PREFACE

The present thesis entitled "Study of Properties of Certain Multi-Variable and Mixed Type Special Functions" is an outcome of the studies made by the author at the Department of Mathematics, Aligarh Muslim University, Aligarh, India.

Special functions are the solutions of a wide class of mathematically and physically relevant functional equations. The special functions of mathematical physics, such as the classical orthogonal polynomials (the Laguerre, Hermite and Jacobi polynomials), the spherical, cylindrical and hypergeometric functions, arise in the solutions of many theoretical and applied problems of physics, engineering, statistics, biology, economics and other most diverse areas of natural, life and social sciences. The theory of special functions with its numerous beautiful formulae is very well suited to an algorithmic approach to mathematics. Although, special functions can be defined in different ways such as (Rodrigue's formulae, generating functions, summation formulae, integral representations et cetera), but it is usually shown to be expressible as a series, because this is frequently the most practical way to obtain numerical values for the functions.

Special functions have centuries of history with immense literature as constructed in the works of Chebyshev, Euler, Gauss, Hardy, Hermite, Legendre, Ramanujan, Watson and other classical authors. A number of books consisting of the theory and applications of special functions are available, see for example Andrews [1,2], Andrews et al. [3], Erdélyi et al. [52,53,56], Iwasaki et al. [68], Lebedev [79], Rainville [99], Sneddon [112] et cetera.

Various generalizations of the special functions of mathematical physics have witnessed a significant evolution during the recent years. This further advancement in the theory of special functions serves as an analytical foundation for the majority of problems in mathematical physics that have been solved exactly and finds broad practical applications. For some physical problems the use of new classes of special functions provided solutions hardly achievable with conventional analytical and numerical means. For example, the use of generalized Bessel functions is now a well established tool to
treat synchrotron radiation [42] and crystallographic [90] problems. Further the importance of generalized Hermite polynomials has been recognized [26,35] and has been exploited to deal with quantum mechanical and optical beam transport problems. The usefulness of the generalized Laguerre polynomials to treat radiation physics problems such as wave propagation and quantum beam life time in storage rings due to the quantum fluctuations is a well established fact, see [127].

Dattoli and his co-workers have given several contributions of many sets of special functions in several variables, see for example Dattoli [21-25], Dattoli and Khan [29,30], Dattoli et al. [26,27,31,34-36,43,44]. Dattoli and Torre [44-46]. Some contributions on two dimensional polynomials and functions are given by Wünsche [128-131] and the monographs by Suetin [125] and Dunkl and Yuan [51] are useful in the study of orthogonal polynomials of two and several variables respectively.

The theory of binomial enumeration is variously called the calculus of finite differences or the umbral calculus. The umbral calculus is a kind of symbolic calculus whose starting point is the study of polynomials of binomial type. An elementary introduction to the modern umbral calculus is given by Roman [102]. Gustafson has made an attempt to show how umbral calculus can be used to enrich the theory of special functions, see for example [61,62]. Some connections between the classical invariant theory for $SL_2$ and special functions are discussed in [63], with the purpose to provide a point of unification between the representation theory of $SL_2$ and special functions and also to indicate possible generalizations of hypergeometric series and classical orthogonal polynomials to several variables.

The operational techniques (including differential and integral operators) provide a systematic and analytic approach to study special functions. These techniques are useful to derive the properties of special functions of mathematical physics. The operational techniques combined with the monomiality principle open new possibilities to deal with the theoretical foundations of special polynomials and also to introduce new families of polynomials. The concepts and the formalism associated with the monomiality treatment can be exploited in different ways. On one side, they can be used to simplify the derivation of the properties of ordinary or generalized special polynomials.
and to introduce new families of polynomials. On the other side, they can be useful to establish rules of operational nature, framing the special polynomials within the context of particular solutions of generalized forms of partial differential equations of evolution type. The importance of the use of operational techniques in the study of special functions and their applications has been recognized by Dattoli and his co-workers, see for example [9,21-36,38,41,43-47]. Most of the interest is relevant to operational identities associated with ordinary and multi-variable forms of Hermite and Laguerre polynomials.

Methods connected with the use of integral transforms have been successfully applied to the solution of differential and integral equations, the study of generalized special functions and the evaluation of integrals. An important advantage of the method of integral transforms lies in the possibility of compiling tables of the direct and inverse transforms of the various functions commonly encountered in applications. In research fields like classical and quantum optics the use of integral operators has provided powerful and efficient method of investigation. Among the integral transforms, Laplace’s occupies a special place, mainly because of its usefulness in solving differential equations of functions of exponential order with initial value conditions or semi-infinite boundary value conditions. It has applications in various areas of science and engineering. An operator treatment of the operational calculus, using the Laplace transform is given by Diktin and Prudnikov [49] and Plessner [94]. Integral transforms have become an extensively used tool and their practical applications depend largely on tables of transform pairs. A number of books consisting of tables of formulae for integral transforms are available, see for example Brychkov [14], Ditkin and Prudnikov [49], Erdélyi et al. [54,55], Magnus et al. [82] and Prudnikov et al. [96-98].

The objectives of this thesis are:

(i) To introduce new families of mixed type polynomials associated with Appell; and Sheffer sequences of polynomials;

(ii) To establish summation formulae for multi-variable forms of Hermite polynomials;
(iii) To obtain the evaluations of Euler type integrals in terms of beta function and generalized hypergeometric series and

(iv) To obtain the evaluations of integrals in terms of extended beta function and various multi-variable and mixed type special polynomials.

The thesis consists of six chapters.

In Chapter 1, we review various notations, known definitions, concepts and results which are used in the presentation of the work of the subsequent chapters.

In Chapter 2, we use the concepts and the formalism associated with monomiality principle to introduce the family of Laguerre-based Appell polynomials and discuss their properties. Further, we establish a correspondence between Appell and Laguerre-Appell polynomials and apply it to derive several identities for Laguerre-based Bernoulli and Euler polynomials. Furthermore, we derive some results for Laguerre-Genocchi and Laguerre-modified Laguerre polynomials.

In Chapter 3, we use the concepts and the formalism associated with monomiality principle and Sheffer sequences to generate the family of Hermite-based Sheffer polynomials and derive their properties. Further, we establish operational rules providing a correspondence between Sheffer and Hermite-Sheffer polynomials. Furthermore, by using these operational rules, we derive several relations, identities and expansions for Hermite-Sheffer polynomials from the corresponding results of Sheffer polynomials.

In Chapter 4, we establish summation formulae for 2-variable Hermite-Kampé de-Fériet polynomials \( H_n(x, y) \) and Gould-Hopper generalized Hermite polynomials \( g_n^m(x, y) \) by using different analytical means on their respective generating functions. Further, we derive the summation formulae for polynomials related to \( H_n(x, y) \) and \( g_n^m(x, y) \) as applications. Furthermore, we extend this process to derive a summation formula for multi-variable generalized Hermite polynomials \( H_n^{(s)}(\{x_s\}_M) \).

In Chapter 5, we obtain the evaluations of certain Euler type integrals and derive some Laplace transforms. Further, we establish the equivalence of integrals involving the product of two \( \Psi \) functions with integrals involving hypergeometric function \( _2F_1 \).
Furthermore, we derive a number of integrals, summation formulae and representations as applications of these results.

In Chapter 6, we establish a theorem on extended beta function $B(\alpha, \beta; b)$ and apply it to obtain the evaluations of Euler type integrals in terms of certain mixed type special polynomial families. Further, we obtain the evaluations of Euler type integrals in terms of certain multi-variable special polynomials. Furthermore, we derive a number of new integrals as applications of these results.

Six papers based on the research contained in this thesis have been communicated for publication in international journals. Two of them which contains the work of Chapter 2 and Chapter 5 have been accepted for publication in journals entitled “Mathematical and Computer Modelling” and “International Journal of Mathematical Analysis” respectively.

In the end, we give a comprehensive list of references of books, monographs, edited volumes, proceedings and research papers used in carrying out this research work.