8b)$$

The properties of $2VHKdP H_n(x,y)$ are derived by using the monomiality principle, according to which the differential equation and the generating function for $H_n(x,y)$ are given as: [21, p. 149(1.10) and (1.14)]

$$\left(2y \frac{\partial^2}{\partial x^2} + x \frac{\partial}{\partial x} - n\right) H_n(x,y) = 0 \quad (3.1.9)$$

and

$$\exp(x t + y t^2) = \sum_{n=0}^{\infty} H_n(x,y) \frac{t^n}{n!}, \quad (3.1.10)$$

respectively.

In view of definition (3.1.7), we note that

$$H_n(2x,-1) = H_n(x) \quad \text{and} \quad H_n(x,-\frac{1}{2}) = H_{cn}(x) \quad (3.1.11)$$

with $H_n(x)$ or $H_{cn}(x)$ being ordinary Hermite polynomials (1.4.15). Also

$$H_n(x,0) = x^n. \quad (3.1.12)$$

Now, since we have

$$\frac{\partial}{\partial y} H_n(x,y) = \frac{\partial^2}{\partial x^2} H_n(x,y), \quad (3.1.13)$$

which in view of Eq. (3.1.12), gives the following operational definition for $H_n(x,y)$:

$$H_n(x,y) = \exp \left(y \frac{\partial^2}{\partial x^2}\right) \{x^n\}. \quad (3.1.14)$$
Next, we recall that the 3-variable Hermite polynomials (3VHP) $H_n(x, y, z)$ \cite{22, p. 114(22)} are defined as:

$$H_n(x, y, z) = n! \sum_{r=0}^{[n/3]} \frac{z^r H_{n-3r}(x, y)}{r! (n - 3r)!}, \quad (3.1.15)$$

which are quasi-monomials under the action of the operators

$$\hat{M} = x + 2y \frac{\partial}{\partial x} + 3z \frac{\partial^2}{\partial x^2}, \quad (3.1.16a)$$

$$\hat{P} = \frac{\partial}{\partial x}. \quad (3.1.16b)$$

The polynomials $H_n(x, y, z)$ satisfy the following differential equation:

$$\left(3z \frac{\partial^3}{\partial x^3} + 2y \frac{\partial^2}{\partial x^2} + x \frac{\partial}{\partial x} - n\right) H_n(x, y, z) = 0 \quad (3.1.17)$$

and are specified by the generating function

$$\exp(x^t + yt + zt^3) = \sum_{n=0}^{\infty} H_n(x, y, z) \frac{t^n}{n!}. \quad (3.1.18)$$

These polynomials satisfy the following relations

$$\frac{\partial}{\partial y} H_n(x, y, z) = \frac{\partial^2}{\partial x^2} H_n(x, y, z) \quad (3.1.19a)$$

and

$$\frac{\partial}{\partial z} H_n(x, y, z) = \frac{\partial^3}{\partial x^3} H_n(x, y, z). \quad (3.1.19b)$$

Now, in view of Eqs. (3.1.12), (3.1.13), (3.1.19b) and the initial condition

$$H_n(x, y, 0) = H_n(x, y), \quad (3.1.20)$$

we get the following operational definition for $H_n(x, y, z)$:

$$H_n(x, y, z) = \exp \left( y \frac{\partial^2}{\partial x^2} + z \frac{\partial^3}{\partial x^3} \right) \{x^n\}. \quad (3.1.21)$$

In this Chapter, we use the concepts and the formalism associated with monomiality principle and Sheffer sequences to introduce and study Hermite-based Sheffer
polynomials. In Section 3.2, we generate family of Hermite-based Sheffer polynomials associated with 3VHP $H_n(x, y, z)$ and discuss their properties. In Section 3.3, we derive the generating functions and series expansions of some members of Hermite-Sheffer family. In Section 3.4, we derive several new relations, identities and expansions for Hermite-Sheffer polynomials. Finally, we give some concluding remarks in Section 3.5.

3.2. HERMITE-BASED SHEFFER POLYNOMIALS

To generate Hermite-based Sheffer polynomials associated with 3VHP $H_n(x, y, z)$, we introduce the generating function

$$Q(x, y, z; t) = A(t) e^{\hat{M} H(t)},$$

or, equivalently

$$Q(x, y, z; t) = A(t) \exp \left( x + 2y \frac{\partial}{\partial x} + 3z \frac{\partial^2}{\partial x^2} \right) H(t),$$

which is the result of replacement of $x$ in the l.h.s. of definition (3.1.1) by the multiplicative operator $\hat{M}$ of $H_n(x, y, z)$ given by Eq. (3.1.16a).

Now, decoupling the exponential operator in the r.h.s. of Eq. (3.2.1), by using the Berry decoupling identity [41]

$$e^C + \hat{D} = e^{m^2/2} e^{- \left( \frac{m}{2} \hat{C}^2 + \hat{C} \right)} e^{\hat{D}}, \quad \left[ \hat{C}, \hat{D} \right] = m \hat{C}^{1/2},$$

we get the generating function for Hermite-based Sheffer polynomials $HS_n(x, y, z)$ in the form:

$$Q(x, y, z; t) = A(t) \exp \left( xH(t) + yH^2(t) + zH^3(t) \right) = \sum_{n=0}^{\infty} HS_n(x, y, z) \frac{t^n}{n!}.$$  (3.2.3)

Differentiating Eq. (3.2.3) partially with respect to $x$, $y$ and $z$, we observe that $HS_n(x, y, z)$ are solutions of the equations

$$\frac{\partial}{\partial y} HS_n(x, y, z) = \frac{\partial^2}{\partial x^2} HS_n(x, y, z),$$

$$\frac{\partial}{\partial z} HS_n(x, y, z) = \frac{\partial^3}{\partial x^3} HS_n(x, y, z).$$

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Now, proceeding on the same lines with the multiplicative operator \( \hat{M} \) given in Eq. (3.1.8a) of the 2VHKdFP \( H_n(x,y) \), we get the generating function for Hermite-based Sheffer polynomials \( H_{Sn}(x,y) \) in the form:

\[
\mathcal{R}(x,y; t) = A(t) \exp \left( xH(t) + yH^2(t) \right) = \sum_{n=0}^{\infty} H_{Sn}(x,y) \frac{t^n}{n!}. \quad (3.2.5)
\]

From Eq. (3.2.5), it follows that

\[
\frac{\partial}{\partial y} H_{Sn}(x,y) = \frac{\partial^2}{\partial x^2} H_{Sn}(x,y). \quad (3.2.6)
\]

Now, in view of Eqs. (3.2.3), (3.2.5) and (3.1.1), we have

\[
H_{Sn}(x,y,0) = H_{Sn}(x,y), \quad (3.2.7)
\]

\[
H_{Sn}(x,0,0) = H_{Sn}(x,0) = s_n(x). \quad (3.2.8)
\]

Finally, solving Eqs. (3.2.6) and (3.2.4b) using Eqs. (3.2.7) and (3.2.8), we get the following operational definition for \( H_{Sn}(x,y,z) \):

\[
H_{Sn}(x,y,z) = \exp \left( y \frac{\partial^2}{\partial x^2} + z \frac{\partial^3}{\partial x^3} \right) \{s_n(x)\}. \quad (3.2.9)
\]

Thus the Hermite-Sheffer polynomials \( H_{Sn}(x,y,z) \) can be generated from the corresponding Sheffer polynomials \( s_n(x) \) by employing the operational rule (3.2.9).

A simple computation shows that the operational rule (3.2.9) can be written in the following general form:

\[
H_{Sn}(m(x+w),m^2y,m^3z) = \exp \left( y \frac{\partial^2}{\partial x^2} + z \frac{\partial^3}{\partial x^3} \right) \{s_n(m(x+w))\}. \quad (3.2.10)
\]

For \( m = 1 \), Eq. (3.2.10) gives

\[
H_{Sn}(x+w,y,z) = \exp \left( y \frac{\partial^2}{\partial x^2} + z \frac{\partial^3}{\partial x^3} \right) \{s_n(x+w)\} \quad (3.2.11)
\]

and for \( w = 0 \), Eq. (3.2.10) gives

\[
H_{Sn}(mx,m^2y,m^3z) = \exp \left( y \frac{\partial^2}{\partial x^2} + z \frac{\partial^3}{\partial x^3} \right) \{s_n(mx)\}. \quad (3.2.12)
\]
3.3. PROPERTIES OF SOME MEMBERS OF HERMITE-SHEFFER FAMILY

First, we recall some members of the Sheffer and associated Sheffer family:

I. Generalized Hermite Polynomials $H_{n,m,\nu}(x)$ [77]:

The generalized Hermite polynomials $H_{n,m,\nu}(x)$ are Sheffer for

$$g(t) = e^{(\frac{t}{\nu})^m}, \quad f(t) = \frac{t}{\nu}. \quad (3.3.1)$$

The generating function for the generalized Hermite polynomials $H_{n,m,\nu}(x)$ is given as:

$$\exp(\nu xt - t^m) = \sum_{n=0}^{\infty} H_{n,m,\nu}(x) \frac{t^n}{n!}. \quad (3.3.2)$$

Using Eqs. (3.1.5a), (3.1.5b), we find that the multiplicative and derivative operators for $H_{n,m,\nu}(x)$ are given by

$$M(X, D) = \frac{m}{\nu^m-1} D^{m-1} \quad (3.3.3a)$$

and

$$P(D) = \frac{D}{\nu}. \quad (3.3.3b)$$

Also, in view of Eqs. (3.1.6a), (3.1.6b), we have

$$A(t) = e^{-t^m}, \quad H(t) = \nu t. \quad (3.3.4)$$

Taking $m = \nu = 2$ and using the relation

$$H_{n,2,2}(x) = H_n(x), \quad (3.3.5)$$

we obtain the corresponding results for the ordinary Hermite polynomials $H_n(x)$.

II. Generalized Laguerre Polynomials $L_n^{(\alpha)}(x)$ [1,99]:

The generalized Laguerre polynomials $L_n^{(\alpha)}(x)$ are Sheffer for

$$g(t) = (1 - t)^{-\alpha-1}, \quad f(t) = \frac{t}{t-1}. \quad (3.3.6)$$

The generating function for the generalized Laguerre polynomials $L_n^{(\alpha)}(x)$ is given by Eq. (1.4.21).
The multiplicative and derivative operators for \( L_n^{(\alpha)}(x) \) are given by
\[
\hat{M}(X,D) = -XD^2 + (2X - \alpha - 1)D - X + \alpha + 1 \quad (3.3.7a)
\]
and
\[
\hat{P}(D) = \frac{D}{D - 1}. \quad (3.3.7b)
\]

Also, we have
\[
A(t) = \frac{1}{(1-t)^{\alpha+1}}, \quad H(t) = \frac{t}{t-1}. \quad (3.3.8)
\]

Taking \( \alpha = 0 \), we obtain the corresponding results for the classical Laguerre polynomials \( L_n(x) \).

Further, we note that the polynomials \( L_n^{(-1)}(x) \) are the associated Sheffer for
\[
g(t) = 1, \quad f(t) = \frac{t}{t-1}. \quad (3.3.9)
\]

### III. Pidduck Polynomials \( P_n(x) \) [8], (see also [12], [56]):

The Pidduck polynomials \( P_n(x) \) are Sheffer for
\[
g(t) = \frac{2}{e^t - 1}, \quad f(t) = \frac{e^t - 1}{e^t + 1}. \quad (3.3.10)
\]

The generating function for the Pidduck polynomials \( P_n(x) \) is given as:
\[
\frac{t}{1-t} \left( \frac{1+t}{1-t} \right)^x = \sum_{n=0}^{\infty} P_n(x) \frac{t^n}{n!}. \quad (3.3.11)
\]

The multiplicative and derivative operators for \( P_n(x) \) are given by
\[
\hat{M}(X,D) = \frac{X(1+eD)^2}{2e^D} - \frac{(1+eD)^2}{2(1-eD)} \quad (3.3.12a)
\]
and
\[
\hat{P}(D) = \frac{e^D - 1}{e^D + 1}. \quad (3.3.12b)
\]

Also, we have
\[
A(t) = \frac{t}{1-t}, \quad H(t) = \ln \left( \frac{1+t}{1-t} \right). \quad (3.3.13)
\]
The Mittag-Leffler polynomials $M_n(x)$ [8] are the associated Sheffer for

$$g(t) = 1 \quad , \quad f(t) = \frac{e^t - 1}{e^t + 1} . \quad (3.3.14)$$

The generating function for Mittag-Leffler polynomials $M_n(x)$ is given as:

$$\left( \frac{1+t}{1-t} \right)^x = \sum_{n=0}^{\infty} M_n(x) \frac{t^n}{n!} . \quad (3.3.15)$$

The multiplicative and derivative operators for $M_n(x)$ are given by

$$\hat{M}(X, D) = \frac{X(1 + e^D)^2}{2e^D} \quad (3.3.16a)$$

and

$$\hat{P}(D) = \frac{e^D - 1}{e^D + 1} . \quad (3.3.16b)$$

Also, we have

$$A(t) = 1 \quad , \quad H(t) = \ln \left( \frac{1+t}{1-t} \right) . \quad (3.3.17)$$

IV. Acturial Polynomials $a_n^{(\beta)}(x)$ [12]:

The acturial polynomials $a_n^{(\beta)}(x)$ are Sheffer for

$$g(t) = (1-t)^{-\beta} \quad , \quad f(t) = \ln(1-t) . \quad (3.3.18)$$

The generating function for the acturial polynomials $a_n^{(\beta)}(x)$ is given as:

$$\exp \left( \beta t + x(1 - e^t) \right) = \sum_{n=0}^{\infty} a_n^{(\beta)}(x) \frac{t^n}{n!} . \quad (3.3.19)$$

The multiplicative and derivative operators for $a_n^{(\beta)}(x)$ are given by

$$\hat{M}(X, D) = XD - X + \beta \quad (3.3.20a)$$

and

$$\hat{P}(D) = \ln(1 - D) . \quad (3.3.20b)$$

Also, we have

$$A(t) = e^{\beta t} \quad , \quad H(t) = 1 - e^t . \quad (3.3.21)$$
The exponential polynomials $\phi_n(x)$ [10] are the associated Sheffer for

$$g(t) = 1, \quad f(t) = \ln(1 + t).$$

(3.3.22)

The generating function for the exponential polynomials $\phi_n(x)$ is given as:

$$\exp \left( x(e^t - 1) \right) = \sum_{n=0}^{\infty} \phi_n(x) \frac{t^n}{n!}.$$  \hspace{1em} (3.3.23)

The multiplicative and derivative operators for $\phi_n(x)$ are given by

$$\hat{M}(X, D) = X(1 + D)$$ \hspace{1em} (3.3.24a)

and

$$\hat{P}(D) = \ln(1 + D).$$ \hspace{1em} (3.3.24b)

Also, we have

$$A(t) = 1, \quad H(t) = e^t - 1.$$ \hspace{1em} (3.3.25)

Further, in view of Eqs. (3.3.22) and (3.3.23), one can verify that $\phi_n(-x)$ are the associated Sheffer for

$$g(t) = 1, \quad f(t) = \ln(1 - t).$$ \hspace{1em} (3.3.26)

V. Poisson-Charlier Polynomials $c_n(x; a)$ [69], (see also [53], [126]):

The Poisson-Charlier polynomials $c_n(x; a)$ are Sheffer for

$$g(t) = \exp \left( a(e^t - 1) \right), \quad f(t) = a(e^t - 1).$$

(3.3.27)

The generating function for the Poisson-Charlier polynomials $c_n(x; a)$ is given as:

$$e^{-t} \left( 1 + \frac{t}{a} \right)^x = \sum_{n=0}^{\infty} c_n(x; a) \frac{t^n}{n!}.$$ \hspace{1em} (3.3.28)

The multiplicative and derivative operators for $c_n(x; a)$ are given by

$$\hat{M}(X, D) = \frac{X}{ae^D} - 1$$ \hspace{1em} (3.3.29a)

and

$$\hat{P}(D) = a(e^D - 1).$$ \hspace{1em} (3.3.29b)

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Also, we have
\[ A(t) = e^{-t} \quad , \quad H(t) = \ln \left(1 + \frac{t}{a}\right). \]  

\[ \text{(3.3.30)} \]

**VI. Peters Polynomials \( s_n(x; \lambda, \mu) \) [12]:**

The Peters polynomials \( s_n(x; \lambda, \mu) \) are Sheffer for
\[ g(t) = \left(1 + e^{\lambda t}\right)\mu \quad , \quad f(t) = e^t - 1. \]  

\[ \text{(3.3.31)} \]

The generating function for the Peters polynomials \( s_n(x; \lambda, \mu) \) is given as:
\[ \left(1 + (1 + t)^\lambda\right)^\mu (1 + t)^x = \sum_{n=0}^{\infty} s_n(x; \lambda, \mu) \frac{t^n}{n!}. \]  

\[ \text{(3.3.32)} \]

The multiplicative and derivative operators for \( s_n(x; \lambda, \mu) \) are given by
\[ \hat{M}(X, D) = X e^{-D} - \frac{\mu \lambda e^{(\lambda-1)D}}{1 + e^{\lambda D}} \]  

\[ (3.3.33a) \]

and
\[ \hat{P}(D) = e^{D} - 1. \]  

\[ (3.3.33b) \]

Also, we have
\[ A(t) = \left(1 + (1 + t)^\lambda\right)^\mu \quad , \quad H(t) = \ln(1 + t). \]  

\[ \text{(3.3.34)} \]

For \( \mu = 1 \), Peters polynomials \( s_n(x; \lambda, \mu) \) reduce to Boole polynomials \( s_n(x; \lambda) \) [12, p. 37], [69]. Thus, replacing \( \mu \) by 1 in Eqs. (3.3.31)-(3.3.34), we obtain the corresponding expressions for Boole polynomials \( s_n(x; \lambda) \).

**VII. Bernoulli Polynomials of Second Kind \( b_n(x) \) [69]:**

The Bernoulli polynomials of the second kind \( b_n(x) \) are Sheffer for
\[ g(t) = \frac{t}{e^t - 1} \quad , \quad f(t) = e^t - 1. \]  

\[ \text{(3.3.35)} \]

The generating function for the Bernoulli polynomials of the second kind \( b_n(x) \) is given as:
\[ \frac{t}{\ln(1 + t)} (1 + t)^x = \sum_{n=0}^{\infty} b_n(x) \frac{t^n}{n!}. \]  

\[ \text{(3.3.36)} \]
The multiplicative and derivative operators for $b_n(x)$ are given by

$$
\hat{M}(X, D) = X e^{-D} + \frac{D + e^{-D} - 1}{D(e^D - 1)}
$$

and

$$
\hat{P}(D) = e^D - 1.
$$

Also, we have

$$
A(t) = \frac{t}{\ln(1 + t)} , \quad H(t) = \ln(1 + t).
$$

VIII. Related Polynomials $r_n(x)$ [69]:

The related polynomials $r_n(x)$ are Sheffer for

$$
g(t) = \frac{1}{2}(1 + e^t) , \quad f(t) = e^t - 1.
$$

The generating function for the related polynomials $r_n(x)$ is given as:

$$
\frac{2}{2 + t}(1 + t)^2 = \sum_{n=0}^{\infty} r_n(x) \frac{t^n}{n!}.
$$

The multiplicative and derivative operators for $r_n(x)$ are given by

$$
\hat{M}(X, D) = X e^{-D} - \frac{1}{e^D + 1}
$$

and

$$
\hat{P}(D) = e^D - 1.
$$

Also, we have

$$
A(t) = \frac{2}{2 + t} , \quad H(t) = \ln(1 + t).
$$

The lower factorial polynomials $(x)_n$ [102] are the associated Sheffer for

$$
g(t) = 1 , \quad f(t) = e^t - 1.
$$

The generating function for the lower factorial polynomials $(x)_n$ is given as:

$$
(1 + t)^2 = \sum_{n=0}^{\infty} (x)_n \frac{t^n}{n!}.
$$
The multiplicative and derivative operators for \( (x)_n \) are given by

\[
\hat{M}(X, D) = X e^{-D} \tag{3.3.45a}
\]

and

\[
\hat{P}(D) = e^D - 1. \tag{3.3.45b}
\]

Also, we have

\[
A(t) = 1, \quad H(t) = \ln(1 + t). \tag{3.3.46}
\]

**IX. Hahn Polynomials \( R_n(x) \) [11]:**

The Hahn polynomials \( R_n(x) \) are Sheffer for

\[
g(t) = \sec t, \quad f(t) = \tan t. \tag{3.3.47}
\]

The generating function for the Hahn polynomials \( R_n(x) \) is given as:

\[
\frac{1}{\sqrt{1 + t^2}} \exp(x \arctan(t)) = \sum_{n=0}^{\infty} R_n(x) \frac{t^n}{n!}. \tag{3.3.48}
\]

The multiplicative and derivative operators for \( R_n(x) \) are given by

\[
\hat{M}(X, D) = (X - \tan D) \cos^2 D \tag{3.3.49a}
\]

and

\[
\hat{P}(D) = \tan D. \tag{3.3.49b}
\]

Also, we have

\[
A(t) = \frac{1}{\sqrt{1 + t^2}}, \quad H(t) = \arctan(t). \tag{3.3.50}
\]

**X. Shively's Pseudo-Laguerre Polynomials \( R_n(a,x) \) [99]:**

The Shively's pseudo-Laguerre polynomials \( R_n(a,x) \) are Sheffer for

\[
g(t) = \frac{1 + t}{(1 - t)^a}, \quad f(t) = \frac{1}{4} - \frac{1}{4} \left( \frac{1 + t}{1 - t} \right)^2. \tag{3.3.51}
\]
The generating function for the Shively’s pseudo-Laguerre polynomials $R_n(a,x)$ is given as:

$$
(1 - 4t)^{-\frac{1}{2}} \left( \frac{2}{1 + \sqrt{1 - 4t}} \right)^{a-1} \exp \left( \frac{-4xt}{(1 + \sqrt{1 - 4t})^2} \right) = \sum_{n=0}^{\infty} R_n(a,x)t^n. \quad (3.3.52)
$$

The multiplicative and derivative operators for $R_n(a,x)$ are given by

$$
\hat{M}(X, D) = \frac{-X(1 - D)^3}{(1 + D)} + \frac{(1 - D)^3}{(1 + D)^2} + \frac{a(1 - D)^2}{(1 + D)}
$$

and

$$
\hat{P}(D) = \frac{1}{4} \left( \frac{1 + D}{1 - D} \right)^2. \quad (3.3.53b)
$$

Also, we have

$$
A(t) = (1 - 4t)^{-\frac{1}{2}} \left( \frac{2}{1 + \sqrt{1 - 4t}} \right)^{a-1}, \quad H(t) = \frac{-4t}{(1 + \sqrt{1 - 4t})^2}. \quad (3.3.54)
$$

**XI. Bessel Polynomials** $p_n(x)$ [15] (see also [75]):

The Bessel polynomials $p_n(x)$ are the associated Sheffer for

$$
g(t) = 1, \quad f(t) = -\frac{1}{2} t^2 + t. \quad (3.3.55)
$$

The generating function for the Bessel polynomials $p_n(x)$ is given as:

$$
\exp \left( x(1 + \sqrt{1 - 2t}) \right) = \sum_{n=0}^{\infty} p_n(x) \frac{t^n}{n!}. \quad (3.3.56)
$$

The multiplicative and derivative operators for $p_n(x)$ are given by

$$
\hat{M}(X, D) = \frac{X}{1 - D} \quad (3.3.57a)
$$

and

$$
\hat{P}(D) = -\frac{1}{2} D^2 + D. \quad (3.3.57b)
$$

Also, we have

$$
A(t) = 1, \quad H(t) = 1 - \sqrt{1 - 2t}. \quad (3.3.58)
$$
Now, we derive the generating functions for some members belonging to Hermite-Sheffer family by taking $A(t)$ and $H(t)$ of the corresponding Sheffer polynomials (or by choosing only $H(t)$ in the case of the associated Sheffer polynomials) in Eq. (3.2.3). First we consider the Sheffer polynomials given in examples I-V.

For $A(t) = e^{-t}$ and $H(t) = \nu t$, i.e. corresponding to the generating function (3.3.2) for the generalized Hermite polynomials $H_{n,m,\nu}(x)$, we get the following generating function for Hermite-generalized Hermite polynomials $H_{n,m,\nu}(x,y,z)$:

$$e^{-t} \exp(\nu xt + \nu^2 yt^2 + \nu^3 zt^3) = \sum_{n=0}^{\infty} H_{n,m,\nu}(x,y,z) \frac{t^n}{n!},$$

(3.3.59)

which for $m = \nu = 2$, gives the generating function for Hermite-Hermite polynomials $Hn(x,y,z)$:

$$e^{-t^2} \exp(2xt + 4yt^2 + 8zt^3) = \sum_{n=0}^{\infty} H_n(x,y,z) \frac{t^n}{n!}.$$  

(3.3.60)

Similarly, for $A(t) = \frac{1}{(1-t)^{a+1}}$ and $H(t) = \frac{-t}{1-t}$, i.e. corresponding to the generating function (1.4.21) for the generalized Laguerre polynomials $L_n^{(a)}(x)$, we get the following generating function for Hermite-generalized Laguerre polynomials $HL_n^{(a)}(x,y,z)$:

$$\frac{1}{(1-t)^{a+1}} \exp\left(-\frac{xt}{1-t} + \frac{yt^2}{(1-t)^2} - \frac{zt^3}{(1-t)^3}\right) = \sum_{n=0}^{\infty} HL_n^{(a)}(x,y,z) \frac{t^n}{n!},$$

(3.3.61)

which for $a = 0$, reduces to the generating function for Hermite-Laguerre polynomials $HL_n(x,y,z)$ [74, p. 763(4.4)].

Next, for $A(t) = \frac{t}{1-t}$ and $H(t) = \ln\left(\frac{1+t}{1-t}\right)$, i.e. corresponding to the generating function (3.3.11) for Pidduck polynomials $P_n(x)$, we get the following generating function for Hermite-Pidduck polynomials $HP_n(x,y,z)$:

$$\frac{t}{1-t} \exp\left(x \ln\left(\frac{1+t}{1-t}\right) + y \ln^2\left(\frac{1+t}{1-t}\right) + z \ln^3\left(\frac{1+t}{1-t}\right)\right) = \sum_{n=0}^{\infty} HP_n(x,y,z) \frac{t^n}{n!},$$

(3.3.62)
For $A(t) = 1$, Eq. (3.3.62) gives the generating function for Hermite-Mittag-Leffler polynomials $H M_n(x, y, z)$:

$$\exp \left( x \ln \left( \frac{1+t}{1-t} \right) + y \ln^2 \left( \frac{1+t}{1-t} \right) + z \ln^3 \left( \frac{1+t}{1-t} \right) \right) = \sum_{n=0}^{\infty} H M_n(x, y, z) \frac{t^n}{n!}. \quad (3.3.63)$$

Again, for $A(t) = e^{\beta t}$ and $H(t) = 1 - e^t$, i.e. corresponding to the generating function (3.3.19) for the acturial polynomials $a_n^{(\beta)}(x)$, we get the following generating function for Hermite-acturial polynomials $H a_n^{(\beta)}(x, y, z)$:

$$e^{\beta t} \exp \left( x(1-e^t) + y(1-e^t)^2 + z(1-e^t)^3 \right) = \sum_{n=0}^{\infty} H a_n^{(\beta)}(x, y, z) \frac{t^n}{n!}, \quad (3.3.64)$$

which for $\beta = 0$ and using the relation $a_n^{(0)}(x) = \phi_n(-x)$ then replacing $x$ by $-x$ and $z$ by $-z$, gives the generating function for Hermite-exponential polynomials $H \phi_n(x, y, z)$:

$$\exp \left( x(e^t-1) + y(e^t-1)^2 + z(e^t-1)^3 \right) = \sum_{n=0}^{\infty} H \phi_n(x, y, z) \frac{t^n}{n!}. \quad (3.3.65)$$

Also, for $A(t) = e^{-t}$ and $H(t) = \ln \left( 1 + \frac{t}{a} \right)$, i.e. corresponding to the generating function (3.3.28) for the Poisson-Charlier polynomials $c_n(x; a)$, we get the following generating function for Hermite-Poisson-Charlier polynomials $H c_n(x, y, z; a)$:

$$\exp \left( x \ln \left( 1 + \frac{t}{a} \right) + y \ln^2 \left( 1 + \frac{t}{a} \right) + z \ln^3 \left( 1 + \frac{t}{a} \right) \right) = \sum_{n=0}^{\infty} H c_n(x, y, z; a) \frac{t^n}{n!}. \quad (3.3.66)$$

Further, in view of examples VI-VIII, i.e. corresponding to the generating functions (3.3.32), (3.3.36) and (3.3.40), we get the generating functions for the Hermite-Peters polynomials $H s_n(x, y, z; \lambda, \mu)$, Hermite-Bernoulli polynomials of the second kind $H b_n(x, y, z)$ and Hermite-related polynomials $H r_n(x, y, z)$ as:

$$(1 + (1 + t)^\lambda)^{-\mu} \exp(x \ln(1+t) + y \ln^2(1+t) + z \ln^3(1+t)) = \sum_{n=0}^{\infty} H s_n(x, y, z; \lambda, \mu) \frac{t^n}{n!}, \quad (3.3.67)$$

$$\frac{t}{\ln(1+t)} \exp(x \ln(1+t) + y \ln^2(1+t) + z \ln^3(1+t)) = \sum_{n=0}^{\infty} H b_n(x, y, z) \frac{t^n}{n!}. \quad (3.3.68)$$
and
\[ \frac{2}{2+t} \exp(x \ln(1+t) + y \ln^2(1+t) + z \ln^3(1+t)) = \sum_{n=0}^{\infty} Hr_n(x, y, z) \frac{t^n}{n!}, \quad (3.3.69) \]
respectively.

We note that for \( \mu = 1 \), Eq. (3.3.67) gives the generating functions for Hermite-Boole polynomials \( Hs_n(x, y, z; \lambda) \) as:
\[ \frac{1}{(1 + (1 + t)^\lambda)} \exp(x \ln(1+t) + y \ln^2(1+t) + z \ln^3(1+t)) = \sum_{n=0}^{\infty} Hs_n(x, y, z; \lambda) \frac{t^n}{n!}. \quad (3.3.70) \]

Also, we note that if we consider generating functions (3.3.32), (3.3.36) and (3.3.40), with \( A(t) = 1 \), then we get the following generating function for Hermite-lower factorial polynomials \( H(x, y, z)_n \):
\[ \exp(x \ln(1+t) + y \ln^2(1+t) + z \ln^3(1+t)) = \sum_{n=0}^{\infty} H(x, y, z)_n \frac{t^n}{n!}. \quad (3.3.71) \]

Next, we explore the possibility of using operational definition (3.2.9) to derive the results for the Hermite-Sheffer polynomials from the ones given for Sheffer polynomials. To give an example, we recall the following explicit formula for the actuarial polynomials \( a_n^{(\beta)}(x) \) [102, p. 123]:
\[ a_n^{(\beta)}(x) = \sum_{k=0}^{n} \binom{\beta}{k} \sum_{j=k}^{n} S(n, j) (j)_k (-x)^{j-k}, \quad (3.3.72) \]
where \( S(n, k) \) denotes the Stirling numbers of the second kind [20] defined by Eqs. (2.3.30a), (2.3.30b).

Now, operating \( \exp \left( y \frac{\partial^2}{\partial x^2} + z \frac{\partial^3}{\partial x^3} \right) \) on both sides of Eq. (3.3.72), we have
\[ \exp \left( y \frac{\partial^2}{\partial x^2} + z \frac{\partial^3}{\partial x^3} \right) \{ a_n^{(\beta)}(x) \} = \sum_{k=0}^{n} \binom{\beta}{k} \sum_{j=k}^{n} S(n, j) (j)_k \times \exp \left( y \frac{\partial^2}{\partial x^2} + z \frac{\partial^3}{\partial x^3} \right) \{ (-x)^{j-k} \}, \quad (3.3.73) \]
which on using the operational definitions (3.2.9) and (3.1.21) in the l.h.s. and r.h.s.
respectively, yields the following explicit representation for the Hermite-actuarial polynomials
$H_{\alpha}^{(\beta)}(x, y, z)$ in terms of 3VHP $H_n(x, y, z)$:

\[
H_{\alpha}^{(\beta)}(x, y, z) = \sum_{k=0}^{n} \binom{\beta}{k} \sum_{j=k}^{n} (-1)^{j-k} S(n, j) (j)_k H_{j-k}(x, y, z). \tag{3.3.74}
\]

Similarly, corresponding to the series defining the generalized Laguerre polynomials $L_n^{(\alpha)}(x)$ (1.4.22), we obtain the series definition for Hermite-generalized Laguerre polynomials $H L_n^{(\alpha)}(x, y, z)$ as:

\[
H L_n^{(\alpha)}(x, y, z) = \sum_{k=0}^{n} (-1)^k \frac{(n!)^2}{k!} \binom{n + \alpha}{n - k} H_k(x, y, z), \tag{3.3.75}
\]

which for $\alpha = 0$, gives the series definition for Hermite-Laguerre polynomials $H L_n(x, y, z)$ as:

\[
H L_n(x, y, z) = \sum_{k=0}^{n} (-1)^k \frac{(n!)^2}{k!} \binom{n}{k} H_k(x, y, z). \tag{3.3.76}
\]

Next, we recall that the Poisson-Charlier polynomials $c_n(x; a)$ are defined by means of the following series [102, p. 120]

\[
c_n(x; a) = \sum_{j=0}^{n} \binom{n}{k} (-1)^{n-k} a^{-k} s(k, j) x^j, \tag{3.3.77}
\]

where $s(n, k)$ are the Stirling numbers of the first kind defined through the recurrence relation [102, p. 61] (see also [100], [101])

\[
s(n + 1, k) = s(n, k - 1) - ns(n, k), \tag{3.3.78}
\]

together with the initial conditions

\[
s(n, 0) = 0 \ (n > 0), \ s(0, k) = 0 \ (k > 0) \ \text{and} \ s(0, 0) = 0,
\]

where $s(n, n) = 1$.

Also, we note that the following connection between Stirling numbers of the first kind $s(n, k)$ and the Stirling numbers of the second kind $S(n, k)$ [102, p. 67]

\[
s(n, i) = \sum_{k=i}^{n} \sum_{j=0}^{k} s(n, k) s(k, j) S(j, i). \tag{3.3.79}
\]
Operating \( \exp \left( y \frac{\partial^2}{\partial x^2} + z \frac{\partial^2}{\partial y^2} \right) \) on Eq. (3.3.77) and using the appropriate operational definitions, we get the series definition for Hermite-Poisson-Charlier polynomials \( H c_n(x, y, z; a) \):

\[
H c_n(x, y, z; a) = \sum_{j=0}^{n} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} a^{-k} s(k, j) H_j(x, y, z). \tag{3.3.80}
\]

Again, corresponding to the series defining the exponential polynomials \( \phi_n(x) \) [102, p. 64]

\[
\phi_n(x) = \sum_{k=0}^{n} S(n, k) x^k, \quad n \geq 0, \tag{3.3.81}
\]

we find the series definition for Hermite-exponential polynomials \( H\phi_n(x, y, z) \):

\[
H\phi_n(x, y, z) = \sum_{n=0}^{\infty} S(n, k) H_k(x, y, z). \tag{3.3.82}
\]

Also, corresponding to the series defining the Bernoulli polynomials of the second kind \( b_n(x) \) [102, p. 115]

\[
b_n(x) = b_n(0) + \sum_{k=0}^{n} \frac{n}{k} s(n - 1, k - 1) x^k, \tag{3.3.83}
\]

where \( b_n(0) \) denotes the Bernoulli numbers of the second kind defined by the following definite integral [102, p. 114]

\[
b_n(0) = \int_{0}^{1} (u)_n \, du, \tag{3.3.84}
\]

we get the series definition for Hermite-Bernoulli polynomials of the second kind \( Hb_n(x, y, z) \):

\[
Hb_n(x, y, z) = b_n(0) + \sum_{k=1}^{n} \frac{n}{k} s(n - 1, k - 1) H_k(x, y, z). \tag{3.3.85}
\]

### 3.4. APPLICATIONS

The operational formalism developed in the previous sections can be used to derive the results for Hermite-Sheffer polynomials \( Hs_n(x, y, z) \) from the results of the corresponding Sheffer polynomials. Now, we use the correspondence between Sheffer and
Hermite-Sheffer polynomials to derive several new relations, identities and expansions for Hermite-Sheffer polynomials.

To achieve this, we perform the following operation:

(B): Operating $\exp \left( y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \right)$ on both sides of a given relation.

Roman [102] characterized Sheffer sequences in several ways. One of which is the Sheffer identity (1.5.18), which is equivalently written as:

$$s_n(x + y) = \sum_{k=0}^{n} \binom{n}{k} s_k(x) p_{n-k}(y),$$

(3.4.1)

where the sequence $p_n(y)$ is an associated Sheffer for $f(t)$.

Now, we recall the following Sheffer identities for generalized Laguerre polynomials $L_n^{(\alpha)}(x)$, Pidduck polynomials $P_n(x)$, actuarial polynomials $a_n^{(\beta)}(x)$, Poisson-Charlier polynomials $C_n(x; a)$, Peters polynomials $s_n(x; \lambda, \mu)$, Bernoulli polynomials of the second kind $b_n(x)$ and related polynomials $r_n(x)$, respectively.

$$L_n^{(\alpha+\beta+1)}(x + w) = \sum_{k=0}^{n} \binom{n}{k} L_k^{(\alpha)}(x) L_{n-k}^{(\beta)}(w),$$

$$P_n(x + w) = \sum_{k=0}^{n} \binom{n}{k} P_k(x) M_{n-k}(w),$$

$$a_n^{(\beta)}(x + w) = \sum_{k=0}^{n} \binom{n}{k} a_k^{(\beta)}(x) \phi_{n-k}(-w),$$

$$c_n(x + w; a) = \sum_{k=0}^{n} \binom{n}{k} a^{k-n} c_k(x; a) (w)_{n-k},$$

$$s_n(x + w; \lambda, \mu) = \sum_{k=0}^{n} \binom{n}{k} s_k(x; \lambda, \mu) (w)_{n-k},$$

$$b_n(x + w) = \sum_{k=0}^{n} \binom{n}{k} b_k(x) (w)_{n-k},$$

$$r_n(x + w) = \sum_{k=0}^{n} \binom{n}{k} r_k(x) (w)_{n-k}. $$

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Performing the operation \((B)\) on the above identities and using the appropriate operational definitions on the resultant equations, we get the following identities for Hermite-generalized Laguerre polynomials \(H_{\alpha+\beta+1}(x, y, z)\), Hermite-Pidduck polynomials \(H_{\alpha}(x, y, z)\), Hermite-actuarial polynomials \(H_{\alpha}^{(\beta)}(x, y, z)\), Hermite-Poisson-Charlier polynomials \(H_{\alpha}^{(\beta)}(x, y, z; a)\), Hermite-Peters polynomials \(H_{\alpha}^{(\beta)}(x, y, z; \lambda, \mu)\), Hermite-Bernoulli polynomials of the second kind \(H_{\alpha}^{(\beta)}(x, y, z; \lambda, \mu)\) and Hermite-related polynomials \(H_{\alpha}^{(\beta)}(x, y, z; \lambda, \mu)\) as:

\[
H_{\alpha+\beta+1}(x + w, y, z) = \sum_{k=0}^{n} \binom{n}{k} H_{\alpha}^{(\beta)}(x, y, z) L_{n-k}^{(\beta)}(w),
\]

\[
H_{\alpha}(x + w, y, z) = \sum_{k=0}^{n} \binom{n}{k} H_{\alpha}^{(\beta)}(x, y, z) M_{n-k}(w),
\]

\[
H_{\alpha}^{(\beta)}(x + w, y, z) = \sum_{k=0}^{n} \binom{n}{k} H_{\alpha}^{(\beta)}(x, y, z) \phi_{n-k}(w),
\]

\[
H_{\alpha}^{(\beta)}(x + w, y, z; a) = \sum_{k=0}^{n} \binom{n}{k} \alpha^{k-n} H_{\alpha}^{(\beta)}(x, y, z; a) (w)_{n-k},
\]

\[
H_{\alpha}^{(\beta)}(x + w, y, z; \lambda, \mu) = \sum_{k=0}^{n} \binom{n}{k} H_{\alpha}^{(\beta)}(x, y, z; \lambda, \mu) (w)_{n-k},
\]

\[
H_{\alpha}^{(\beta)}(x + w, y, z; \lambda, \mu) = \sum_{k=0}^{n} \binom{n}{k} H_{\alpha}^{(\beta)}(x, y, z; \lambda, \mu) (w)_{n-k},
\]

\[
H_{\alpha}^{(\beta)}(x + w, y, z; \lambda, \mu) = \sum_{k=0}^{n} \binom{n}{k} H_{\alpha}^{(\beta)}(x, y, z; \lambda, \mu) (w)_{n-k},
\]

respectively.

Further, we recall the Binomial identity (1.5.18) in the following equivalent form:

\[
p_{n}(x + y) = \sum_{k=0}^{n} \binom{n}{k} p_{k}(x) p_{n-k}(y) \quad (y \in \mathbb{C}).\]

We consider the following binomial identities for lower factorial polynomials \((x)_{n}\), exponential polynomials \(p_{n}(x)\), Bessel polynomials \(p_{n}(x)\) and Mittag-Leffler polynomials \(M_{n}(x)\), respectively.
\[(x + w)_n = \sum_{k=0}^{n} \binom{n}{k} (x)_k (w)_{n-k} , \]
\[
\phi_n(x + w) = \sum_{k=0}^{n} \binom{n}{k} \phi_k(x) \phi_{n-k}(w) , \]
\[
p_n(x + w) = \sum_{k=0}^{n} \binom{n}{k} p_k(x) p_{n-k}(w) , \]
\[
M_n(x + w) = \sum_{k=0}^{n} \binom{n}{k} M_k(x) M_{n-k}(w) , \]
which on performing the operation (B) and using the appropriate operational definitions on the resultant equations, yield the following identities for Hermite-lower factorial polynomials \(H(x, y, z)_n\), Hermite-exponential polynomials \(H\phi_n(x, y, z)\), Hermite-Bessel polynomials \(Hp_n(x, y, z)\) and Hermite-Mittag-Leffler polynomials \(H M_n(x, y, z)\) as:
\[
H(x + w, y, z)_n = \sum_{k=0}^{n} \binom{n}{k} H(x, y, z)_k (w)_{n-k} , \quad (3.4.10)
\]
\[
H\phi_n(x + w, y, z) = \sum_{k=0}^{n} \binom{n}{k} H\phi_k(x, y, z) \phi_{n-k}(w) , \quad (3.4.11)
\]
\[
Hp_n(x + w, y, z) = \sum_{k=0}^{n} \binom{n}{k} Hp_k(x, y, z) p_{n-k}(w) , \quad (3.4.12)
\]
\[
H M_n(x + w, y, z) = \sum_{k=0}^{n} \binom{n}{k} H M_k(x, y, z) M_{n-k}(w) . \quad (3.4.13)
\]
respectively.

Next, we recall the following relations between Bernoulli polynomials of the first and second kinds \(B_n(x)\) and \(b_n(x)\), respectively [102, pp. 117-118].
\[
b_n(x) = (1 - n)b_n(0) + n(2 - n)b_{n-1}(0) + nb_{n-1}(0)B_1(x)
\]
\[
+ \sum_{k=2}^{n} \frac{n(n - 1)}{k(k - 1)} s(n - 2, k - 2) B_k(x) \quad (3.4.14)
\]
and

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\[ B_n(x) = (1 - n)B_n(0) - nB_{n-1}(0) + nB_{n-1}(0) b_1(x) + \sum_{k=2}^{n} \frac{n(n-1)}{k(k-1)} S(n-2, k-2) b_k(x). \] (3.4.15)

The Bernoulli polynomials of the first kind \( B_n(x) \) are defined through the generating function [99]
\[
\frac{t}{e^t - 1} \exp(xt) = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}
\] (3.4.16)
and thus belong to the Appell family.

Again, performing the operation \((\mathcal{B})\) on Eqs. (3.4.14) and (3.4.15) and using the appropriate operational definitions on the resultant equations, we get the following identities for corresponding Hermite-based polynomials:

\[ Hb_n(x, y, z) = (1 - n)bn(0) + n(2 - n)b_{n-1}(0) + nb_{n-1}(0) Hb_1(x, y, z) \]
\[ + \sum_{k=2}^{n} \frac{n(n-1)}{k(k-1)} S(n-2, k-2) Hb_k(x, y, z) \] (3.4.17)

and

\[ HB_n(x, y, z) = (1 - n)B_n(0) - nB_{n-1}(0) + nB_{n-1}(0) b_1(x, y, z) \]
\[ + \sum_{k=2}^{n} \frac{n(n-1)}{k(k-1)} S(n-2, k-2) b_k(x, y, z), \] (3.4.18)
respectively.

Further, replacing \( x \) by \( mx \) in Eqs. (3.4.14), (3.4.15) and using the multiplication theorem for Bernoulli polynomials of the first kind \( B_n(x) \) [52], (see Section 2.4 of Chapter 2) we get

\[ b_n(mx) = (1 - n)b_n(0) + n(2 - n)b_{n-1}(0) + nb_{n-1}(0) \sum_{j=0}^{m-1} B_1(x + \frac{j}{m}) \]
\[ + \sum_{k=2}^{n} \sum_{j=0}^{m-1} \frac{n(n-1)}{k(k-1)} m^{k-1} S(n-2, k-2) B_k(x + \frac{j}{m}) \quad (k \in \mathbb{N}_0; \ m \in \mathbb{N}) \] (3.4.19)
and

\[ B_n(mx) = (1 - n)B_n(0) - nB_{n-1}(0) + nB_{n-1}(0) b_1(mx) \]
\[ + \sum_{k=2}^{n} \frac{n(n-1)}{k(k-1)} S(n-2, k-2) b_k(mx) \quad (k \in \mathbb{N}_0; \ m \in \mathbb{N}), \quad (3.4.20) \]

which on performing the operation \( \mathcal{B} \) and using the appropriate operational definitions on the resultant equations, yield the following results for Hermite-Bernoulli polynomials of the second kind \( h_\mathcal{B}n(x, y, z) \) in terms of Hermite-Bernoulli polynomials of the first kind \( h_Bn(x, y, z) \) and vice-versa as:

\[ h_\mathcal{B}n(mx, m^2y, m^3z) = (1 - n)h_\mathcal{B}n(0) + n(2 - n)h_\mathcal{B}n-1(0) + n h_\mathcal{B}n-1(0) \sum_{j=0}^{m-1} h_B1(x + \frac{j}{m}, y, z) \]
\[ + \sum_{k=2}^{n} \sum_{j=0}^{m-1} \frac{n(n-1)}{k(k-1)} m^{k-1} s(n-2, k-2) h_Bk(x + \frac{j}{m}, y, z) \quad (k \in \mathbb{N}_0; \ m \in \mathbb{N}) \]
\[ (3.4.21) \]

and

\[ h_Bn(mx, m^2y, m^3z) = (1 - n)h_Bn(0) - n h_Bn(0) + n h_Bn-1(0) h_\mathcal{B}1(mx, m^2y, m^3z) \]
\[ + \sum_{k=2}^{n} \frac{n(n-1)}{k(k-1)} S(n-2, k-2) h_\mathcal{B}k(mx, m^2y, m^3z) \quad (k \in \mathbb{N}_0; \ m \in \mathbb{N}), \quad (3.4.22) \]

respectively. For \( m = 1 \), Eqs. (3.4.21) and (3.4.22) reduce to Eqs. (3.4.17) and (3.4.18) respectively.

Next, we recall the following relations [99, p. 298(2) and p. 207(3)]:

\[ R_n(a, x) = \frac{1}{(a - 1)_n} \sum_{k=0}^{n} \frac{(a - 1)_{n+k} L_{n-k}(x)}{k!} \]

and

\[ H_n(x) = 2^n (1 + \alpha)_n \sum_{k=0}^{n} \binom{2F_2}{-\frac{1}{2}(n - k), \ -\frac{1}{2}(n - k - 1); \ -\frac{1}{2}(\alpha + n), \ -\frac{1}{2}(\alpha + n - 1); \ -\frac{1}{4}} \frac{(-n)_k L_k^{(\alpha)}(x)}{(1 + \alpha)_k}, \]

which on performing the operation \( \mathcal{B} \) and using the appropriate operational definitions on the resultant equations, yield the following relations for the corresponding Hermite-based Sheffer polynomials.
\[ H R_n(a, x, y, z) = \frac{1}{(a - 1)^n} \sum_{k=0}^{n} \binom{n}{k} (a - 1)^{n+k} H L_{n-k}(x, y, z) \]  
(3.4.23)

and

\[ H H_n(x, y, z) = 2^n (1 + \alpha)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{n!}{k!} \frac{\Gamma\left(\frac{1}{2}(n-k)\right)}{\Gamma\left(\frac{1}{2}(n-k-1)\right)} \]
\[ \times \frac{\Gamma\left(\frac{1}{2}(\alpha + n)\right)}{\Gamma\left(\frac{1}{2}(\alpha + n-1)\right)} \]

\[ \times \binom{\alpha}{k} \frac{2^n (1 + \alpha)^{-\frac{n}{2}}}{(1 + \alpha_k)^{-\frac{n}{2}}} \]  
(3.4.24)

respectively.

Also, corresponding to the functional equation involving generalized Laguerre polynomials \( L_n^{(\alpha)}(x) \) [99, p. 209(5)]

\[ L_n^{(\alpha)}(wx) = \sum_{k=0}^{n} \left( \frac{1 + \alpha}{n-k} \right)^{\frac{n}{2}} \left( \frac{1}{1+\alpha_k} \right)^{\frac{n}{2}} L_k^{(\alpha)}(x), \]  
(3.4.25)

we find the following identity involving Hermite-generalized Laguerre polynomials \( H L_n^{(\alpha)}(x, y, z) \):

\[ H L_n^{(\alpha)}(wx, w^2y, w^3z) = \sum_{k=0}^{n} \binom{n}{k} \frac{(1 + \alpha)^n (1 - w)^{-k} (w^k)^k}{(1 + \alpha_k)^{-k}} H L_k^{(\alpha)}(x, y, z), \]  
(3.4.26)

which for \( \alpha = 0 \), becomes

\[ H L_n(wx, w^2y, w^3z) = \sum_{k=0}^{n} \binom{n}{k} \frac{n!(1 - w)^{-k} (w^k)^k}{k!} H L_k(x, y, z). \]  
(3.4.26)

To provide further examples, we consider the summation formulae [102, p. 81 and p. 124]:

\[ x^n = \sum_{k \geq n - \frac{n}{2}}^{\infty} \binom{\frac{n}{2}}{n - k} \frac{n!}{k!} \left( \frac{-1}{2} \right)^{n-k} p_k(x) \]

and

\[ a_n^{(\beta+1)}(x) = \sum_{k=0}^{n} \binom{n}{k} a_k^{(\beta)}(x), \]

which yield the following results:
\[ H_n(x, y, z) = \sum_{k \geq \frac{n}{2}} \binom{k}{n-k} \frac{n!}{k!} \left( \frac{-1}{2} \right)^{n-k} H_{p_k}(x, y, z) \]  \hspace{1cm} (3.4.27)

and

\[ H_{a_n^{(p+1)}}(x, y, z) = \sum_{k=0}^{n} \binom{n}{k} H_{a_k^{(p)}}(x, y, z), \]  \hspace{1cm} (3.4.28)

respectively.

Again, corresponding to the following summation formulae [1, p. 169 and p. 176]:

\[ x^n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!}{2^n k!(n-2k)!} H_{n-2k}(x) \]

and

\[ H_n(x + w) = \frac{1}{(2)^{\frac{n}{2}}} \sum_{k=0}^{n} \binom{n}{k} H_{n-k}(\sqrt{2}x) H_k(\sqrt{2}w), \]

we find the following results:

\[ H_n(x, y, z) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!}{2^n k!(n-2k)!} H_{n-2k}(x, y, z) \]  \hspace{1cm} (3.4.29)

and

\[ H H_n(x + w, y, z) = \frac{1}{(2)^{\frac{n}{2}}} \sum_{k=0}^{n} \binom{n}{k} H H_{n-k}(\sqrt{2}x, 2y, 2\sqrt{2}z) H_k(\sqrt{2}w), \]  \hspace{1cm} (3.4.30)

respectively.

Also, corresponding to the recurrence relations [102, p. 77 and p. 121]:

\[ M_n(x + 1) - M_n(x) = n \left[ M_{n-1}(x + 1) + M_{n-1}(x) \right] \]

and

\[ a c_n(x + 1; a) - a c_n(x; a) - n c_{n-1}(x; a) = 0, \]

we find the following recurrence relations for the corresponding Hermite-based polynomials:

\[ H M_n(x + 1, y, z) - H M_n(x, y, z) = n \left[ H M_{n-1}(x + 1, y, z) + H M_{n-1}(x, y, z) \right] \]  \hspace{1cm} (3.4.31)

and

\[ a H c_n(x + 1, y, z; a) - a H c_n(x, y, z; a) - n H c_{n-1}(x, y, z; a) = 0, \]  \hspace{1cm} (3.4.32)
respectively.

Finally, we mention special cases of some of the results derived in this section.

As for example taking $\beta = -1$ in Eq. (3.4.2), we get

$$ H_L^{(\alpha)}(x + w, y, z) = \sum_{k=0}^{n} \binom{n}{k} H_L^{(\alpha)}(x, y, z) L_{n-k}^{(-1)}(w), $$  \hspace{1cm} (3.4.33)

which on taking $\alpha = 0$, gives the following expansion:

$$ H_L(x + w, y, z) = \sum_{k=0}^{n} \binom{n}{k} H_L(x, y, z) L_{n-k}^{(-1)}(w). $$  \hspace{1cm} (3.4.34)

Again, taking $\alpha = \beta = -\frac{1}{2}$ in Eq. (3.4.2), we get

$$ H_L(x + w, y, z) = \sum_{k=0}^{n} \binom{n}{k} H_L^{(-\frac{1}{2})}(x, y, z) L_{n-k}^{(-\frac{1}{2})}(w). $$  \hspace{1cm} (3.4.35)

Further, taking $w = 1$ in Eqs. (3.4.5)-(3.4.8) and (3.4.10), we get the following expansions:

$$ H_C(x + 1, y, z; a) = \sum_{k=0}^{n} \frac{n!}{k!} a^{k-n} H_C(x, y, z; a), $$  \hspace{1cm} (3.4.36)

$$ H_S(x + 1, y, z; \lambda, \mu) = \sum_{k=0}^{n} \frac{n!}{k!} H_S(x, y, z; \lambda, \mu), $$  \hspace{1cm} (3.4.37)

$$ H_b(x + 1, y, z) = \sum_{k=0}^{n} \frac{n!}{k!} H_b(x, y, z), $$  \hspace{1cm} (3.4.38)

$$ H_r(x + 1, y, z) = \sum_{k=0}^{n} \frac{n!}{k!} H_r(x, y, z), $$  \hspace{1cm} (3.4.39)

and

$$ H(x + 1, y, z) = \sum_{k=0}^{n} \frac{n!}{k!} H(x, y, z)_k. $$  \hspace{1cm} (3.4.40)

Also, taking $\beta = 0$ in Eq. (3.4.28) and using the relation $a^{(\alpha)}(x) = \phi_n(-x)$, we get

$$ H_n^{(1)}(x, y, z) = \sum_{k=0}^{n} \binom{n}{k} \phi_k(-x, y, -z). $$  \hspace{1cm} (3.4.41)

The above examples show that the operational formalism developed here is useful to derive results for Hermite-Sheffer polynomials.
3.5. CONCLUDING REMARKS

In the previous sections, we have used the concepts and the formalism associated with monomiality principle and Sheffer sequences to introduce family of Hermite-Sheffer polynomials. The approach presented is general and we have developed an operational formalism, providing a correspondence between Sheffer and Hermite-Sheffer polynomials. These operational rules can be used to derive the results for Hermite-Sheffer polynomials from the results of the corresponding Sheffer polynomials.

Here, we explore the possibility of introducing family of Laguerre-based Sheffer polynomials.

To generate Laguerre-based Sheffer polynomials associated with 2VLP \( L_n(x, y) \) (2.1.5), (2.1.6), we introduce the generating function

\[
\mathcal{L}(x, y; t) = A(t) \exp(\hat{M}H(t)),
\]

which on replacing \( \hat{M} \) by the multiplicative operator (2.1.1a) of 2VLP \( L_n(x, y) \) becomes

\[
\mathcal{L}(x, y; t) = A(t) \exp\left((y - \hat{D}_x^{-1}) H(t)\right). \tag{3.5.1}
\]

Now, decoupling the exponential operator in the r.h.s. of Eq. (3.5.1), by using the Weyl identity [41], (see Eq. (2.2.2)), we find

\[
\mathcal{L}(x, y; t) = A(t) \exp(y H(t)) \exp(-D_x^{-1} H(t)).
\]

Further, on expanding the second exponential in the r.h.s. of the above equation and using Eq. (2.1.2b), we find

\[
\mathcal{L}(x, y; t) = A(t) \exp(y H(t)) \sum_{r=0}^\infty \frac{(-1)^r (x H(t))^r}{(r!)^2}.
\]

Finally, in view of definition (1.4.40), we get the generating function for Laguerre-based Sheffer polynomials \( L_S(x, y) \) in the form

\[
\mathcal{L}(x, y; t) = A(t) \exp(y H(t)) C_0(x H(t)) = \sum_{n=0}^\infty L_S(x, y) \frac{t^n}{n!}. \tag{3.5.2}
\]
Now, for the generating function given in Eq. (3.3.2), we get the following generating function for Laguerre-generalized Hermite polynomials $L_{H_n,m,v}(x,y)$:

$$\exp(-t^n) \exp(uyt) C_0(vxt) = \sum_{n=0}^{\infty} L_{H_n,m,v}(x,y) \frac{t^n}{n!}.$$  \hspace{1cm} (3.5.3)

which for $m = v = 2$, gives the generating function for Laguerre-Hermite polynomials $L_{H_n}(x,y)$:

$$\exp(-t^2) \exp(2yt) C_0(2xt) = \sum_{n=0}^{\infty} L_{H_n}(x,y) \frac{t^n}{n!}.$$  \hspace{1cm} (3.5.4)

Similarly, for the generating function given in Eq. (1.4.21), we get the following generating function for Laguerre-generalized Laguerre polynomials $L_{Ln}(x,y)$:

$$\frac{1}{(1-t)^{n+1}} \exp\left(-\frac{yt}{1-t}\right) C_0\left(-\frac{xt}{1-t}\right) = \sum_{n=0}^{\infty} L_{Ln}(x,y) t^n.$$ \hspace{1cm} (3.5.5)

which for $\alpha = 0$, gives the generating function for Laguerre-Laguerre polynomials $L_{L_n}(x,y)$:

$$\frac{1}{1-t} \exp\left(-\frac{yt}{1-t}\right) C_0\left(-\frac{xt}{1-t}\right) = \sum_{n=0}^{\infty} L_{L_n}(x,y) t^n.$$ \hspace{1cm} (3.5.6)

Next, for the generating function given in Eq. (3.3.11), we get the following generating function for Laguerre-Pidduck polynomials $L_{P_n}(x,y)$:

$$\frac{t(1+t)^y}{(1-t)^{y+1}} C_0\left(x \ln \left(\frac{1+t}{1-t}\right)\right) = \sum_{n=0}^{\infty} L_{P_n}(x,y) \frac{t^n}{n!}.$$ \hspace{1cm} (3.5.7)

which for $A(t) = 1$, gives the generating function for Laguerre-Mittag-Leffler polynomials $L_{M_n}(x,y)$:

$$\left(\frac{1+t}{1-t}\right)^y C_0\left(x \ln \left(\frac{1+t}{1-t}\right)\right) = \sum_{n=0}^{\infty} L_{M_n}(x,y) \frac{t^n}{n!}.$$ \hspace{1cm} (3.5.8)

Again, for the generating function given in Eq. (3.3.32), we get the following generating function for Laguerre-Peters polynomials $L_{S_n}(x,y; \lambda, \mu)$:

$$(1 + (1 + t)^\lambda)^{-\mu}(1 + t)^y C_0(x \ln(1 + t)) = \sum_{n=0}^{\infty} L_{S_n}(x,y; \lambda, \mu) \frac{t^n}{n!},$$ \hspace{1cm} (3.5.9)
which for \( \mu = 1 \), gives the generating function for Laguerre-Boole polynomials \( L_n(x, y; \lambda) \): 
\[
\frac{(1 + t)^y}{(1 + (1 + t)\lambda)} C_0(x \ln(1 + t)) = \sum_{n=0}^{\infty} H_n(x, y; \lambda) \frac{t^n}{n!}. 
\] (3.5.10)

Further, corresponding to the generating functions (3.3.36) and (3.3.40), we find
\[
t\frac{(1 + t)^y}{\ln(1 + t)} C_0(x \ln(1 + t)) = \sum_{n=0}^{\infty} b_n(x, y) \frac{t^n}{n!},
\] (3.5.11)
and
\[
\frac{2(1 + t)^y}{2 + t} C_0(x \ln(1 + t)) = \sum_{n=0}^{\infty} r_n(x, y) \frac{t^n}{n!},
\] (3.5.12)
respectively.

For \( A(t) = 1 \), Eqs. (3.5.11) and (3.5.12) give the following generating functions for Laguerre-lower factorial polynomials \( L(x, y)_n \):
\[
(1 + t)^y C_0(x \ln(1 + t)) = \sum_{n=0}^{\infty} L(x, y)_n \frac{t^n}{n!}. 
\] (3.5.13)

The Sheffer polynomials, which include Appell polynomials as a special case along with the underlying operational formalism, offer a powerful tool for investigation of the properties of a wide class of polynomials.