CHAPTER 1
PRELIMINARIES

1.1. INTRODUCTION

The role of special functions in the solution of differential equations was exploited by Newton and Leibnitz and the subject of special functions has been in continuous development ever since. Special functions crop up in all branches of mathematics and in almost all areas of application and thus arise in a wide range of problems which exist in classical and quantum physics, engineering and applied mathematics. The procedure followed in most investigations on these topics (e.g., quantum mechanics, electrodynamics, modern physics, classical mechanics, heat conduction, communication systems, electro-optics, electro-magnetic theory, electro-circuit theory et cetera) is to formulate the problem as a differential equation that is related to one of several special differential equations (Hermite, Bessel’s, Tricomi’s, Laguerre’s, Legendre’s et cetera). There are classes of special functions whose members are orthogonal (the inner product of distinct members is zero). Orthogonal functions include spherical harmonics and Walsh functions, but arguably the most important are the systems of orthogonal polynomials, which include Chebyshev, Hermite, Jacobi, Laguerre and Legendre polynomials.

The theory of special functions with its numerous beautiful formulae is very well suited to an algorithmic approach to mathematics. Although, special functions can be defined in different ways such as (Rodrique's formulae, generating functions, summation formulae, integral representations et cetera), but it is usually shown to be expressible as a series, because this is frequently the most practical way to obtain numerical values for the functions.

Special functions have centuries of history with immense literature as constructed in the works of Chebyshev, Euler, Gauss, Hardy, Hermite, Legendre, Ramanujan, Watson and other classical authors. A number of books consisting of the theory and applications of special functions are available, see for example Andrews [1,2], Andrews et al. [3], Erdélyi et al. [52,53,56], Iwasaki et al. [68], Lebedev [79], Rainville [99], Sneddon [112] et cetera.
The importance of special functions has been further stressed by their various generalizations. Dattoli and his co-workers have given several contributions of many sets of special functions in several variables, see for example Dattoli [21-25], Dattoli and Khan [29,30], Dattoli et al. [26,27,31,34-36,43,44]; see also Khan [71]. The monographs by Suetin [125] and Dunkl and Yuan [51] are useful in the study of orthogonal polynomials of two and several variables.

An increasing interest has grown around operational techniques and special functions. Recently, Dattoli and his co-workers have shown that by using operational techniques, many properties of ordinary and multi-variable special functions are simply derived and framed in a more general context, see for example [9,21-36,38,41,43-47]. Operational techniques provide a general framework to derive generating relations and summation formulae involving multi-variable special functions.

Operational techniques are important because they are closer to implementations and language definitions than more abstract mathematical techniques. It is well known from the literature that operational techniques include integral, differential and exponential operators and provide a systematic and analytic approach to study special functions, see for example Srivastava and Manocha [119]. The operational techniques are based upon single, double and multiple integral transforms and upon certain operators involving derivatives. Methods connected with the use of integral transforms have been successfully applied to the solution of differential and integral equations, the study of generalized special functions and the evaluation of integrals.

The umbral calculus is a kind of symbolic calculus whose starting point is the study of polynomials of binomial type. An elementary introduction to the modern umbral calculus is given by Roman [102]. Gustafson has made an attempt to show how umbral calculus can be used to enrich the theory of special functions, see for example [61,62]. Some connections between the classical invariant theory for \( SL_2 \) and special functions are discussed in [63], with the purpose to provide a point of unification between the representation theory of \( SL_2 \) and special functions and also to indicate possible generalizations of hypergeometric series and classical orthogonal polynomials to several variables. Operational methods can be exploited to simplify the derivation
of the properties associated with ordinary and generalized polynomials. On the other hand, they may provide a fairly unique tool to treat various polynomials from a unified point of view.

This Chapter contains the necessary background material of special functions. In Section 1.2, we review the definitions of gamma, beta and related functions. In Section 1.3, we review the definitions of hypergeometric functions of single and several variables. In Section 1.4, we review the definitions of some other special functions expressible in terms of hypergeometric functions. In Section 1.5, we discuss in brief, the Sheffer polynomials, umbral calculus and monomiality principle. In Section 1.6, we review the concepts of generating functions. However, the definitions given here are only those which are needed in carrying out the work of this thesis.

1.2. GAMMA, BETA AND RELATED FUNCTIONS

A fairly wide range of special functions can be represented in terms of the hypergeometric and confluent hypergeometric functions. We first give the definitions and important properties of some elementary functions such as gamma, beta and related functions.

**Gamma Function**

The gamma function is a generalization of the factorial function from the domain of positive integers to the domain of all real and complex numbers (except for $z > 0$ and $\Re(z) > 0$, respectively). For a complex number $z$ with positive real part the gamma function is defined by

$$
\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt \quad (\Re(z) > 0).
$$

(1.2.1)

The gamma function is distinguished by being analytic (except at the non-positive integers) and by being the most useful solution in practice and can be characterized in several ways. However, the integral representation of $\Gamma(z)$ given in Eq. (1.2.1) is the most common way in which the gamma function is now defined.

It appears occasionally by itself in physical applications, but much of its importance stems from its usefulness in developing other functions such as hypergeometric
functions and Bessel functions, which have more direct physical application. Further, the gamma function is a component in various probability-distribution functions and as such it is applicable in the fields of probability and statistics, as well as combinatorics.

Upon integration by parts, definition (1.2.1) yields the recurrence relation for $\Gamma(z)$:

$$\Gamma(z + 1) = z \Gamma(z), \quad (1.2.2)$$

which enables us to use definition (1.2.1) to define $\Gamma(z)$ on the entire $z$-plane except when $z$ is zero or a negative integer as follows:

$$\Gamma(z) = \begin{cases} 
\int_0^\infty t^{z-1}e^{-t} \, dt & (\operatorname{Re}(z) > 0), \\
\frac{\Gamma(z + 1)}{z} & (\operatorname{Re}(z) < 0; \ z \neq -1, -2, -3, \ldots). 
\end{cases} \quad (1.2.3)$$

The recurrence relation (1.2.2) yields the useful result

$$\Gamma(n + 1) = n! \quad (1.2.4)$$

**Pochhammer’s Symbol and the Factorial Function**

The Pochhammer symbol $(\lambda)_n$ is defined by

$$(\lambda)_n = \begin{cases} 
1 & (n = 0), \\
\lambda(\lambda + 1)\ldots(\lambda + n - 1) & (n = 1, 2, \ldots). 
\end{cases} \quad (1.2.5)$$

Since $(1)_n = n!$, the symbol $(\lambda)_n$ is also referred to as the factorial function.

In terms of gamma functions, we have

$$(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} \quad (\lambda \neq 0, -1, -2, \ldots), \quad (1.2.6)$$

which can easily be verified. Furthermore, the binomial coefficient may now be expressed as

$$\binom{\lambda}{n} = \frac{\lambda(\lambda - 1)\ldots(\lambda - n + 1)}{n!} = \frac{(-1)^n(-\lambda)_n}{n!} \quad (1.2.7)$$
or, equivalently, as
\[
\binom{\lambda}{n} = \frac{\Gamma(\lambda + 1)}{n! \Gamma(\lambda - n + 1)}.
\] (1.2.8)

It follows from (1.2.7) and (1.2.8) that
\[
\frac{\Gamma(\lambda + 1)}{\Gamma(\lambda - n + 1)} = (-1)^n (-\lambda)_n,
\] (1.2.9)

which, for \( \lambda = \alpha - 1 \), yields
\[
\frac{\Gamma(\alpha - n)}{\Gamma(\alpha)} = \frac{(-1)^n}{(1 - \alpha)_n} \quad (\alpha \neq 0, \pm 1, \pm 2, \ldots).
\] (1.2.10)

Eqs. (1.2.6) and (1.2.10) suggest the definition:
\[
(\lambda)_n = \frac{(-1)^n}{(1 - \lambda)_n} \quad (n = 1, 2, \ldots; \; \lambda \neq 0, \pm 1, \pm 2, \ldots). \quad (1.2.11)
\]

Eq. (1.2.6) also yields
\[
(\lambda)_{m+n} = (\lambda)_m (\lambda + m)_n, \quad (1.2.12)
\]

which, in conjunction with (1.2.11), gives
\[
(\lambda)_{n-k} = \frac{(-1)^k (\lambda)_n}{(1 - \lambda - n)_k} \quad (0 \leq k \leq n). \quad (1.2.13)
\]

For \( \lambda = 1 \), we have
\[
(n-k)! = \frac{(-1)^k n!}{(-n)_k} \quad (0 \leq k \leq n), \quad (1.2.14)
\]

which may alternatively be written in the form:
\[
(-n)_k = \begin{cases} \frac{(-1)^k n!}{(n-k)!} & (0 \leq k \leq n), \\ 0 & (k > n). \end{cases} \quad (1.2.15)
\]
Legendre’s Duplication Formula

The Legendre’s duplication formula for the gamma function is given by

\[ \sqrt{\pi} \Gamma(2z) = 2^{2z-1} \Gamma(z) \Gamma(z + \frac{1}{2}) \quad (z \neq 0, -\frac{1}{2}, -1, -\frac{3}{2}, \ldots). \quad (1.2.16) \]

In view of definition (1.2.5), it is not difficult to show that

\[ (\lambda)_{2n} = 2^{2n} \left( \frac{1}{2} \lambda \right)_n \left( \frac{1}{2} (\lambda + 1) \right)_n \quad (n = 0, 1, 2, \ldots), \quad (1.2.17) \]

which follows also from Eq. (1.2.16).

Gauss’s Multiplication Theorem

For every positive integer \( m \), we have

\[ (\lambda)_m = m^m \prod_{j=1}^{m} \left( \frac{\lambda + j - 1}{m} \right)_n \quad (n = 0, 1, 2, \ldots), \quad (1.2.18) \]

which reduces to Eq. (1.2.17) when \( m = 2 \).

Starting from Eq. (1.2.18) with \( \lambda = mz \), it can be proved that

\[ \Gamma(mz) = (2\pi)^{\frac{1}{2}(1-m)} (m)^{mz - \frac{1}{2}} \prod_{j=1}^{m} \Gamma \left( z + \frac{j - 1}{m} \right) \]

\[ \left( z \neq 0, -\frac{1}{m}, -\frac{2}{m}, \ldots; \ m = 1, 2, 3, \ldots \right), \quad (1.2.19) \]

which is known as Gauss’s multiplication theorem for the gamma function.

The Beta Function

The beta function \( B(\alpha, \beta) \) is a function of two complex variables \( \alpha \) and \( \beta \), defined by the Eulerian integral of the first kind

\[ B(\alpha, \beta) = \int_0^1 t^{\alpha-1} (1 - t)^{\beta-1} \, dt \quad (\text{Re}(\alpha), \text{Re}(\beta) > 0). \quad (1.2.20) \]

The beta function is closely related to the gamma function; in fact, we have

\[ B(\alpha, \beta) = \frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha + \beta)} \quad (\alpha, \beta \neq 0, -1, -2, \ldots). \quad (1.2.21) \]
In view of Eq. (1.2.21), we obtain the symmetry property

\[ B(\alpha, \beta) = B(\beta, \alpha). \]  

(1.2.22)

The Incomplete Gamma and Beta Functions

The incomplete gamma function \( \gamma(z, \alpha) \) and its complement \( \Gamma(z, \alpha) \) (also known as Prym’s function) are defined by

\[ \gamma(z, \alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt \quad (\text{Re}(z) > 0; \ |\arg(\alpha)| < \pi), \]  

(1.2.23)

\[ \Gamma(z, \alpha) = \int_\alpha^\infty t^{\alpha-1} e^{-t} dt \quad (|\arg(\alpha)| < \pi). \]  

(1.2.24)

We note that

\[ \gamma(z, \alpha) + \Gamma(z, \alpha) = \Gamma(z). \]  

(1.2.25)

The incomplete beta function \( B_x(\alpha, \beta) \) is defined by

\[ B_x(\alpha, \beta) = \int_0^x t^{\alpha-1}(1 - t)^{\beta-1} dt. \]  

(1.2.26)

1.3. HYPERGEOMETRIC FUNCTIONS OF SINGLE AND SEVERAL VARIABLES

A hypergeometric series, in the most general sense, is a power series in which the ratio of successive coefficients indexed by \( n \) is a rational function of \( n \). The series, if convergent, will define a hypergeometric function, which may then turn out to be defined over a wider domain of the argument by analytic continuation. Hypergeometric functions have many particular special functions as special cases, including many elementary functions, the Bessel functions, the incomplete gamma function, the error function, the elliptic integrals and the classical orthogonal polynomials. This phenomenon is in part because the hypergeometric functions are solutions to the hypergeometric differential equation, which is a fairly general second-order ordinary differential equation. The term hypergeometric series also refers to a specific type of these series, also known as Gauss series (Carl Friedrich Gauss), which were the object of a great deal of interest in the 19th century.
The term “hypergeometric” was first used by Wallis in 1655 in his work “Arithmetica Infinitorum”, when referring to any series which could be regarded as a generalization of the ordinary geometric series

$$\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \ldots.$$ 

Gauss’s series were studied by Euler in 1769, he established an integral representation, a series expansion, a differential equation and several other properties including reduction and transformation formulae for hypergeometric function. But the first full systematic treatment is found in Gauss’s seminal paper of 1812, who introduced the hypergeometric series into analysis and gave $F$-notation for it. Gauss’s work was of great historical importance because it initiated for reaching development in many branches of analysis not only in infinite series, but also in the general theories of linear differential equations and functions of a complex variable. The hypergeometric function has retained its significance in modern mathematics because of its powerful unifying influence since many of the principal special functions of higher analysis are also related to it.

Studies in the nineteenth century included those of Ernst Kummer, and the fundamental characterization by Bernhard Riemann of the $F$-function by means of the differential equation it satisfies. Riemann showed that the second-order differential equation (in $z$) for the $\, _2F_1$, examined in the complex plane, could be characterized (on the Riemann sphere) by its three regular singularities, that effectively the entire algorithmic side of the theory was a consequence of basic facts and the use of Möbius transformations as a symmetry group. The cases where the solutions are algebraic functions were found by H. A. Schwarz (Schwarz’s list).

There are several varieties of functions of the hypergeometric type, but the most common are the standard hypergeometric function and the confluent hypergeometric function. Also, a natural generalization of these functions is the generalized hypergeometric functions, which is accomplished by the introduction of an arbitrary number of numerator and denominator parameters.
Hypergeometric Function

The hypergeometric function $2F_1[a, b; c; z]$ is defined by

$$2F_1[a, b; c; z] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n z^n}{(c)_n n!} \quad (|z| < 1; \ c \neq 0, -1, -2, \ldots), \quad (1.3.1)$$

where $a, b, c$ are real or complex parameters.

By d’Alembert’s ratio test, it is easily seen that the hypergeometric series in Eq. (1.3.1) converges absolutely within the unit circle, that is, when $|z| < 1$, provided that the denominator parameter $c$ is neither zero nor a negative integer. However, if either or both of the numerator parameters $a$ and $b$ in Eq. (1.3.1) is zero or a negative integer, the hypergeometric series terminates.

When $|z| = 1$ (that is, on the unit circle), the hypergeometric series is:

1. Absolutely convergent, if $\Re (c - a - b) > 0$;
2. Conditionally convergent, if $-1 < \Re (c - a - b) \leq 0$, $z \neq 1$;
3. Divergent, if $\Re (c - a - b) \leq -1$.

$2F_1[a, b; c; z]$ is a solution of the differential equation

$$z(1-z)\frac{d^2 u}{dz^2} + (c-(a+b+1)z)\frac{du}{dz} - abu = 0, \quad (1.3.2)$$

in which $a, b$ and $c$ are real or complex parameters. This is a homogeneous linear differential equation of the second order and is called the hypergeometric equation. It has at most three singularities $0, \infty$ and $1$ which are all regular [95]. This function has the following integral representation:

$$2F_1[a, b; c; z] = \frac{\Gamma(c)}{\Gamma(a) \Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-zt)^{-b} dt \quad (\Re (c) > \Re (a) > 0; \ |\arg(1-z)| < \pi). \quad (1.3.3)$$
Confluent Hypergeometric Function

If in hypergeometric equation (1.3.2), we replace \( z \) by \( z/b \), the resulting equation will have three singularities at \( z = 0, b, \infty \).

By letting \( |b| \to \infty \), this transformed equation reduces to

\[
\frac{d^2 u}{dz^2} + \left( c - z \right) \frac{du}{dz} - au = 0. \tag{1.3.4}
\]

Eq. (1.3.4) has a regular singularity at \( z = 0 \) and an irregular singularity at \( z = \infty \) which is formed by the confluence of two regular singularities at \( b \) and \( \infty \) of Eq. (1.3.2) with \( z \) replaced by \( z/b \).

Consequently, Eq. (1.3.4) is called the confluent hypergeometric equation or Kummer’s differential equation after E.E. Kummer who presented a detailed study of its solutions in 1836, see [76].

The simplest solution of Eq. (1.3.4) is confluent hypergeometric function or Kummer’s function \( \, _1F_1[a; c; z] \) which is given as

\[
\, _1F_1[a; c; z] = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{(c)_n n!} \, (c \neq 0, -1, -2, \ldots; \, |z| < \infty), \tag{1.3.5}
\]

which can also be deduced as a special case of hypergeometric function \( \, _2F_1[a, b; c; z] \).

In fact, we have

\[
\lim_{|b| \to \infty} \, _2F_1[a, b; c; \frac{z}{b}] = \, _1F_1[a; c; z]. \tag{1.3.6}
\]

Other popular notation for the series solution (1.3.5) is \( \Phi(a; c; z) \) introduced by Humbert [65,66].

The function

\[
\Psi(a, c; z) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-z t} t^{a-1} (1 + t)^{c-a-1} \, dt \quad (\text{Re}(a), \, \text{Re}(z) > 0), \tag{1.3.7}
\]

defines an alternative form of solution of Kummer’s equation (1.3.4) in the half plane: \( \text{Re}(z) > 0 \). The \( \Psi \) function was introduced by F.G. Tricomi in 1927.
We note the following relation between $\Phi$ or $(1F_1)$ and $\Psi$ functions:

$$
\Psi[a, c; z] = \frac{\Gamma(1-c)}{\Gamma(a-c+1)} \, _1F_1[a; c; z] + \frac{\Gamma(c-1)}{\Gamma(a)} \, z^{1-c} \, _1F_1[a-c+1; 2-c; z]
$$

\((c \neq 0, \pm 1, \pm 2\ldots)\). (1.3.8)

Also, we note the following relation between the hypergeometric function $\, _2F_0[a, b; \_; z]$ and $\Psi$ function

$$
\Psi[a, a-b+1; z] = z^{-a} \, _2F_0\left[a, b; \_, \_; \frac{1}{z}\right]
$$

(1.3.9)

Further, we note the following relations between elementary functions and hypergeometric function:

$$
\exp(z) = \, _1F_1[a; a; z] = _0F_0[\_, \_; z]
$$

(1.3.10)

and

$$
(1-z)^{-a} = \, _1F_0[a; \_, \_; z] \quad (|z| < 1).
$$

(1.3.11)

Generalized Hypergeometric Function

A natural generalization of the Gaussian hypergeometric series $\, _2F_1[a, b; c; z]$, is accomplished by introducing any arbitrary number of numerator and denominator parameters. Thus, the resulting series

$$
_pF_q\left[\begin{array}{c}
\alpha_1, \alpha_2, \ldots, \alpha_p \\
\beta_1, \beta_2, \ldots, \beta_q
\end{array}; z\right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n(\alpha_2)_n\ldots(\alpha_p)_n}{(\beta_1)_n(\beta_2)_n\ldots(\beta_q)_n} \frac{z^n}{n!}
$$

(1.3.12)

is known as the generalized Gauss series, or simply, the generalized hypergeometric series. Here $p$ and $q$ are positive integers or zero and we assume that the variable $z$, the numerator parameters $\alpha_1, \alpha_2, \ldots, \alpha_p$ and the denominator parameters $\beta_1, \beta_2, \ldots, \beta_q$ take on complex values, provided that

$$
\beta_j \neq 0, -1, -2, \ldots \; ; \; j = 1, 2, \ldots, q.
$$

Supposing that none of numerator parameters is zero or a negative integer and for $\beta_j \neq 0, -1, -2, \ldots; j = 1, 2, \ldots, q$, we note that the $\, _pF_q$ series defined by Eq. (1.3.12):
(i) converges for $|z| < \infty$, if $p \leq q$,

(ii) converges for $|z| < 1$, if $p = q + 1$ and

(iii) diverges for all $z$, $z \neq 0$, if $p > q + 1$.

Furthermore, if we set

$$w = \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j$$

it is known that the $_pF_q$ series, with $p = q + 1$, is

(I.) absolutely convergent for $|z| = 1$, if $\text{Re}(w) > 0$,

(II.) conditionally convergent for $|z| = 1$, $|z| \neq 1$, if $-1 < \text{Re}(w) \leq 0$ and

(III.) divergent for all $|z| = 1$, if $\text{Re}(w) \leq -1$.

**Hypergeometric Functions of Two and More Variables**

Continuation to the great success of the theory of hypergeometric functions in a single variable has stimulated the development of a corresponding theory in two and more variables. A multiple Gaussian hypergeometric series is a hypergeometric series in two and more variables, which reduces to the familiar Gaussian hypergeometric series (1.3.1), whenever only one variable is non-zero. Fourteen distinct double Gaussian series exist: Appell [5] introduced $F_1, F_2, F_3, F_4$, but the set was not completed until after Horn [64] gave the remaining ten series $G_1, G_2, G_3, H_1, H_2, \ldots, H_7$. Lauricella [78] introduced 14 triple Gaussian series $F_1, F_2, \ldots, F_{14}$. More precisely, he defined four $n$-dimensional series $F_A^{(n)}, F_B^{(n)}, F_C^{(n)}$ and $F_D^{(n)}$ and ten further triple series $F_3, F_4, F_6, F_7, F_8, F_{10}, \ldots, F_{14}$. Saran [106] initiated a systematic study of these ten triple Gaussian series of Lauricella’s set, and gave his notations $F_E, F_F, \ldots, F_T$. Some additional triple Gaussian series have been introduced, for example, $G_A$ and $G_B$ by Pandey [91], $H_A, H_B$ and $H_C$ by Srivastava [113,114] and $G_C$ by Srivastava [116]. We give the definitions of the functions which are used in our work.
Appell Functions

A formal extension of the hypergeometric function \(_2F_1\) to two variables, resulting in four kinds of functions called Appell functions and denoted by \(F_1\), \(F_2\), \(F_3\) and \(F_4\). Appell defined the functions in 1880 [5] and they were subsequently studied by Picard in 1881. Appell functions are defined as:

\[
F_1[a, b, b'; c; x, y] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n}{(c)_{m+n}} \frac{x^m y^n}{m! n!} \quad (\max\{|x|, |y|\} < 1), \quad (1.3.13)
\]

\[
F_2[a, b, b'; c, c'; x, y] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n}{(c)_m(c')_n} \frac{x^m y^n}{m! n!} \quad (\max\{|x| + |y|\} < 1), \quad (1.3.14)
\]

\[
F_3[a, a', b, b'; c, c'; x, y] = \sum_{m,n=0}^{\infty} \frac{(a')_m(a)_n(b)_m(b')_n}{(c)_m(c')_n} \frac{x^m y^n}{m! n!} \quad \left(\max\{|x|, |y|\} < 1\right), \quad (1.3.15)
\]

\[
F_4[a, b; c, c'; x, y] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n}{(c)_m(c')_n} \frac{x^m y^n}{m! n!} \quad \left(\max\{|x|, |y|\} < 1\right), \quad (1.3.16)
\]

where, as usual, the denominator parameters \(c\) and \(c'\) are neither zero nor a negative integer.

The standard work on the theory of Appell functions is the monograph by Appell and Kampé de Fériet [6]. For a review of the subsequent work on the subject, see Erdélyi et al. [52,53], Bailey [7], Slater [110], Exton [57] and Srivastava and Karlsson [118].

Humbert Functions

Seven confluent forms of the four Appell functions were defined in 1920 by Humbert [65] and he denoted these confluent hypergeometric functions of two variables by

\[
\Phi_1, \Phi_2, \Phi_3, \Psi_1, \Psi_2, \Xi_1, \Xi_2,
\]

five of them are given below:

\[
\Phi_1[\alpha, \beta; \gamma; x, y] = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m}{(\gamma)_{m+n}} \frac{x^m y^n}{m! n!} \quad (|x| < 1; |y| < \infty), \quad (1.3.17)
\]
\[ \Phi_2[\beta, \beta'; \gamma; x, y] = \sum_{m,n=0}^{\infty} \frac{(\beta)_m (\beta')_n}{(\gamma)_{m+n}} \frac{x^m y^n}{m! n!} \quad (|x| < \infty; |y| < \infty), \]  
(1.3.18)

\[ \Phi_3[\beta; \gamma; x, y] = \sum_{m,n=0}^{\infty} \frac{(\beta)_m}{(\gamma)_{m+n}} \frac{x^m y^n}{m! n!} \quad (|x| < \infty; |y| < \infty), \]  
(1.3.19)

\[ \Psi_1[\alpha, \beta; \gamma, \gamma'; x, y] = \sum_{m,n=0}^{\infty} \frac{\alpha_{m+n} (\beta)_m}{(\gamma)_m (\gamma')_n} \frac{x^m y^n}{m! n!} \quad (|x| < 1; |y| < \infty), \]  
(1.3.20)

\[ \Psi_2[\alpha; \gamma, \gamma'; x, y] = \sum_{m,n=0}^{\infty} \frac{\alpha_{m+n}}{(\gamma)_m (\gamma')_n} \frac{x^m y^n}{m! n!} \quad (|x| < \infty; |y| < \infty). \]  
(1.3.21)

**Kampé de Fériet Function**

The four Appell functions were unified and generalized by Kampé de Fériet [70] who defined a general hypergeometric function of two variables, see Appell and Kampé de Fériet [6, p. 150(29)].

Kampé de Fériet function is defined by

\[
F_{l,m;n}^{p,q;k}(\alpha_p : (b_q); (c_k) ; x, y) = \sum_{r,s=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_{r+s} \prod_{j=1}^{q} (b_j)_r \prod_{j=1}^{k} (c_j)_s}{\prod_{j=1}^{l} (\alpha_j)_{r+s} \prod_{j=1}^{m} (\beta_j)_r \prod_{j=1}^{n} (\gamma_j)_s} \frac{x^r y^s}{r! s!}, \quad (1.3.22)
\]

where, for convergence,

(i) \( p + q < l + m + 1, \) \( p + k < l + n + 1, \) \(|x| < \infty, \) \(|y| < \infty, \) or

(ii) \( p + q = l + m + 1, \) \( p + k = l + n + 1, \) and

\[
\begin{align*}
|x|^{\frac{1}{p-l}} + |y|^{\frac{1}{p-l}} &< 1, \quad \text{if } p > l, \\
\max\{|x|, |y|\} &< 1, \quad \text{if } p \leq l.
\end{align*}
\]

The notation \( F_{l,m;n}^{p,q;k} \) for Kampé de Fériet general double hypergeometric series of superior order applies successfully to the Appell double hypergeometric series \( F_1, F_2, F_3, F_4 \)
and their confluent forms $\Phi_1$, $\Phi_2$, $\Phi_3$, $\Psi_1$, $\Psi_2$, $\Xi_1$, $\Xi_2$ defined by Humbert. Thus, for example, we have

$$F_1 = F^{1:1:1}_{1:0:0}, \quad F_2 = F^{1:1:1}_{0:1:1}, \quad F_3 = F^{0:2:2}_{1:0:0}, \quad F_4 = F^{2:0:0}_{0:1:1}$$

$$\Phi_1 = F^{1:1:0}_{1:0:0}, \quad \Phi_2 = F^{0:1:1}_{1:0:0}, \ldots, \Xi_2 = F^{0:2:0}_{1:0:0}. \quad (1.3.23)$$

### Lauricella Functions of $n$ Variables

Lauricella [78] further generalized the four Appell functions $F_1, F_2, F_3$ and $F_4$ to functions of $n$ variables. Let $n$ be the number of variables, then the Lauricella functions are defined as follows:

$$F^{(n)}_A[a, b_1, b_2, \ldots, b_n; c_1, c_2, \ldots, c_n; x_1, x_2, \ldots, x_n]$$

$$= \sum_{m_1, m_2, \ldots, m_n=0}^{\infty} \frac{(a)_m b_1 b_2 \ldots b_n (c)_n m_1 m_2 \ldots m_n}{m_1! m_2! \ldots m_n!} x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n}$$

\[ (|x_1| + |x_2| + \ldots + |x_n| < 1), \quad (1.3.24) \]

$$F^{(n)}_B[a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n; c; x_1, x_2, \ldots, x_n]$$

$$= \sum_{m_1, m_2, \ldots, m_n=0}^{\infty} \frac{(a_1)_m b_1 b_2 \ldots b_n (c)_n m_1 m_2 \ldots m_n}{m_1! m_2! \ldots m_n!} x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n}$$

\[ \max\{|x_1|, |x_2|, \ldots, |x_n|\} < 1, \quad (1.3.25) \]

$$F^{(n)}_C[a, b; c_1, c_2, \ldots, c_n; x_1, x_2, \ldots, x_n]$$

$$= \sum_{m_1, m_2, \ldots, m_n=0}^{\infty} \frac{(a)_m b_1 b_2 \ldots b_n (c)_n m_1 m_2 \ldots m_n}{m_1! m_2! \ldots m_n!} x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n}$$

\[ \sqrt{|x_1|} + \sqrt{|x_2|} + \ldots + \sqrt{|x_n|} < 1, \quad (1.3.26) \]

$$F^{(n)}_D[a, b_1, b_2, \ldots, b_n; c_1, x_1, x_2, \ldots, x_n]$$

$$= \sum_{m_1, m_2, \ldots, m_n=0}^{\infty} \frac{(a)_m b_1 b_2 \ldots b_n (c)_n m_1 m_2 \ldots m_n}{m_1! m_2! \ldots m_n!} x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n}$$

\[ \max\{|x_1|, |x_2|, \ldots, |x_n|\} < 1. \quad (1.3.27) \]
Clearly, if \( n = 2 \), then these functions reduce to the Appell hypergeometric functions \( F_2, F_3, F_4 \) and \( F_1 \), respectively. If \( n = 1 \), all four become the Gauss hypergeometric function \( _2F_1 \).

Two important confluent forms of Lauricella functions in \( n \) variables are the functions \( \Phi_2^{(n)} \) and \( \Psi_2^{(n)} \) which are defined by

\[
\Phi_2^{(n)}[b_1, b_2, \ldots, b_n; c; x_1, x_2, \ldots, x_n] = \sum_{m_1, m_2, \ldots, m_n=0}^{\infty} \frac{(b_1)_{m_1} (b_2)_{m_2} \cdots (b_n)_{m_n}}{(c)_{m_1+m_2+\ldots+m_n}} \times \frac{x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}}{m_1! \cdot m_2! \cdots m_n!},
\]

and

\[
\Psi_2^{(n)}[a; c_1, c_2, \ldots, c_n; x_1, x_2, \ldots, x_n] = \sum_{m_1, m_2, \ldots, m_n=0}^{\infty} \frac{(a)_{m_1+m_2+\ldots+m_n}}{(c_1)_{m_1} (c_2)_{m_2} \cdots (c_n)_{m_n}} \times \frac{x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}}{m_1! \cdot m_2! \cdots m_n!}.
\]

Clearly, we have

\[
\Phi_2^{(2)} = \Phi_2, \quad \Psi_2^{(2)} = \Psi_2,
\]

where \( \Phi_2 \) and \( \Psi_2 \) are Humbert’s confluent hypergeometric functions of two variables.

**Srivastava’s Triple Hypergeometric Function**

A unification of Lauricella’s fourteen hypergeometric functions \( F_1, F_2, \ldots, F_{14} \) and the additional functions \( H_A, H_B, H_C \) was introduced by Srivastava [113-115], who defined a general triple hypergeometric series \( F^{(3)}[x, y, z] \) (see also [106]) as:

\[
F^{(3)}[x, y, z] \equiv F^{(3)} \left[ \begin{array}{c}
(a) \,::\, (b); (b'); (b'') \,::\, (c); (c'); (c''); \\
(c) \,::\, (g); (g'); (g'') \,::\, (h); (h'); (h'');
\end{array} \right]_{x, y, z}
\]

\[
= \sum_{m, n, p=0}^{\infty} \frac{A}{E} \prod_{j=1}^{A} (a_j)_{m+n+p} \prod_{j=1}^{B} (b_j)_m \prod_{j=1}^{B'} (b_j')_{n+p} \prod_{j=1}^{B''} (b_j'')_{p+m} \prod_{j=1}^{C} (c_j)_m \prod_{j=1}^{C'} (c_j')_n \prod_{j=1}^{C''} (c_j'')_p \prod_{j=1}^{G} (g_j)_m \prod_{j=1}^{G'} (g_j')_{n+p} \prod_{j=1}^{G''} (g_j'')_{p+m} \prod_{j=1}^{H} (h_j)_m \prod_{j=1}^{H'} (h_j')_n \prod_{j=1}^{H''} (h_j'')_p
\]

\[
\times \frac{x^m y^n z^p}{m! \cdot n! \cdot p!},
\]

(1.3.30)
where \( (a) \) abbreviates the array of \( A \) parameters \( a_1, a_2, \ldots, a_A \) with similar interpretations for \((b), (b'), (b''), \) \textit{et cetera}. The triple hypergeometric series in Eq. (1.3.30) converges absolutely when

\[
\begin{align*}
1 + E + G + G' + G'' + H - A - B - B'' - C & \geq 0, \\
1 + E + G + G' + H' - A - B - B' - C' & \geq 0, \\
1 + E + G' + G'' + H'' - A - B' - B'' - C'' & \geq 0,
\end{align*}
\]

where the equalities hold true for suitably constrained values of \(|x|, |y|\) and \(|z|\).

1.4. SPECIAL FUNCTIONS EXPRESSIBLE IN TERMS OF HYPERGEOMETRIC FUNCTIONS

The Gauss hypergeometric function \( _2F_1 \) and the confluent hypergeometric function \( _1F_1 \) form the core of the special functions and include as special cases most of the commonly used functions. The \( _2F_1 \) includes as its special cases, many elementary functions, Legendre functions of the first and second kinds, the incomplete beta function, complete elliptic integrals of the first and second kinds, Jacobi polynomials, Gegenbauer (or ultraspherical) polynomials, Legendre (or spherical) polynomials, Tchebycheff polynomials of the first and second kinds \textit{et cetera} [119, pp. 34-36]. On the other hand \( _1F_1 \) includes as its special cases, Bessel functions, Whittaker functions, incomplete gamma functions, error functions, parabolic cylinder (or Weber) functions, Bateman’s \( k \)-function, Hermite polynomials, Laguerre polynomials and functions, Poisson-Charlier polynomials \textit{et cetera} [119, pp. 39-41].

In this Section, we give the definitions of some special functions and discuss their relationship between the hypergeometric functions (we consider only those special functions which will be used in our work).

I. Legendre Polynomials and Functions

The Legendre polynomials are closely associated with physical phenomena for which spherical geometry is important. In particular, these polynomials first arose in the problem of expressing the Newtonian potential of a conservative force field.
in an infinite series involving the distance variables of two points and their included central angle. Other similar problems dealing with either gravitational potentials or electrostatic potentials also lead to Legendre polynomials, as do certain steady-state heat-conduction problems in spherical-shaped solids, and so forth.

The Legendre (or spherical) polynomials \( P_n(x) \) are defined by means of the generating function

\[
(1 - 2xt + t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x) t^n \quad (|t| < 1; \ |x| \leq 1).
\]

The Legendre polynomials \( P_n(x) \) are also defined by the series

\[
P_n(x) = \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \frac{(2n - 2k)!}{2^n k! (n - k)! (n - 2k)!} x^n.
\]

The Legendre polynomials \( P_n(x) \) are solutions of the differential equation

\[
(1 - x^2) \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + n(n + 1)y = 0,
\]

which is called Legendre’s differential equation.

We note that

\[
P_n(x) = _2F_1 \left[ -n, n + 1; 1; \frac{1-x}{2} \right].
\]

The orthogonality property of the Legendre polynomials \( P_n(x) \) is given by

\[
\int_{-1}^{1} P_n(x) P_m(x) dx = 0, \quad m \neq n
\]

i.e. the set of Legendre polynomials \( P_n(x) \) is orthogonal with respect to the weight function unity on the interval \(-1 < x < 1\).

For \( m = n \), we have the following additional property of \( P_n(x) \):

\[
\int_{-1}^{1} [P_n(x)]^2 dx = \frac{2}{2n+1} \quad (n = 0, 1, 2, \ldots).
\]

The associated Legendre functions of the first and second kinds, \( P_n^m(x) \) and \( Q_n^m(x) \), respectively are defined by

\[
P_n^m(x) = (1 - x^2)^{m/2} \frac{d^m}{dx^m} P_n(x) \quad (m = 0, 1, 2, \ldots, n)
\]

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and

\[ Q_n^m(x) = (1 - x^2)^{m/2} \frac{d^m}{dx^m} Q_n(x) \quad (m = 0, 1, 2, \ldots, n). \]  

(1.4.8)

The associated Legendre functions of the first and second kinds are solutions of the differential equation

\[
(1 - x^2) \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + \left( n(n + 1) - \frac{m^2}{1 - x^2} \right) y = 0 \]  

(1.4.9)

which is called associated Legendre equation. It arises in applications involving either the Laplace or the Helmholtz equation in spherical, oblate spheroidal, or prolate spheroidal coordinates.

For \( m = 0 \), we get the special case

\[ P_n^0(x) = P_n(x). \]  

(1.4.10)

We note that

\[ P_n^m(x) = \frac{1}{\Gamma(1-m)} \left( \frac{x+1}{x-1} \right)^{m/2} \binom{-n}{n-m} \binom{1/2}{n+1/2} \]

(1.4.11)

and

\[ Q_n^m(x) = \frac{\sqrt{\pi} \exp(im\pi) \Gamma(m+n+1)}{(2)^{n+1} \Gamma(n+3/2)} \frac{(x^2-1)^{m/2}}{(x)^{m+n+1}} \]

\[ \times \binom{1/2}{n+1/2} \binom{1/2}{n+1} \binom{3/2}{1} \binom{1}{x^2} . \]  

(1.4.12)

The orthogonality property of the associated Legendre functions \( P_n^m(x) \) is given by

\[
\int_{-1}^{1} P_n^m(x) P_k^m(x) \, dx = 0, \quad k \neq n
\]  

(1.4.13)

For \( k = n \), we have

\[
\int_{-1}^{1} |P_n^m(x)|^2 \, dx = \frac{2(n+m)!}{(2n+1)(n-m)!}.
\]  

(1.4.14)
II. Hermite Polynomials

The Hermite polynomials play an important role in problems involving Laplace’s equation in parabolic coordinates, in various problems in quantum mechanics and in probability theory.

The Hermite polynomials $H_n(x)$ are defined by means of the generating function

$$\exp(2xt - t^2) = \sum_{n=0}^{\infty} \frac{H_n(x) t^n}{n!} \quad (|t| < \infty; \quad |x| < \infty). \tag{1.4.15}$$

The Hermite polynomials $H_n(x)$ are also defined by the series

$$H_n(x) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k n! (2x)^{n-2k}}{k! (n-2k)!}. \tag{1.4.16}$$

The Hermite polynomials $H_n(x)$ are solutions of the differential equation

$$\frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + 2ny = 0, \tag{1.4.17}$$

which is called Hermite’s equation.

We note that

$$H_n(x) = (2x)^n \sum_{k=0}^{n} \frac{(-1)^k n! (2x)^{n-2k}}{k! (n-2k)!}. \tag{1.4.18}$$

The orthogonality property of the Hermite polynomials $H_n(x)$ is given by

$$\int_{-\infty}^{+\infty} e^{-x^2} H_n(x)H_m(x) \, dx = 0, \quad m \neq n, \tag{1.4.19}$$

i.e. the set of Hermite polynomials $H_n(x)$ is orthogonal with respect to the weight function $e^{-x^2}$ on the interval $-\infty < x < \infty$.

For $m = n$, we have the following additional property of $H_n(x)$:

$$\int_{-\infty}^{+\infty} e^{-x^2} [H_n(x)]^2 \, dx = 2^n n! \sqrt{\pi}. \tag{1.4.20}$$
III. Laguerre Polynomials

In many applications, particularly in quantum-mechanical problems, a generalization of the Laguerre polynomials \( L_n(x) \) called the associated Laguerre polynomials is needed.

The associated Laguerre polynomials \( L_n^{(m)}(x) \) are defined by means of the generating function

\[
\frac{1}{(1-t)^{m+1}} \exp \left( \frac{-xt}{1-t} \right) = \sum_{n=0}^{\infty} L_n^{(m)}(x) t^n \quad (|t| < 1; \; 0 \leq x < \infty). \tag{1.4.21}
\]

The associated Laguerre polynomials \( L_n^{(m)}(x) \) are also defined by the series

\[
L_n^{(m)}(x) = \sum_{k=0}^{n} \frac{(-1)^k (n+m)!}{(n-k)! (m+k)! k!} x^k \quad (m = 0, 1, 2, \ldots). \tag{1.4.22}
\]

The associated Laguerre polynomials \( L_n^{(m)}(x) \) are solutions of the differential equation

\[
x \frac{d^2 y}{dx^2} + (1 + m - x) \frac{dy}{dx} + ny = 0, \tag{1.4.23}
\]

which is called the associated Laguerre’s equation.

We note that

\[
L_n^{(m)}(x) = \frac{\Gamma(n+m+1)}{\Gamma(n+1) \Gamma(m+1)} \, _1F_1[-n; \, m+1; \, x]. \tag{1.4.24}
\]

When \( m = 0 \), the resultant polynomial is denoted by \( L_n(x) \). Thus, we have

\[
L_n^{(0)}(x) = L_n(x), \tag{1.4.25}
\]

where \( L_n(x) \) are called the Laguerre polynomials.

The orthogonality property of the associated Laguerre polynomials \( L_n^{(m)}(x) \) is given by

\[
\int_0^{\infty} x^m e^{-x} L_n^{(m)}(x) L_k^{(m)}(x) \, dx = 0, \quad k \neq n \quad (\text{Re}(m) > -1), \tag{1.4.26}
\]

i.e. the set of associated Laguerre polynomials \( L_n^{(m)}(x) \) is orthogonal with respect to weight function \( x^m e^{-x} \) on the interval \( 0 < x < \infty \), if \( \text{Re}(m) > -1 \).
For \( k = n \), we have the following additional property of \( L_n^{(m)}(x) \):

\[
\int_0^{\infty} x^m e^{-x} [L_n^{(m)}(x)]^2 dx = \frac{\Gamma(n + m + 1)}{n!} \quad (\text{Re}(m) > -1).
\] (1.4.27)

We note that for \( m = 0 \), Eqs. (1.4.26) and (1.4.27) reduce to the corresponding results for the Laguerre polynomials \( L_n(x) \).

IV. Jacobi Polynomials

The Jacobi polynomials \( P_n^{(\alpha,\beta)}(x) \) are defined by means of the generating function

\[
\frac{2^{\alpha+\beta}}{R} (1 - t + R)^{-\alpha} (1 + t + R)^{-\beta} = \sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x) t^n
\]

\( (R := (1 - 2xt + t^2)^{1/2}; \quad \alpha, \beta > -1). \) (1.4.28)

The Jacobi polynomials \( P_n^{(\alpha,\beta)}(x) \) are also defined by the series

\[
P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} \frac{(n + \alpha) (n + \beta) (x - 1)^k}{n!} \left[ \frac{x + 1}{2} \right]^{n-k}.
\] (1.4.29)

The Jacobi polynomials \( P_n^{(\alpha,\beta)}(x) \) are solutions of the differential equation

\[
(1 - x^2) \frac{d^2y}{dx^2} + (\beta - \alpha - (\alpha + \beta + 2)x) \frac{dy}{dx} + n(n + \alpha + \beta + 1)y = 0.
\] (1.4.30)

We note that

\[
P_n^{(\alpha,\beta)}(x) = \frac{(\alpha + 1)n}{n!} \, \, _2F_1 \left[ -n, n + \alpha + \beta + 1; \alpha + 1; \frac{1 - x}{2} \right].
\] (1.4.31)

The orthogonality property of the Jacobi polynomials \( P_n^{(\alpha,\beta)}(x) \) is given by

\[
\int_{-1}^{1} (1 - x)^{\alpha} (1 + x)^{\beta} P_n^{(\alpha,\beta)} P_m^{(\alpha,\beta)}(x) dx = 0, \quad m \neq n \quad (\text{Re}(\alpha), \text{Re}(\beta) > -1),
\] (1.4.32)

i.e. the set of Jacobi polynomials \( P_n^{(\alpha,\beta)}(x) \) is orthogonal with respect to weight function \((1 - x)^{\alpha}(1 + x)^{\beta}\) on the interval \(-1 < x < 1\), if \(\text{Re}(\alpha), \text{Re}(\beta) > -1\).
For \( m = n \), we have the following additional property of \( P_n^{(\alpha,\beta)}(x) \):
\[
\int_{-1}^{1} (1 - x)^\alpha (1 + x)^\beta |P_n^{(\alpha,\beta)}(x)|^2 \, dx = \frac{2^{\alpha+\beta+1} \Gamma(n + \alpha + 1) \Gamma(n + \beta + 1)}{(2n + \alpha + \beta + 1) \Gamma(n + \alpha + \beta + 1)n!} \quad (\text{Re}(\alpha), \text{Re}(\beta) > -1).
\] (1.4.33)

V. Bessel Functions

Bessel functions are closely associated with problems possessing circular or cylindrical symmetry. For example, they arise in the study of free vibrations of a circular membrane and in finding the temperature distribution in a circular cylinder. They also occur in electromagnetic theory and numerous other areas of physics and engineering. Because of their close association with cylindrical-shaped domains, all solutions of Bessel’s equation are collectively called cylinder functions.

The Bessel functions \( J_n(x) \) are defined by means of the generating relation
\[
\exp \left( \frac{x}{2} \left( t - \frac{1}{t} \right) \right) = \sum_{n=-\infty}^{\infty} J_n(x) t^n \quad (t \neq 0; |x| < \infty). \] (1.4.34)

The Bessel functions \( J_n(x) \) are also defined by the series
\[
J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \left( \frac{x}{2} \right)^{n+2k}}{k! \Gamma(1 + n + k)} \quad (-\infty < x < \infty), \] (1.4.35)
where \( n \) is a positive integer or zero and
\[
J_n(x) = (-1)^n J_{-n}(x), \] (1.4.36)
where \( n \) is a negative integer.

The Bessel functions \( J_n(x) \) are solutions of the differential equation
\[
x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + (x^2 - n^2)y = 0.
\] (1.4.37)
which is called Bessel's equation. Among other areas of application, it arises in the solution of various partial differential equations of mathematical physics, particularly those problems displaying either circular or cylindrical symmetry.
We note that
\[ J_n(x) = \frac{(x/2)^n}{\Gamma(1+n)} \, {}_0F_1 \left[ -; n + 1; -\frac{x^2}{4} \right]. \] (1.4.38)

Next, we recall the definition of Tricomi function \( C_n(x) \), which is a Bessel-like function. The Tricomi functions \( C_n(x) \) are defined by means of the generating relation
\[ \exp \left( t - \frac{x}{t} \right) = \sum_{n=-\infty}^{\infty} C_n(x) t^n \quad (t \neq 0; \ |x| < \infty). \] (1.4.39)

The Tricomi functions \( C_n(x) \) are also defined by the series
\[ C_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k! \,(n+k)!} \quad (n = 0, 1, 2, \ldots). \] (1.4.40)

Further, Tricomi functions are characterized by the following link with the ordinary Bessel functions \( J_n(x) \):
\[ C_n(x) = x^{-\frac{n}{2}} J_n(2\sqrt{x}). \] (1.4.41)

The Tricomi functions \( C_n(x) \) are solutions of the differential equation
\[ \left( x \frac{d^2}{dx^2} + (n + 1) \frac{d}{dx} + 1 \right) C_n(x) = 0. \] (1.4.42)

We note that
\[ C_n(x) = \frac{1}{\Gamma(1+n)} \, {}_0F_1 \left[ -; n + 1; -x \right]. \] (1.4.43)

Next, we give the definitions of modified Bessel functions of the first and second kinds.

**Modified Bessel Functions**

The modified Bessel functions of the first kind \( I_n(x) \) of order \( n \), are defined by means of the generating function
\[ \exp \left( \frac{1}{2} x \left( t + \frac{1}{t} \right) \right) = \sum_{n=-\infty}^{\infty} I_n(x) \, t^n \quad (t \neq 0; \ |x| < \infty). \] (1.4.44)

Modified Bessel functions of the first kind \( I_n(x) \) of order \( n \) are also defined by the series
\[ I_n(x) = i^{-n} J_n(ix) = \sum_{k=0}^{\infty} \frac{(\frac{x}{2})^{2k+n}}{k! \, \Gamma(k+n+1)}, \] (1.4.45)
for \( n \) a positive integer or zero. For negative \( n \), we define

\[
I_{-n}(x) = \sum_{k=0}^{\infty} \frac{(\frac{x}{2})^{2k-n}}{k! \Gamma(k - n + 1)} \quad (n > 0),
\]

from which we deduce

\[
I_n(x) = I_{-n}(x) \quad (n = 0, 1, 2, \ldots).
\]

Modified Bessel functions \( I_n(x) \) are solutions of the differential equation

\[
x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - (x^2 + n^2)y = 0 \quad (n \geq 0),
\]

which is known as Bessel's modified equation.

We note that

\[
I_n(x) = \frac{(\frac{x}{2})^n}{\Gamma(n+1)} \mathbf{F}_1 \left[-n; n + 1; \frac{x^2}{4}\right].
\]

The modified Bessel function of the second kind \( K_n(x) \) of order \( n \) (or Macdonald's function), is defined by

\[
K_n(x) = \frac{\pi}{2} \frac{I_{-n}(x) - I_n(x)}{\sin(n\pi)}.
\]

We note that

\[
K_n(x) = \sqrt{\pi}(2x)^n \exp(-x)\Psi \left[n + \frac{1}{2}, 2n + 1; 2x\right].
\]

VI. Error Functions

The error function \( \text{erf}(x) \) is defined by the integral

\[
\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2) \, dt \quad (-\infty < x < \infty).
\]

This function is encountered in probability theory, the theory of errors, the theory of heat conduction and various branches of mathematical physics.
The error function \( \text{erf}(x) \) is also defined by the series

\[
\text{erf}(x) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n! (2n+1)} \quad (|x| < \infty).
\]  

(1.4.53)

In some applications it is useful to introduce the complementary error function \( \text{erfc}(x) \) which is defined by

\[
\text{erfc}(x) = 1 - \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty \exp(-t^2) \, dt.
\]  

(1.4.54)

The alternative notations

\[
\text{Erf}(x) = \frac{\sqrt{\pi}}{2} \text{erf}(x), \quad \text{Erfc}(x) = \frac{\sqrt{\pi}}{2} \text{erfc}(x)
\]  

(1.4.55)

are sometimes used for the error functions.

We note that

\[
\text{Erf}(x) = \frac{\sqrt{\pi}}{2} \text{erf}(x) = \frac{1}{2} \gamma \left( \frac{1}{2}, x^2 \right) = x \, {}_1F_1 \left[ \frac{1}{2}, \frac{3}{2}; -x^2 \right]
\]  

(1.4.56)

and

\[
\text{Erfc}(x) = \frac{\sqrt{\pi}}{2} \text{erfc}(x) = \frac{1}{2} \Gamma \left( \frac{1}{2}, x^2 \right) = \frac{1}{2} \exp(-x^2) \Psi \left[ \frac{1}{2}, \frac{1}{2}; x^2 \right].
\]  

(1.4.57)

VII. Whittaker Functions and Parabolic Cylinder Functions

Whittaker’s functions of the first and second kinds, \( M_{k,\mu}(x) \) and \( W_{k,\mu}(x) \) are defined by

\[
M_{k,\mu}(x) = x^{\mu+\frac{1}{2}} \exp \left( -\frac{1}{2} x \right) {}_1F_1 \left[ \mu - k + \frac{1}{2}, 2\mu + 1; x \right]
\]  

\[
= x^{\mu+\frac{1}{2}} \exp \left( \frac{1}{2} x \right) {}_1F_1 \left[ \mu + k + \frac{1}{2}, 2\mu + 1; -x \right]
\]  

\[\quad (2\mu \neq -1, -2, -3, \ldots)\]  

(1.4.58)

and
\[
W_{k, \mu}(x) = x^{\mu + \frac{1}{2}} \exp \left( -\frac{1}{2} x \right) \Psi \left[ \mu - k + \frac{1}{2}, 2 \mu + 1; x \right] = W_{k, -\mu}(x)
\]

\[
(2\mu \neq -1, -2, -3, \ldots).
\] (1.4.59)

We note the following special cases of the Whittaker’s functions \( M_{k, \mu}(x) \) and \( W_{k, \mu}(x) \):

\[
\frac{2}{\sqrt{\pi x}} \exp \left( -\frac{x^2}{2} \right) M_{-\frac{1}{4}, \frac{1}{4}}(x^2) = \text{erf}(x)
\] (1.4.60)

and

\[
\frac{1}{\sqrt{\pi x}} \exp \left( -\frac{x^2}{2} \right) W_{-\frac{1}{4}, \frac{1}{4}}(x^2) = \text{erfc}(x)
\] (1.4.61)

The parabolic cylinder (or Weber) function \( D_{\nu}(x) \) is defined by

\[
D_{\nu}(x) = 2^{\nu/2} \exp \left( -\frac{1}{4} x^2 \right) \Psi \left[ -\frac{1}{2} \nu, \frac{1}{2}; \frac{1}{2} x^2 \right]
\]

\[
= 2^{(\nu-1)/2} x \exp \left( -\frac{1}{4} x^2 \right) \Psi \left[ \frac{1}{2} - \frac{1}{2} \nu, \frac{3}{2}; \frac{1}{2} x^2 \right].
\] (1.4.62)

The parabolic cylinder function \( D_{-\nu}(x) \) has the following integral representation

\[
D_{-\nu}(x) = \frac{1}{\Gamma(\nu)} \exp \left( -\frac{1}{4} x^2 \right) \int_0^\infty t^{-\nu-1} \exp \left( -x t - \frac{1}{2} t^2 \right) dt \quad (x \in \mathbb{C}; \text{Re}(\nu) > 0).
\] (1.4.63)

The parabolic cylinder function \( D_{\nu}(x) \) satisfies the following differential equation

\[
\frac{d^2y}{dx^2} + \left( \nu + \frac{1}{2} - \frac{x^2}{4} \right) y = 0.
\] (1.4.64)

We note that

\[
D_{\nu}(x) = 2^{\frac{1}{2}(\nu+\frac{1}{2})} x^{-1/2} W_{\frac{1}{2}(\nu+\frac{1}{2}), -\frac{1}{4}} \left( \frac{x^2}{2} \right)
\]

\[
= 2^{-\nu/2} \exp \left( -\frac{x^2}{4} \right) H_n \left( \frac{x}{\sqrt{2}} \right)
\]

\[(\nu = 0, \pm 1, \pm 2, \ldots).\] (1.4.65)
1.5. SHEFFER POLYNOMIALS, UMBRAL CALCULUS AND MONOMIALITY PRINCIPLE

Sequences of polynomials play a fundamental role in applied mathematics. Such sequences can be described in various ways, for example,

(1) By orthogonality conditions:

\[ \int_a^b p_n(x)p_m(x)w(x)\,dx = \delta_{n,m}, \quad (1.5.1) \]

where \( w(x) \) is a weight function and \( \delta_{n,m} = 0 \) or 1 according as \( n \neq m \) or \( n = m \), respectively.

(2) As solutions to differential equations: for instance, the Hermite polynomials \( H_n(x) \) satisfy the second-order linear differential equation (1.4.17).

(3) By generating functions: for example, the associated Laguerre polynomials \( L_n^{(m)}(x) \) are characterized by equation (1.4.21).

(4) By the recurrence relations: for example, the Legendre polynomials \( P_n(x) \) satisfy the following pure and differential recurrence relations:

\[ (n + 1)P_{n+1}(x) - (2n + 1) x P_n(x) + n P_{n-1}(x) = 0 \quad (1.5.2a) \]

and

\[ (1 - x^2) \frac{d}{dx} P_n(x) = -nx P_n(x) + nP_{n-1}(x), \quad (1.5.2b) \]

\[ (1 - x^2) \frac{d}{dx} P_n(x) = (n + 1)x P_n(x) - (n + 1)P_{n+1}(x), \quad (1.5.2c) \]

respectively.

(5) By operational formulas: for example, the associated Laguerre polynomials \( L_n^{(m)}(x) \) satisfy

\[ L_n^{(m)}(x) = x^{-m} e^{x} D^{n} e^{-x} x^{n+m}. \quad (1.5.3) \]

One of the simplest classes of polynomial sequences, yet still large enough to include many important instances, is the class of Sheffer sequences (also known, in a
slightly different form, as sequences of Sheffer A-type zero or poweroids). This class
may be defined in many ways, most commonly by a generating function and by a
differential recurrence relation. A sequence \( s_n(x) \) is a Sheffer sequence if and only if
its generating function has the form
\[
\sum_{k=0}^{\infty} \frac{s_k(x)}{k!} t^k = A(t) \exp(xH(t)),
\]
where
\[
A(t) = A_0 + A_1 t + A_2 t^2 + \ldots ~ (A_0 \neq 0)
\]
and
\[
H(t) = H_1 t + H_2 t^2 + \ldots ~ (H_1 \neq 0).
\]

The Sheffer class contains such important sequences as those formed by
(1) The Hermite polynomials, which play an important role in applied mathematics
and physics (such as Brownian motion and the Schrödinger wave equation).
(2) The Laguerre polynomials, which also play a key role in applied mathematics and
physics (they are involved in solutions to the wave equation of the hydrogen atom).
(3) The Bernoulli polynomials, which find applications, for example, in number theory
(evaluation of the Hurwitz zeta function, a generalization of the famous Riemann
zeta function).

A detailed study of these polynomials and more will be discussed in Chapter 3.

The modern classical umbral calculus can be described as a systematic study of
the class of Sheffer sequences, made by employing the simplest techniques of modern
algebra.

More explicitly, if \( P \) is the algebra of polynomials in a single variable, the set \( P^* \)
of all linear functionals on \( P \) is usually thought of as a vector space (under pointwise
operations). However, it is well known that a linear functional on \( P \) can be identified
with a formal power series. In fact, there is one-to-one correspondence between linear
functionals on \( P \) and formal power series in a single variable. For example, we may
associate to each linear functional $L$ the power series $\sum_{k=0}^{\infty} L(x^k) t^k / k!$. But the set of formal power series is usually given the structure of an algebra (under formal addition and multiplication). This algebra structure, the additive part of which “agrees” with the vector space structure on $P^*$, can be “transferred” to $P^*$. The algebra so obtained is called the umbral algebra, and the umbral calculus is the study of this algebra.

Now, since $P^*$ has the structure of an algebra, we may consider, for two linear functionals $L$ and $M$, the geometric sequence $M, ML, ML^2, ML^3, \ldots$. Then under mild conditions on $L$ and $M$, the equations

$$ML^k(s_n(x)) = n! \delta_{n,k}$$

(1.5.5)

for $n, k \geq 0$ uniquely determine a sequence $s_n(x)$ of polynomials which turns out to be of Sheffer type and conversely, for any sequence $s_n(x)$ of Sheffer type there are linear functionals $L$ and $M$ for which the above equations hold. Thus we may characterize the class of Sheffer sequences by means of the umbral algebra. The resulting interplay between the umbral algebra and the algebra of polynomials allows for the natural development of some powerful adjointness properties wherein lies the real strength of the theory. The umbral calculus is, to be sure, formal mathematics. This means that limiting processes, such as the convergence of infinite series, play no role.

Let $C$ be a field of characteristic zero. Let $\mathcal{F}$ be the set of all formal power series in the variable $t$ over $C$. Thus, an element of $\mathcal{F}$ has the form

$$f(t) = \sum_{k=0}^{\infty} a_k t^k \quad (a_k \in C).$$

(1.5.6)

Two formal power series are equal if and only if the coefficients of like powers of $t$ are equal. It is well known that if addition and multiplication are defined formally,

$$\sum_{k=0}^{\infty} a_k t^k + \sum_{k=0}^{\infty} b_k t^k = \sum_{k=0}^{\infty} (a_k + b_k) t^k,$$

and

$$\left(\sum_{k=0}^{\infty} a_k t^k\right) \left(\sum_{k=0}^{\infty} b_k t^k\right) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} a_j b_{k-j}\right) t^k,$$

then $\mathcal{F}$ is an algebra (with no zero divisors).
The order $o(f(t))$ of a power series $f(t)$ is the smallest integer $k$ for which the coefficient of $t^k$ does not vanish. We take $o(f(t)) = +\infty$ if $f(t) = 0$. It is easy to see that

$$o(f(t)g(t)) = o(f(t)) + o(g(t)),$$

$$o(f(t) + g(t)) \geq \min\{o(f(t)), o(g(t))\}. \tag{1.5.7}$$

The series $f(t)$ has a multiplicative inverse, denoted by $f(t)^{-1}$ or $\frac{1}{f(t)}$, if and only if $o(f(t)) = 0$. We shall then say that $f(t)$ is invertible.

If $o(f(t)) = 1$, then the formal power series $f(t)$ has a compositional inverse $\bar{f}(t)$ satisfying $f(\bar{f}(t)) = \bar{f}(f(t)) = t$. A series $f(t)$ for which $o(f(t)) = 1$ will be called a delta series.

Let $P$ be the algebra of polynomials in the single variable $x$ over the field $C$ of characteristic zero. Let $P^*$ be the vector space of all linear functionals on $P$. We use the notation $\langle L \mid p(x) \rangle$, borrowed from physics, to denote the action of a linear functional $L$ on a polynomial $p(x)$ and we recall that the vector space operations on $P^*$ are defined by

$$\langle L + M \mid p(x) \rangle = \langle L \mid p(x) \rangle + \langle M \mid p(x) \rangle \tag{1.5.9}$$

and

$$\langle cL \mid p(x) \rangle = c \langle L \mid p(x) \rangle \quad (c \in C). \tag{1.5.10}$$

Since a linear functional is uniquely determined by its action on a basis, $L$ is uniquely determined by the sequence of constants $\langle L \mid x^n \rangle$. Further, let $\mathcal{F}$ denote the algebra of formal power series in the variable $t$ over the field $C$. The formal power series

$$f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \tag{1.5.11}$$

defines a linear functional on $P$ by setting

$$\langle f(t) \mid x^n \rangle = a_n \quad (n \geq 0). \tag{1.5.12}$$
In particular
\[ \langle \ell^k | x^n \rangle = n! \delta_{n,k} . \]

Any linear functional \( L \) in \( P^* \) has the form (1.5.11). For if
\[
f_L(t) = \sum_{k=0}^{\infty} \frac{\langle L | x^k \rangle}{k!} t^k, \tag{1.5.13}
\]
then from (1.5.12), we get
\[ \langle f_L(t) | x^n \rangle = \langle L | x^n \rangle \tag{1.5.14} \]
and so as linear functional \( L = f_L(t) \).

Also, we note that the map \( L \to f_L(t) \) is a vector space isomorphism from \( P^* \) onto \( \mathcal{F} \).

Henceforth, \( \mathcal{F} \) will denote both the algebra of formal power series in the variable \( t \) and the vector space of all linear functionals on \( P \) and so an element \( f(t) \) of \( \mathcal{F} \) will be thought of as both a formal power series and a linear functional. It is important to note that two elements \( f(t) \) and \( g(t) \) of \( \mathcal{F} \) are equal as formal power series if and only if they are equal as linear functionals. This follows directly from (1.5.12) and the corresponding definitions of equality. Thus, an algebra structure on the vector space of all linear functionals on \( P \), namely, the algebra of formal power series is defined. We shall call \( \mathcal{F} \) the umbral algebra.

We use the notation \( t^k \) for the \( k \)th derivative operator on \( P \), that is
\[
t^k x^n = \begin{cases} 
(n)_k x^{n-k}, & k \leq n \\
0, & k > n.
\end{cases} \tag{1.5.15}
\]

With this notation, any power series (1.5.11) is a linear operator on \( P \) defined by
\[ f(t) x^n = \sum_{k=0}^{\infty} \binom{n}{k} a_k x^{n-k}. \tag{1.5.16} \]

Notice that we use juxtaposition \( f(t)p(x) \) to denote the action of the operator \( f(t) \) on the polynomial \( p(x) \).

Thus an element of \( \mathcal{F} \) plays three roles in the umbral calculus – it is a formal power series, a linear functional and a linear operator.
By a sequence $s_n(x)$ of polynomials we shall always imply that $\deg s_n(x) = n$.

Roman [102] characterized Sheffer sequences in several ways. First we recall the following result [102, p.17], which can be viewed as an alternate definition of Sheffer sequences:

Let $f(t)$ be a delta series and let $g(t)$ be an invertible series. Then there exists a unique sequence $s_n(x)$ of polynomials satisfying the orthogonality conditions

$$\langle g(t) f(t)^k | s_n(x) \rangle = n! \delta_{n,k} \quad (\forall n, k \geq 0). \quad (1.5.17)$$

The sequence $s_n(x)$ in Eq. (1.5.17) is the Sheffer sequence for the pair $(g(t), f(t))$, or that $s_n(x)$ is Sheffer for $(g(t), f(t))$. Notice that $g(t)$ must be invertible and $f(t)$ must be a delta series.

There are two types of Sheffer sequences that deserve special consideration. The Sheffer sequence for $(1, f(t))$ is the associated sequence for $f(t)$ and $s_n(x)$ is associated to $f(t)$. The Sheffer sequence for $(g(t), t)$ is the Appell sequence for $g(t)$ and $s_n(x)$ is Appell for $g(t)$.

According to the Sheffer identity [102, p. 21], a sequence $s_n(x)$ is Sheffer for $(g(t), f(t))$, for some invertible $g(t)$, if and only if

$$s_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} p_k(y) s_{n-k}(x) \quad (y \in C), \quad (1.5.18)$$

where $p_n(x)$ is associated to $f(t)$.

Further, according to the Appell identity [102, p. 27], a sequence $A_n(x)$ is an Appell sequence if and only if

$$A_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} A_k(y) x^{n-k} \quad (y \in C). \quad (1.5.19)$$

Furthermore, we recall that [102, p. 27], a sequence $A_n(x)$ is Appell for $g(t)$ if and only if

$$t A_n(x) = n A_{n-1}(x), \quad (1.5.20)$$

which in view of Eq. (1.5.15) can be written as

$$A'_n(x) = n A_{n-1}(x). \quad (1.5.21)$$
The theory of binomial enumeration is variously called the calculus of finite differences or the umbral calculus. This theory studies the analogies between various sequences of polynomials $p_n$ and the powers sequence $x^n$. The subscript $n$ in $p_n$ was thought of as the shadow ("umbra" means "shadow" in Latin, whence the name umbral calculus) of the superscript $n$ in $x^n$ and many parallels were discovered between such sequences.

The umbral calculus in the form that we know today, is the theory of Sheffer polynomials. The history of Sheffer polynomials goes back to 1880 when Appell studied sequences of polynomials $(p_n)_n$ satisfying $p'_n(x) = n p_{n-1}(x)$, which is same as Eq. (1.5.21) (see [5]). These sequences are nowadays called Appell polynomials. Although this class was widely studied (see the bibliography in [48]), it was not until 1939 that Sheffer noticed the similarities with monomial $x^n$. These similarities led him to extend the class of Appell polynomials which he called polynomials of type zero (see [107]), but which nowadays are called Sheffer polynomials. This class already appeared in [84]. Although Sheffer uses operators to study his polynomials, his theory is mainly based on formal power series. In 1941 the Danish actuary Steffensen also published a theory of Sheffer polynomials based on formal power series [121]. Steffensen uses the name poweroids for Sheffer polynomials (see also [108,109,121-124]).

Operators methods are used to free umbral calculus from its mystery [104]. In [85] the ideas from [104] are extended to give a beautiful theory combining enriched functions, umbral methods and operator methods. However, only the subclass of polynomials of binomial type are treated in [85]. The extension to Sheffer polynomials is accomplished in [105]. The latter paper is much more geared towards special functions, while the former paper is a combinatorial paper. The papers [85] and [105] were soon followed by papers that reacted directly on the new umbral calculus.

The idea of monomiality traces back to the early forties of the last century, when J.F. Steffensen, in a largely unnoticed paper [121], suggested the concept of poweroid. A new interest in this subject was by the work of G. Dattoli and his collaborators [21,41].
It turns out that all polynomial families and in particular all special polynomials, are essentially the same, since it is possible to obtain each of them transforming a basic monomial set by means of suitable operators (called the derivative and multiplication operator of the considered family). This was shown by a theoretical proof in [18,19], and can be viewed as the basis of the umbral calculus [103] – a term invented by Sylvester – since the exponent, for example in $x^n$, is transformed into his “shadow” in $p_n(x)$.

The notion of quasi-monomiality has been exploited within different contexts to deal with isospectral problems [111] and to study the properties of new families of special functions [44]. The monomiality principle is a fairly useful tool for treating various families of special polynomials as well as their new and known generalizations.

The concept of quasi-monomiality is fairly straightforward and can be summarized as follows:

The polynomial set \( \{p_n(x)\}_{n \in \mathbb{N}} \) is quasi-monomial, if there exist two operators \( \hat{P} \) and \( \hat{M} \), called respectively derivative operator and multiplicative operator, satisfying (\( \forall n \in \mathbb{N} \)) the identities
\[
\hat{P}\{p_n(x)\} = np_{n-1}(x),
\]
\[
\hat{M}\{p_n(x)\} = p_{n+1}(x).
\] (1.5.22)

The operators \( \hat{P} \) and \( \hat{M} \) are satisfy the commutation property
\[
[\hat{P}, \hat{M}] = \hat{P} \hat{M} - \hat{M} \hat{P} = \hat{1}
\] (1.5.23)
and thus display a Weyl group structure.

If the considered polynomial set \( \{p_n(x)\} \) is quasi-monomial, its properties can be easily derived from the structure of operators \( \hat{P} \) and \( \hat{M} \). In fact

(a). If \( \hat{P} \) and \( \hat{M} \) have a differential realization, then the polynomials \( p_n(x) \) satisfy the differential equation
\[
\hat{M} \hat{P}\{p_n(x)\} = np_n(x).
\] (1.5.24)
(b). Assuming here and in the following \( p_0(x) = 1 \), then the polynomials \( p_n(x) \) can be explicitly constructed by the relation

\[
p_n(x) = \hat{M}^n \{1\}.
\]  

(1.5.25)

(c). The last identity implies that the exponential generating function of the polynomials \( p_n(x) \) can be cast in the form

\[
\exp(t\hat{M})\{1\} = \sum_{n=0}^{\infty} p_n(x) \frac{t^n}{n!} \quad (|t| < \infty).
\]

(1.5.26)

The concepts and the formalism associated with the monomiality treatment can be exploited in different ways. On one side, they can be used to study the properties of ordinary or generalized special polynomials by means of a formalism closer to that of natural monomials. On the other side, they can be useful to establish rules of operational nature, framing the special polynomials within the context of particular solutions of generalized forms of partial differential equations.

Most of the properties of families of polynomials, recognized as quasi monomials, can be deduced, quite straightforwardly, by using operational rules associated with the relevant derivative and multiplicative operators. Furthermore they suggest that we can define families of isospectral problems by exploiting the correspondence

\[
\begin{align*}
\hat{P} & \iff \frac{\partial}{\partial x}, \\
\hat{M} & \iff x, \\
p_n(x) & \iff x^n.
\end{align*}
\]  

(1.5.27)

In the case of multi-variable generalized special functions, the use of operational techniques, combined with the principle of monomiality [21] provides new means of analysis for the solutions of a wide class of partial differential equations often encountered in physical problems.

There is a continuous demand of operational techniques in research fields like classical and quantum optics and in these fields the use of operational techniques has provided powerful and efficient means of investigation. Most of the interest is relevant
to operational identities associated with ordinary and multi-variable forms of Hermite and Laguerre polynomials, see for example [21,29,30,43].

1.6. GENERATING FUNCTIONS

The research into Leibniz’s analogy led Laplace to the calculus of generating functions. Laplace formulated the calculus of generating functions in 1779, and later he returned to it in several occasions. In 1812, Laplace introduced the concept of ‘generating function’. Since then the theory of generating functions has been developed in various directions. A generating function may be used to define a set of functions, to determine differential or pure recurrence relations, to evaluate certain integrals et cetera. Generating relations of special functions arise in a diverse range of applications in quantum physics, molecular chemistry, harmonic analysis, multivariate statistics, number theory et cetera. Generating functions play an important role in the investigation of various useful properties of the sequences which they generate. There are various methods of obtaining generating functions for a fairly wide variety of sequences of special functions (and polynomials), see for example [83] and [119].

**Linear Generating Functions**

Consider a two-variable function $F(x, t)$, which possesses a formal (not necessarily convergent for $t \neq 0$) power series expansion in $t$ such that

$$F(x, t) = \sum_{n=0}^{\infty} f_n(x) t^n, \quad (1.6.1)$$

where each member of the coefficient set $\{f_n(x)\}_{n=0}^{\infty}$ is independent of $t$. Then, the expansion (1.6.1) of $F(x, t)$ is said to have generated the set $\{f_n(x)\}$ and $F(x, t)$ is called a linear generating function (or, simply, a generating function) for the set $\{f_n(x)\}$.

This definition may be extended slightly to include a generating function of the type:

$$G(x, t) = \sum_{n=0}^{\infty} c_n g_n(x) t^n, \quad (1.6.2)$$

where the sequence $\{c_n\}_{n=0}^{\infty}$ may contain the parameters of the set $g_n(x)$, but is independent of $x$ and $t$. 

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If $c_n$ and $g_n(x)$ in expansion (1.6.2) are prescribed and if we can formally determine the sum function $G(x, t)$ in terms of known special functions, then we say that the generating functions $G(x, t)$ has been found.

Further, if the set $\{f_n(x)\}$ is defined for $n = 0, \pm 1, \pm 2, \cdots$, then the definition (1.6.2) may be extended in terms of the Laurent series expansion:

$$ F^*(x, t) = \sum_{n=-\infty}^{\infty} \gamma_n f_n(x) t^n, \quad (1.6.3) $$

where the sequence $\{\gamma_n\}_{n=-\infty}^{\infty}$ is independent of $x$ and $t$.

**Bilinear and Bilateral Generating Functions**

If a three-variable function $F(x, y, t)$ possesses a formal power series expansion in $t$ such that

$$ F(x, y, t) = \sum_{n=0}^{\infty} \gamma_n f_n(x) f_n(y) t^n, \quad (1.6.4) $$

where the sequence $\{\gamma_n\}$ is independent of $x$, $y$ and $t$, then $F(x, y, t)$ is called a bilinear generating function for the set $\{f_n(x)\}$.

More generally, if $F(x, y, t)$ can be expanded in powers of $t$ in the form

$$ F(x, y, t) = \sum_{n=0}^{\infty} \gamma_n f_{\alpha(n)}(x) f_{\beta(n)}(y) t^n, \quad (1.6.5) $$

where $\alpha(n)$ and $\beta(n)$ are functions of $n$ which are not necessarily equal, then also $F(x, y, t)$ is called a bilinear generating function for the set $\{f_n(x)\}$.

Further, suppose that a three-variable function $H(x, y, t)$ has a formal power series expansion in $t$ such that

$$ H(x, y, t) = \sum_{n=0}^{\infty} h_n f_n(x) g_n(y) t^n, \quad (1.6.6) $$

where the sequence $\{h_n\}$ is independent of $x$, $y$ and $t$, and the sets of functions $\{f_n(x)\}_{n=0}^{\infty}$ and $\{g_n(x)\}_{n=0}^{\infty}$ are different. Then $H(x, y, t)$ is called a bilateral generating function for the set of $\{f_n(x)\}$ or $\{g_n(x)\}$. 

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The above definition of a bilateral generating function, may be extended to include bilateral generating function of the type:

\[ H(x, y, t) = \sum_{n=0}^{\infty} \gamma_n f_{\alpha(n)}(x) g_{\beta(n)}(y) t^n \]  

(1.6.7)

where the sequence \( \{\gamma_n\} \) is independent of \( x, y \) and \( t \), the sets of functions \( \{f_n(x)\}_{n=0}^{\infty} \) and \( \{g_n(x)\}_{n=0}^{\infty} \) are different and \( \alpha(n) \) and \( \beta(n) \) are functions of \( n \) which are not necessarily equal.

**Multi-Variable, Multi-Linear, Multi-Lateral and Multiple Generating Functions**

Suppose that \( G(x_1, x_2, \ldots, x_r; t) \) is a function of \( r+1 \) variables, which has a formal expansion in powers of \( t \) such that

\[ G(x_1, x_2, \ldots, x_r; t) = \sum_{n=0}^{\infty} c_n g_n(x_1, x_2, \ldots, x_r) t^n, \]  

(1.6.8)

where the sequence \( \{c_n\} \) is independent of the variables \( x_1, x_2, \ldots, x_r \) and \( t \). Then \( G(x_1, x_2, \ldots, x_r; t) \) is called a generating function for the set \( \{g_n(x_1, x_2, \ldots, x_r)\}_{n=0}^{\infty} \) corresponding to the nonzero coefficients \( c_n \).

Similarly, we extend the definition of bilinear and bilateral generating functions to include such multivariable generating functions as:

\[ F(x_1, x_2, \ldots, x_r, y_1, y_2, \ldots, y_r; t) = \sum_{n=0}^{\infty} \gamma_n f_{\alpha(n)}(x_1, x_2, \ldots, x_r) f_{\beta(n)}(y_1, y_2, \ldots, y_r) t^n \]  

(1.6.9)

and

\[ H(x_1, x_2, \ldots, x_r, y_1, y_2, \ldots, y_s; t) = \sum_{n=0}^{\infty} h_n f_{\alpha(n)}(x_1, x_2, \ldots, x_r) g_{\beta(n)}(y_1, y_2, \ldots, y_s) t^n, \]  

(1.6.10)

respectively.

A multi-variable generating function \( G(x_1, x_2, \ldots, x_r; t) \) given by Eq. (1.6.8) is said to be a multi-linear generating function if

\[ g_n(x_1, x_2, \ldots, x_r) = f_{\alpha_1(n)}(x_1) f_{\alpha_2(n)}(x_2) \cdots f_{\alpha_r(n)}(x_r), \]  

(1.6.11)
where $\alpha_1(n), \alpha_2(n), \cdots, \alpha_r(n)$ are functions of $n$ which are not necessarily equal. More generally, if the functions occurring on the right hand side of (1.6.11) are all different, then the multi-variable generating function (1.6.8) is called a multi-lateral generating function.

A natural further extension of the multi-variable generating function (1.6.8) is a multiple generating function which may be defined formally by

$$
G^*(x_1, x_2, \cdots, x_r; t_1, t_2, \cdots, t_r) = \sum_{n_1, n_2, \cdots, n_r=-\infty}^{\infty} C(n_1, n_2, \cdots, n_r) \Gamma_{n_1, n_2, \cdots, n_r}(x_1, x_2, \cdots, x_r) t_1^{n_1} t_2^{n_2} \cdots t_r^{n_r},
$$

where the multiple sequence $\{C(n_1, n_2, \cdots, n_r)\}$ is independent of the variables $x_1, \ldots, x_r$ and $t_1, t_2, \cdots, t_r$.

Further, definitions (1.6.8) and (1.6.12) may be extended in terms of the Laurent series expansions:

$$
G^*(x_1, x_2, \cdots, x_r; t) = \sum_{n=-\infty}^{\infty} g_n(x_1, x_2, \cdots, x_r) t^n
$$

and

$$
\Psi^*(x_1, x_2, \cdots, x_r; t_1, t_2, \cdots, t_r)
$$

$$
= \sum_{n_1, n_2, \cdots, n_r=-\infty}^{\infty} C(n_1, n_2, \cdots, n_r) \Gamma_{n_1, n_2, \cdots, n_r}(x_1, x_2, \cdots, x_r) t_1^{n_1} t_2^{n_2} \cdots t_r^{n_r},
$$

respectively.