Chapter 4

The Generalized Pebbling Number of Some Graphs

4.1 Introduction

Consider a distribution of pebbles on the vertices of a graph $G$. A generalized pebbling step involves removing $p$ pebbles from a vertex removing $(p - 1)$ pebbles from $G$, and moving the remaining pebble to an adjacent vertex. For any positive $p \geq 2$, is it possible to move a pebble to a root vertex $r$, if we can repeatedly apply generalized pebbling steps. It is answered in affirmative by Chung in [2].

The generalized pebbling number of a vertex $v$ in a graph $G$ is the smallest number $f_{gl}(v, G)$ with the property that from every placement of $f_{gl}(v, G)$ pebbles on $G$, it is possible to move a pebble to $v$ by a sequence of pebbling moves where a pebbling move consists of removing $p$ pebbles from a vertex and placing one pebble on an adjacent vertex. The generalized pebbling number of the graph $G$, denoted by $f_{gl}(G)$, is the maximum of $f_{gl}(v, G)$ over all vertices $v$ in $G$. 
In this Chapter, we determine the generalized pebbling number of some graphs and generalized $t$-pebbling number. The rest of the Chapter deals with the generalized $p$ pebbling property and the generalized pebbling conjecture on products of some graphs $G$ with the graphs $H$ where $H$ satisfies the generalized $p$ pebbling property.

### 4.2 Computation of the generalized pebbling number

**Theorem 4.2.1.** For any graph $G$, $f_{gl}(G) \geq (p - 1)|V(G)| - (p - 2)$ where $p \geq 2$.

**Proof.** Let $G$ be a connected graph on $|V(G)|$ vertices. Let $r$ be a root vertex. Place zero pebbles on $r$ and place $p - 1$ pebbles at every vertex of $G - \{r\}$. We cannot move a pebble to $r$. So, $f_{gl}(r, G) \geq (p - 1)|V(G)| - (p - 2)$.

Therefore $f_{gl}(G) \geq (p - 1)|V(G)| - (p - 2)$.

**Theorem 4.2.2.** For a complete graph $K_n$, $f_{gl}(K_n) = (p - 1)n - (p - 2)$ where $p \geq 2$.

**Proof.** For a complete graph $K_n$, by Theorem 4.2.1, $f_{gl}(K_n) \geq (p - 1)n - (p - 2)$. Let $r$ be a root vertex in $K_n$. Let $(p - 1)n - (p - 2)$ pebbles be placed such that vertex $r$ in $K_n$ has no pebble. There exists at least one vertex $u$ in $K_n$, $u \neq r$ such that $u$ has at least $p$ pebbles. Since $\deg(r) = n - 1$, $r$ is adjacent to all other vertices, in particular to $u$. So, we can move a pebble from $u$ to $r$. So, $f_{gl}(r, K_n) \leq (p - 1)n - (p - 2)$. 

Therefore, \( f_{gl}(K_n) = (p - 1)n - (p - 2) \) since \( r \) is arbitrary.

**Theorem 4.2.3.** If \( G \) contains a cut vertex, then

\[
f_{gl}(G) \geq (p - 1)n + (p^2 - 3p + 3)
\]

where \( n = |V(G)| \) and \( p \geq 2 \).

**Proof.** Let \( u \) be a cut vertex of \( G \) and let \( C_1 \) and \( C_2 \) be two distinct components of \( G - \{u\} \).

Take a vertex \( r \) in \( C_1 \) and a vertex \( y \) in \( C_2 \) (\( r \) is our root vertex). Place \( p - 1 \) pebbles at each vertex of \( V(G) - \{u, y, r\} \) and place \( p^2 - 1 \) pebbles at \( y \). We cannot move a pebble to \( r \). So, \( f_{gl}(G) \geq (p - 1)n + (p^2 - 3p + 3) \).

**Theorem 4.2.4.** For a path of length \( n \), \( f_{gl}(P_n) = p^n \) where \( p \geq 2 \).

**Proof.** We prove the result by induction on \( n \). When \( n = 0 \), the result is true. When \( n = 1 \), let \( (u, v) \) be a path. If \( p \) pebbles are given assume that both are placed at \( u \) (or \( v \)), then we can move a pebble to \( v \) (or \( u \)). If we place at least one pebble at \( u \) and the remaining at \( v \), then the question of moving does not arise. Hence the result is true for \( n = 1 \).

Now, let us assume that the result is true for all paths of length less than \( n \). Consider a path \( P_n = v_0, v_1, v_2, \ldots, v_n \). Suppose \( p^n \) pebbles are placed. Consider the case when no pebble is placed on \( v_n \). Consider the path \( P_{n-1} = v_0, v_1, v_2, \ldots, v_{n-1} \). All the \( p^n \) pebbles have been placed in \( P_{n-1} \). The \( p^n \) pebbles can be arbitrarily grouped into \( p \) disjoint groups of \( p^{n-1} \) pebbles. Then by induction we can move \( p \) pebbles to \( v_{n-1} \). Hence one pebble can be moved to \( v_n \). So, \( f_{gl}(v_n, P_n) \leq p^n \). By placing \( p^n - 1 \) pebbles at \( v_0 \), we cannot move a pebble to \( v_n \). So, \( f_{gl}(v_n, P_n) = p^n \). A similar argument shows that \( f_{gl}(v_0, P_n) = p^n \).
Now consider an internal vertex $v_k (0 < k < n)$. Suppose $p^n$ pebbles are placed on vertices other than $v_k$. By removing $v_k$ from the path, $P_n$ is decomposed into two paths. Let $P_1 = v_0, v_1, v_2, \ldots, v_{k-1}$ and $P_2 = v_{k+1}, v_{k+2}, \ldots, v_n$.

At least one of $P_1$ or $P_2$ contains not less than $p^n - 1$ pebbles. Without loss of generality, assume that $P_1$ has not less than $p^n - 1$ pebbles. $P_1$ is of length at most $n - 2$. But it contains $p^n - 1$ pebbles, by splitting these $p^n - 1$ pebbles into $p$ arbitrary disjoint sets of $p^{n-2}$ pebbles each, we can bring $p$ pebbles to $v_{k-1}$. So, $f_{gl}(v_k, P_n) \leq p^n$ for any internal vertex $v_k$. Hence $f_{gl}(P_n) = p^n$. □

**Theorem 4.2.5.** Let $C_n$ denote a cycle with $n$ vertices. Then,

$$f_{gl}(C_n) = \begin{cases} 
    p^{n/2}, & \text{if } n \text{ is even} \\
    1 + 2\left(\frac{p^n}{p+1}\right)(\lfloor n/2 \rfloor - 1), & \text{if } n \text{ is odd.}
\end{cases}$$

**Proof.** Let $C_n = (v_0, v_1, v_2, \ldots, v_{n-1})$.

**Case 1:** $n$ is even.

The edges of $C_n$ can be partitioned into two disjoint paths say $P_1$ and $P_2$ each from $v_0$ to $v_{n/2}$. By placing $p^{n/2} - 1$ pebbles at $v_{n/2}$ which is the farthest vertex from $v_0$, we cannot move a pebble to $v_0$. So $f_{gl}(v_0, C_n) \geq p^{n/2}$. Let $K$ be the number of pebbles placed at $v_{n/2}$. If $K = p^{n/2}$ then we can move a pebble to $v_0$.

Suppose $K < p^{n/2}$, we place $p^{n/2} - K$ pebbles on other vertices. If we place all pebbles in one path say $P_1$, then $P_1$ has $p^{n/2}$ pebbles. So, we can move a pebble to $v_0$ (Theorem 4.2.4). If we distribute $p^{n/2} - K$ pebbles at the internal vertices of both paths, we have to distribute at least $\lceil (p^{n/2} - K)/2 \rceil$ pebbles at the internal vertices of any one of paths, $P_1$ and $P_2$. Without loss of generality, assume that we distribute at least $\lceil (p^{n/2} - K)/2 \rceil$ pebbles at the internal vertices of the path.
Ch. 4: The Generalized Pebbling Number of Some Graphs

$P_1$. Placing one pebble at $v_{(n/2)} - i$ is equivalent to placing $p^i$ pebbles at $v_{n/2}$. So, placing $\left\lfloor (p^{n/2} - K)/2 \right\rfloor$ pebbles in $v_1, v_2, \ldots, v_{(n/2)-1}$ is equivalent to placing at least $p^{n/2}$ pebbles at $v_{n/2}$. So, we can move a pebble to $v_0$. So, $f_{gl}(v_0, C_n) = p^{n/2}$.

Hence $f_{gl}(C_n) = p^{n/2}$ since $v_0$ is arbitrary.

Case 2: $n$ is odd.

Take $K = 2\lceil(p/p + 1)(p^{[n/2]} - 1)\rceil$.

Suppose $K$ pebbles are placed as follows: $K/2$ pebbles are placed at each of the vertices $v_{[n/2]}$ and $v_{[n/2]} + 1$. We can move $\lceil K/2p \rceil$ pebbles from either of the vertex to the other vertex. So, one of these two vertices has got $p^{[n/2]} - 1$ pebbles. But these two vertices are at a distance $[n/2]$ from $v_0$. Therefore a pebble cannot be moved to $v_0$. So, $f_{gl}(v_0, C_n) \geq 1 + K$. Let us prove that $f_{gl}(v_0, C_n) = 1 + K$.

Suppose that $K + 1$ pebbles are placed at the vertices of $C_n$. Consider the paths $P_1 = (v_0, v_1, v_2, \ldots, v_{[n/2]})$ and $P_2 = (v_{[n/2]} + 1, \ldots, v_{n-1}, v_0)$. In the positioning of $K + 1$ pebbles if $P_1$ or $P_2$ has at least $p^{[n/2]}$ pebbles then we are done. When both $P_1$ and $P_2$ have less than $p^{[n/2]}$ pebbles each, the special case is placing $K/2$ pebbles at either of the vertices $v_{[n/2]}$ and $v_{[n/2]} + 1$ and $(K/2) + 1$ at the other vertex. Now, we can move $\lceil K/2p \rceil$ pebbles from the vertex with $K/2$ pebbles to the vertex with $(K/2) + 1$ pebbles. After this move one of the vertices $v_{[n/2]}$ and $v_{[n/2]} + 1$ has $p^{[n/2]}$ pebbles. Since these two vertices are at a distance $[n/2]$ from $v_0$, a pebble can be moved to $v_0$.

In all other positioning, it is equivalent to placing at least $(K/2)$ pebbles at either of the vertices $v_{[n/2]}$ and $v_{[n/2]} + 1$ and at least $1 + (K/2)$ pebbles at the other vertex because placing one pebble at $v_{[n/2]} + 1$ is equivalent to placing $p$ pebbles at $v_{[n/2]}$. So, one of the vertices $v_{[n/2]}$ and $v_{[n/2]} + 1$ has got at least $p^{[n/2]}$ pebbles. Hence, a pebble can be moved to vertex $v_0$. 


So,  \( f_{gl}(v_0, C_n) = 1 + K \).

Therefore,  \( f_{gl}(C_n) = 1 + K \) since  \( v_0 \) is arbitrary.

\[ \]

**Theorem 4.2.6.** \( f_{gl}(K_1, n) = (p - 1)n + (p^2 - 2p + 2) \) if  \( n > 1 \) and  \( p \geq 2 \).

**Proof.** Let  \( V(K_1, n) = V_1 \cup V_2 \) where  \( V_1 = \{ r \} \) and  \( V_2 = \{ s_1, s_2, \ldots, s_n \} \).

By Theorem 4.2.2, \( f_{gl}(r, K_1, n) = (p - 1)n + 1 \).

To find \( f_{gl}(s_1, K_1, n) \).

We place zero pebbles on  \( r \) and  \( s_1 \). Place  \( p^2 - 1 \) pebbles at  \( s_2 \) and  \( p - 1 \) pebbles at each of the vertices  \( s_3, s_4, \ldots, s_n \). We cannot move a pebble to  \( s_1 \). So \( f_{gl}(s_1, K_1, n) \geq (p - 1)n + (p^2 - 2p + 2) \).

Claim: \( f_{gl}(s_1, K_1, n) = (p - 1)n + (p^2 - 2p + 2) \).

**Case i:** If  \( r \) has only one pebble, the other  \((p - 1)(n - 1 + p)\) pebbles are placed in the  \( n - 1 \) vertices of  \( V_2 - \{ s_1 \} \). So at least  \((p - 1)\) of them will have at least  \( p \) pebbles and hence  \( p - 1 \) pebbles can be moved to  \( r \). Now  \( r \) has  \( p \) pebbles and so one pebble can be moved to  \( s_1 \).

**Case ii:** If  \( r \) has  \( K + 2 \) pebbles where  \( K = 0, 1, 2, \ldots, (p - 3) \), the other  \((p - 1)n + (p^2 - 2p - K) \) \( (p \geq 3) \) pebbles are placed in the  \( n - 1 \) vertices of  \( V_2 - \{ s_1 \} \), at least  \( p - K - 1 \) of them will have at least  \( 2p - 1 \) pebbles, and one of them will have at least  \( 3p - 5 \) pebbles, and hence  \( p - K \) pebbles can be moved to  \( r \). Now  \( r \) has  \( p + 2 \) pebbles and so one pebble can be moved to  \( s_1 \). (By placing  \((p - 1)n + (p^2 - 2p - K)\) pebbles onto the  \( n - 1 \) vertices of  \( V_2 - \{ s_1 \} \), one vertex has got  \( 3p - 5 \) pebbles, and  \( K - 1 \) vertices have got  \( 2p - 4 \) pebbles, and  \( p - K \) vertices have got  \( 2p - 3 \) pebbles, and  \( n - p - 1 \) vertices have got  \( p - 1 \) pebbles.)

**Case iii:** If  \( r \) has  \( p \) pebbles, then we are done.
Case iv: Let $r$ have zero pebbles. If there is a vertex in $V_2 - \{s_1\}$ having $p^2$ pebbles then we are done. Otherwise, We claim that there are $p$ vertices of $V_2 - \{s_1\}$ having at least $p$ pebbles. This is because if $p - 1$ vertices has $2p - 1$ pebbles and all others have $p - 1$ pebbles, then the total count comes to only $(p - 1)n + (p^2 - 2p + 1)$ which contradicts the fact that we have placed $(p - 1)n + (p^2 - 2p + 2)$ pebbles. So $p$ pebbles can be moved to $r$ and then one to $s_1$. So $f_{gl}(s_1, K_{1,n}) = (p - 1)n + (p^2 - 2p + 2)$.

Therefore, $f_{gl}(K_{1,n}) = (p - 1)n + (p^2 - 2p + 2)$ since $s_1$ is arbitrary.

We have determined the generalized pebbling number of complete graph $K_n$, path $P_n$, Cycle $C_n$ and $K_{1,n}$. We find the following definitions in [24].

**Definition 4.2.7** (Path Partition of a Rooted Tree). Let $T$ be a tree and $v$ be a vertex of $T$. Let $T_v$ be the rooted tree obtained from $T$ by directing all edges towards $v$, which becomes the root. For a rooted tree $U$, we shall call a vertex $v$ of $U$ a leaf, if it is of indegree zero. We shall call $v$ a parent of $w$, if there is a directed edge from $w$ to $v$, and an ancestor of $w$, if there is a directed path from $w$ to $v$. A path partition of a rooted tree $U$ is a partition of the edges of $U$ such that each set of edges in the partition forms a directed path. We call $v$, a vertex of level $n$, if the directed path from $v$ to the root has $n$ edges; the height of a tree is the maximum level of its vertices.

**Definition 4.2.8** (Maximum Path Partition of a Rooted Tree). Path partitions of a rooted tree $U$ with height $h$ can be formed in the following way. First we consider the sub tree $U'$ of $U$ induced by all leaves of level $h$ and their ancestors and construct a path partition $P'$ of $U'$ such that every path in $P'$ touches a leaf. Then we
Ch. 4: The Generalized Pebbling Number of Some Graphs

let $U''$ be the subtree of $U$ induced by all leaves of level $h$ or $h-1$ and their ancestors and extend $P'$ to a path partition $P''$ of $U''$ by adding paths, which touch the level $h-1$ leaves of $U$. We continue in this manner until we have a path partition $P$ of all of $U$. A path partition constructed in this way is called maximum.

**Definition 4.2.9 (Path size sequence).** The path-size sequence of a path-partition $(P_1, P_2, \ldots, P_n)$ is an $n$-tuple $(a_1, a_2, \ldots, a_n)$, where $a_j$ is the length of $P_j$ (i.e., the number of edges in it). A pebbling sequence need not move pebbles along the edges of a cycle. In particular, it need not move in both directions on an edge. Thus in a tree, all moves should be towards the root.

**Theorem 4.2.10.** Let $U$ be a rooted tree and let $v$ be the root of $U$. If the path size sequence of some maximum path partitions for $U$ is $(a_1, a_2, \ldots, a_n)$.

Then $f_{gl}(v, U) = \sum_{i=1}^{n} p^{a_i} - n + 1$.

**Proof.** Let $P = (P_1, P_2, \ldots, P_n)$ be a maximum path partition for $U$, rooted at $v$, and let $(a_1, a_2, \ldots, a_n)$ be the path-size sequence of the path-partition.

We give a distribution of $\sum_{i=1}^{n} p^{a_i} - 1$ pebbles, that cannot move a pebble to the root vertex $v$. If some path in $P$ starts at a non leaf vertex, then another path ends there, and the two paths combine to produce a path partition majorizing $P$. Hence in $P$, each path begins at a leaf. For each of length $a_i$ in $P$, we put $p^{a_i} - 1$ pebbles on the starting leaf. Now no pebble can reach the end of its path without help from another path. Since this holds for each, independently no path can acquire a pebble from another path. Hence on each path containing $v$, it is not possible for pebbles to reach $v$. So, $f_{gl}(v, U) \geq \sum_{i=1}^{n} p^{a_i} - n + 1$. Let us use induction on the
number of path partitions $n$ to prove that $f_{gl}(v, U) \leq \sum_{i=1}^{n} p^{a_{i}} - n + 1$. When $n = 1$, by Theorem 4.2.4, $f_{gl}(v, U) = p^{a_{1}}$ where $a_{1}$ denotes the number of edges in a path. Therefore $f_{gl}(v, U) = p^{a_{1}} - 1 + 1$. Assume the result is true for $n' < n$ by placing $p^{a_{1}} + p^{a_{2}} + \cdots + p^{a_{n'}} - n + 1$ pebbles on the tree $U$, we show that every distribution with more than $\sum_{i=1}^{n} p^{a_{i}} - n$ pebbles, a pebble can be moved to a root vertex $v$, using a weight function based on $P$. Let $P_{i}$ be the path in $P$ corresponding to length $a_{i}$.

Given a distribution $D$, let $q_{ij}$ be the number of pebbles on $P_{i}$ at a distance $j$ from the leaf. Define weight of the distribution along $P_{i}$ by $W_{i}(D) = \sum_{j=0}^{a_{i}-1} q_{ij} p^{j}$ where $p \geq 2$. Now we claim that there will be at least one $P_{i}$ (with weight distribution $W_{i}(D)$) has at least $p^{a_{i}}$ pebbles. Otherwise, the total number of pebbles placed will be at most $\sum_{i=1}^{n} p^{a_{i}} - n$ which is a contradiction. So using $p^{a_{i}}$ pebbles, we can put a pebble at the end other than the starting leaf of the path $P_{i}$.

**Case (i):** Suppose $i = 1$. Here the end of the path $P_{i}$ coincides with the root $v$, then there is nothing to prove.

**Case (ii):** suppose $i \neq 1$. We have the end of the path $P_{i}$ is different from root $v$. By the given distribution $D$, remaining $(n-1)$ path partitions receive $p^{a_{1}} + p^{a_{2}} + \cdots + p^{a_{i}-1} + p^{a_{i}+1} + \cdots + p^{a_{n}} - (n-1)$ pebbles. With these, addition of one pebble yield $p^{a_{1}} + p^{a_{2}} + \cdots + p^{a_{i}-1} + p^{a_{i}+1} + \cdots + p^{a_{n}} - (n-1) + 1$ pebbles. By induction, we can move a pebble to a root $v$.

We define the generalized $p$-pebbling property as follows:

**Definition 4.2.11.** If $p^{n+1} - r + 1$ pebbles are assigned to vertices of an $n$-cube, while $r$ vertices have at least one pebble, then $p$ pebbles can be moved to $v$. 
The following lemma describes the number of pebbles that we can transfer from one copy of $H$ to an adjacent copy of $H$ in $G \times H$. It is also called generalized transfer lemma.

**Lemma 4.2.12** (Generalized transfer lemma). Let $(x_i, x_j)$ be an edge in $G$. Suppose that in $G \times H$, we have $a_i$ pebbles on $\{x_i\} \times H$ and $b_i$ of these vertices have $mp + s$, where $1 \leq s \leq p - 1, m = 0, 1, 2, \ldots$ and $p \geq 2$ pebbles. If $b_i \leq k \leq a_i$ and if $k$ and $a_i$ have the same parity, then at most $(p - 1)k$ pebbles can be retained on $\{x_i\} \times H$, while moving at least $\frac{a_i - (p - 1)k}{p}$ pebbles onto $\{x_j\} \times H$. If $k$ and $a_i$ have opposite parity, we must leave at most $(p - 1)(k + 1)$ pebbles on $\{x_i\} \times H$, we can move at least $\frac{a_i - (p - 1)b_i}{p}$ pebbles onto $\{x_j\} \times H$.

**Proof.** For every $p$ pebbles on a vertex in $\{x_i\} \times H$ we can move one pebble to its neighbour in $\{x_j\} \times H$. If we ignore at most $(p - 1)$ pebbles for each vertex with an $(mp + s)$ pebbles we can move $\frac{a_i - (p - 1)b_i}{p}$ pebbles to $\{x_j\} \times H$ from $\{x_i\} \times H$.

The following theorem gives the generalized pebbling number of an $n$-cube, and also it proves that $n$-cube satisfies the generalized $p$-pebbling property.

**Theorem 4.2.13.** In an $n$-cube with a specified vertex, the following are true:

1. If $p^n$ pebbles are assigned to vertices of the $n$-cube, one pebble can be moved to $v$.

2. If $p^{n+1} - b + 1$ pebbles are assigned to vertices of an $n$-cube, while $b$ vertices have at least one pebble, then $p$ pebbles can be moved to $v$. 
Ch. 4: The Generalized Pebbling Number of Some Graphs

Proof. If the vertices of $K_2$ are labeled as $x_1, x_2$ then for any distribution of pebbles on $K_2 \times Q_{n-1}$, we write $a_i$ the number of pebbles on $\{x_i\} \times Q_{n-1}$ and $b_i$ for the number of vertices having at least one pebble. The $n$-cube can be partitioned into two $(n-1)$ cubes, say, $\{x_1\} \times Q_{n-1}$ and $\{x_2\} \times Q_{n-1}$. Assume $v = (x_1, y) \in \{x_1\} \times Q_{n-1}$ for some $y$ is our target vertex. Let $v' = (x_2, y) \in \{x_2\} \times Q_{n-1}$ be adjacent to $v$. Let $a$ denote the total number of pebbles so that $a = a_1 + a_2$.

We will prove (1) by induction an $n$. For $Q_0$, we have one vertex. We need exactly one pebble for $Q_0$ to pebble the target vertex. Hence, $f_{gl}(Q_0) = 1 = p^0$. Suppose it is true for $n' < n$. Suppose there are $a \geq p^n$ pebbles are assigned to vertices of the $n$-cube, where $p \geq 2$. Let us place $p^n - 1$ pebbles at a vertex, which is at a distance $n$ from the target vertex $v = (x_1, y)$. we cannot move a pebble to $v$. So $f_{gl}(v, Q_n) > p^n - 1 \geq p^n$.

Next we have to prove that $f_{gl}(v, Q_n) \leq p^n$. If $a_1 \geq p^n - 1$, then by induction in $\{x_1\} \times Q_{n-1}$, one pebble can be moved to $v$. Assume $a_1 < p^n - 1$. Consider the following two cases:

Case A: Assume $a_1 < b_2$. Consider $\{x_2\} \times Q_{n-1}$.

Since $a_2 = a - a_1 > p^n - p_2$, using (2) in $Q_{n-1}$ (by induction) $p$ pebbles can be moved to $v'$ and hence one pebble can be moved to $v$.

Case B: Assume $b_2 \leq a_1$.

Here we would like to try to move as many pebbles as possible from $\{x_2\} \times Q_{n-1}$ to $\{x_1\} \times Q_{n-1}$ to try to get enough pebbles on $\{x_1\} \times Q_{n-1}$ to satisfy the inductive hypothesis, so that one pebble can be moved to $v$. 
In \( \{x_1\} \times Q_{n-1} \), we have

\[
a_1 + a_2 - \frac{(p-1)b_2}{p} \geq a_1 + a_2 - (p-1)a_1
\]

By Lemma 4.2.12

\[
\geq \frac{a_1 + a_2}{p}
\]

\[
= \frac{a}{p}
\]

\[
\geq p^{n-1} \text{ pebbles}
\]

By induction, we can move one pebble to \( v \). The only distribution from which we cannot pebble the target satisfy the inequality

\[
a_1 + a_2 - \frac{(p-1)b_2}{p} < p^{n-1}
\]  

(4.1)

\[
\frac{a_2 + b_2}{p} \leq p^{n-1}
\]

(4.2)

Multiplying (4.2) by \( (p-1) \) we get

\[
(p-1) \left( \frac{a_2 + b_2}{p} \right) \leq (p-1)p^{n-1}
\]

(4.3)

Adding (4.1) and (4.3) we get \( a_1 + a_2 < p^n \), i.e., \( a < p^n \).

So in any distribution with less than \( p^n \) pebbles we cannot pebble any target.

(2). We will prove by induction on \( n \). Suppose we have \( p-b \) pebbles on \( Q_O \). Since \( b \leq 1 \), we have at least \( p \) pebbles on \( Q_O \) and we are done. Suppose it is true for \( n' < n \).

Assume \( v = (x_1, y) \in \{x_1\} \times Q_{n-1} \) for some \( y \) is our target vertex. Let \( v' = (x_2, y) \in \{x_2\} \times Q_{n-1} \) be adjacent to \( v \). Recall that a denote the total number of pebbles so that each \( \{x_i\} \times Q_{n-1} \) contains \( a_i \) pebbles for \( i = 1, 2 \) with \( b_i \) vertices having at least one pebble. Suppose \( a = a_1 + a_2 = p^{n+1} - b + 1 \) pebbles are assigned to vertices of the \( n \)-cube. We consider the following three possibilities:
Case (A) Suppose \( a_1 > p^n - b_1 \). By induction in \( \{x_1\} \times Q_{n-1} \), \( p \) pebbles can be moved to \( v \).

Case (B): \( p^{n-1} \leq a_1 < p^n - b_1 \). Since \( a_1 \geq p^{n-1} \) one pebble can be moved to \( v \) in \( \{x_1\} \times Q_{n-1} \).

Since \( a_2 = a - a_1 \)
\[
\geq a - (p^n - b_1) \text{ since } a_1 < p^n - b_1
\]
\[
> (p^{n+1} - b) - (p^n - b_1) \text{ by our hypothesis } a > p^{n+1} - b
\]
\[
= p^{n+1} - b_1 - b_2 - p^n + b_1
\]
\[
= (p - 1)p^n - b_2
\]

Therefore, we have \( a_2 > (p - 1)p^n - b_2 \) pebbles in \( \{x_2\} \times Q_{n-1} \) and by induction we can move \((p - 1)\) \( p \) pebbles to \( v' \) by applying generalized pebbling steps in \( \{x_2\} \times Q_{n-1} \).

Then we can move \((p - 1)\) pebbles from \( v' \) to \( v \). Combined with the pebble placed on \( v \) via moves within \( \{x_1\} \times Q_{n-1} \) we have \( p \) pebbles on \( v \) and we are done.

Case (C): \( a_1 < p^{n-1} \). In the final case we will show that we can move enough pebbles from \( \{x_2\} \times Q_{n-1} \) to \( \{x_1\} \times Q_{n-1} \) to move one pebble to \( v \) by applying pebbling moves within \( \{x_1\} \times Q_{n-1} \) and at the same time leave enough pebbles on \( \{x_2\} \times Q_{n-1} \), so that we can move \((p - 1)\) \( p \) pebbles to \( v' \) by applying pebbling moves within \( \{x_2\} \times Q_{n-1} \). By (1) we know that if we have \( p^{n-1} \) pebbles in \( \{x_1\} \times Q_{n-1} \), then we can move one pebble to \( v \). So we need to move \( p^{n-1} - a_1 \) pebbles from \( \{x_2\} \times Q_{n-1} \) to \( \{x_1\} \times Q_{n-1} \). This corresponds to removing \( p(p^{n-1} - a_1) \) pebbles from \( \{x_2\} \times Q_{n-1} \).
This leaves us with at least
\[ a_2 - p(p^{n-1} - a_1) \]
pebbles left in \( \{x_2\} \times Q_{n-1} \) and
\[
\begin{align*}
  a_2 - p(p^{n-1} - a_1) &= a_2 + pa_1 - p^n \\
  &= a_2 + a_1 + (p - 1)a_1 - p^n \\
  &= a + (p - 1)a_1 - p^n \\
  > p^{n+1} - b + (p - 1)a_1 - p^n & \text{ by hypothesis } a > p^{n+1} - b \\
  &= p^{n+1} - b_1 - b_2 + (p - 1)a_1 - p^n \\
  &= (p - 1)p^n - b_1 - b_2 + (p - 1)a_1 \\
  \geq (p - 1)p^n - b_2 & \text{ since } a_1 \geq b_1.
\end{align*}
\]

Note that \( a_1 \geq b_1 \) because every vertex with \( mp + s \) pebbles corresponds to at least one pebble. Also we have more than \( (p - 1)p^n - b_2 \) pebbles on \( \{x_2\} \times Q_{n-1} \), by induction we can move \( (p - 1)p \) pebbles to \( v' \) by pebbling within \( \{x_2\} \times Q_{n-1} \) and then \( (p - 1) \) pebbles to \( v \). At the same time, we can apply pebbling moves using the \( p^{n-1} \) pebbles in \( \{x_1\} \times Q_{n-1} \) and place one additional pebble on \( v \).

**Definition 4.2.14.** We define the wheel graph denoted by \( W_n \) to be the graph with
\[
V(W_n) = \{h, v_1, v_2, \ldots, v_n\}
\]
where \( h \) is called the hub of \( W_n \) and \( E(W_n) = E(C_n) \cup \{hv_1, hv_2, \ldots, hv_n\} \) where \( C_n \) denotes the cycle graph on \( n \) vertices.

**Theorem 4.2.15.** For \( n \geq 4 \), the generalized pebbling number of the wheel graph \( W_n \) is
\[
f_{gl}(W_n) = (p - 1)n + (p^2 - 2p + 1) \text{ where } p \geq 2.
\]

**Proof.** By Theorem 4.2.2, \( f_{gl}(h, W_n) = p + (p - 1)(n - 1) \). Let us now find the generalized pebbling number of \( v_1 \). If we place \( p - 2 \) pebbles at \( v_n \), \( (p^2 - 1) \) pebbles at \( v_{[n/2]} \) and \( (p - 1) \) pebbles at \( W_n - \{v_1, v_n, v_{[n/2]}\} \) then we cannot move
a pebble to \( v_1 \).

So \( f_{gl}(v_1, w_n) > (p^2 - 1) + (p - 1)(n - 3) + (p - 2) \)

\[ \geq (p - 1)n + (p^2 - 2p + 1) \]

Let us now prove that \((p - 1)n + (p^2 - 2p + 1)\) pebbles are sufficient to put a pebble on \( v_1 \). Assume that \( v_1 \) has zero pebbles. Now \( v_1 \) is adjacent with \( h, v_2, v_n \). Hence in the given distribution, any one of \( h, v_2, v_n \) receives \( p \) pebbles, then a pebble can be moved to \( v_1 \). Also any one of the vertices \( \{v_3, v_4, \ldots, v_{n-1}\} \) receives at least \( p^2 \) pebbles then a pebble can be moved to \( v_1 \) through \( h \). Let \( q_i = pm_i + r_i \) where \( 0 \leq r_i \leq p - 1 \) be the number of pebbles on \( v_i \) for \( i = 2 \) to \( n \). Let \( a \) be the number of pebbles on \( h \). Suppose \( a \geq p \), then from \( h \), we can move a pebble to \( v_1 \). Suppose \( a < p \), then let \( b = p - a > 0 \). Let us transfer the pebbles from \( v_i \) \((i = 2 \) to \( n)\) to \( h \).

Let \( m = \sum_{i=2}^{n} m_i \). After this transfer, number of pebbles on \( h \) is \( b + m \). If \( b + m \geq p \), then we can put a pebble on \( v_1 \). So we assume that \( b + m < p \). Therefore \( p - b - m > 0 \). Let \( s = p - b - m \). In order to place \( p - b - m \) pebbles on \( h \) we are in need of \( p(p - b - m) \) pebbles on \( C_n \).

Consider

\[ (p - 1)n + (p^2 - 2p + 1) - b - pm - p^2 + pb + pm = (p - 1)n + b(p - 1) + (1 - 2p). \]

Since \( p \geq 2, n \geq 4, b > 0 \) we get \((p - 1)n + b(p - 1) + (1 - 2p) \geq 2\).

From \( h \), a pebble can be moved to \( v_1 \). If \( h \) has zero pebbles, \( v_2 \) and \( v_n \) have at most \((p - 2)\) pebbles each and no vertex of \( \{v_3, v_4, \ldots, v_{n-1}\} \) has \( p^2 \) pebbles and assume \( n - 3 \geq p \), then there will be at least \( p \) pebbles each, then we can move \( p \) pebbles to \( h \) and so we are done.
Let us assume $n - 3 < p$. Consider

$$(p - 1)n + (p^2 - 2p + 2) - 2(p - 2) = (p - 1)n + (p^2 - 4p + 6).$$

Now, $p^2 + (n - 4)p - (n - 6)$ pebbles are distributed onto $C_n$. Using $p^2$ pebbles we can move a pebble to $v_1$.

**Theorem 4.2.16.** The generalized pebbling number of the fan graph $F_n$ is

$$f_{gl}(F_n) = (p - 1)n + (p^2 - 2p + 1).$$

**Proof.** Fan graph $F_n$ is the spanning sub graph of $W_n$, so $f_{gl}(F_n) \leq f_{gl}(W_n)$.

Hence $f_{gl}(F_n) \leq (p - 1)n + (p^2 - 2p + 1)$.

Suppose that, there are $(p - 1)n + (p^2 - 2p + 1)$ pebbles distributed on the vertices of $F_n$ where $F_n$ is the fan graph with vertices $v_1, v_2, \ldots, v_n, v_{n+1}$ in order. First, let the target vertex be $v_{n+1}$. By Theorem 4.2.2 $f_{gl}(v_{n+1}, F_n) = p + (p - 1)(n - 1)$, if $v_{n+1}$ has zero pebbles then there exists some $v_i$ where $i \in \{1, 2, 3, \ldots, n\}$ with at least $p$ pebbles, so we can move one pebble to $v_{n+1}$ from $v_i$.

Next supposing the target vertex is $v_k$ and assume that $v_k$ has zero pebbles where $k \in \{1, 2, 3, \ldots, n\}$. Suppose $v_{n+1}$ receives at least $p$ pebbles, then a pebble can be moved to $v_k$ or if any one of the vertices of $v_i$ where $i \in \{1, 2, \ldots, n\}$ and $i \neq k$ receives $p^2$ pebbles then from $v_i$ a pebble can be moved to $v_k$ through $v_{n+1}$.

Suppose $v_{n+1}$ receives $m$ where $1 \leq m \leq p - 1$ pebbles and the vertices of $P_n - \{v_k\}$ receive at the most $p^2 - 1$ pebbles, using $p(p - 2)$ pebbles, we can move $(p - 2)$ pebbles to $v_{n+1}$, and the remaining $(p - 1)n$ pebbles are also distributed onto the vertices of $P_n$. Hence there exists a vertex $w$ with at least $p$ pebbles, so a pebble can be moved to $v_{n+1}$ from $w$. Now $v_{n+1}$ receives at least $p$ pebbles, so
a pebble can be moved to \(v_k\) from \(v_{n+1}\). Suppose \(v_{n+1}\) has zero pebbles and all the vertices of \(P_n\) except \(v_k\) receive at the most \(p^2 - 1\) pebbles, then there must be at least one vertex \(v_j\) with at least \(p\) pebbles. If in addition, there are at least two vertices \(v_j\) and \(v_{\ell}\) with \(m\) pebbles in which \(p \leq m \leq p^2 - 1\), then we can move at least \(\lfloor p/2 \rfloor\) pebbles from \(v_{\ell}\) to \(v_{n+1}\). So, \(p\) pebbles can be moved to \(v_{n+1}\). Hence a pebble can be moved to \(v_k\). Otherwise, there is only one vertex \(v_j\) with at least \(p\) pebbles. Therefore all \(v_i\) in which \(1 \leq i \leq n\) and \(i \neq j, k\) have \((p - 1)\) pebbles. Suppose \(j < k\), then using the sequence of pebbling moves \(v_j - v_{j+1} - v_{j+2} - \cdots - v_k\) we can move a pebble to \(v_k\), otherwise using the sequence of moves \(v_j - v_{j-1} - \cdots - v_k\), a pebble can be moved to \(v_k\). Hence in all the cases \(f_{gl}(v_k, F_n) \leq (p - 1)n + (p^2 - 2p + 1)\).

**Theorem 4.2.17.** For \(G = K_{s_1, s_2, \ldots, s_r}^*\) the generalized pebbling number is given by

\[
f_{gl}(G) = \begin{cases} 
p^2 + (p - 1)(s_1 - 2), & \text{if } p \geq n - s_1 \\
p + (p - 1)(n - 2), & \text{if } p < n - s_1.\end{cases}
\]

**Proof.** **Case 1:** Assume \(s_1 < n - p\).

Let the target vertex be \(v\) of \(C_i\) for some \(i = 1, 2, \ldots, r\). Without loss of generality, we assume that \(v\) has zero pebbles on it. If we place \((p - 1)\) pebbles each on \((n - 1)\) vertices of \(G\) other than \(v\), a pebble cannot be moved to \(v\).

So \(f_{gl}(v, G) \geq p + (p - 1)(n - 2)\).

Let us place \(p + (p - 1)(n - 2)\) pebbles on the vertices of \(G\). If there is a vertex \(w\) of \(C_j\) \((j \neq i)\) with at least \(p\) pebbles then a pebble can be moved to \(v\). Otherwise, there is a vertex \(w_i\) of \(C_k\) \((k \neq i)\) with at most \((p - 1)\) pebbles then at least \(p + (p - 1)(n - 3)\) pebbles are distributed onto each of \(n - p - 1\) vertices of
Since \( s_i \leq s_1 < n - p \) then using \((p - 1)p\) pebbles we can move at the most \((p - 1)\) pebbles to \( w_1 \). So \( w_1 \) has at least \( p \) pebbles, then from \( w_1 \) a pebble can be moved to \( v \). Otherwise every vertex of \( G - C_i \) contains zero pebbles on it. Then either there exists a vertex \( w_2 \) of \( C_i \) with at least \( p^2 \) pebbles or all the vertices of \( C_i - \{v\} \) contains at the most \( p^2 - 1 \) pebbles. So \( p \) pebbles can be moved to a vertex \( w_3 \) of \( C_j \) \((j \neq i)\). From \( w_3 \) a pebble can be moved to the vertex \( v \) of \( C_i \).

Hence in all cases \( f_{gl}(v, G) \geq p + (p - 1)(n - 2) \).

Since \( v \) is arbitrary,

\[ f_{gl}(G) \leq p + (p - 1)(n - 2). \]

**Case ii:** Assume \( n - s_1 \leq p \).

Let us choose the vertex class \( C_1 \). Let \( v \in C_1 \) be our target vertex. Without loss of generality assume that vertex \( v \) has zero pebbles on it. Let us place \( p^2 - 1 \) pebbles on one of the \( s_1 \) vertices of \( C_1 \), and place \((p - 1)\) pebbles on each of the remaining \( s_1 - 2 \) vertices of \( C_1 \). Then \((p - 1)\) pebbles can be moved to the vertex \( w \) of \( C_k \) where \( k \neq 1 \). Now all the pebbled vertices in \( G \) receive \((p - 1)\) pebbles. Hence pebbling move is impossible. So

\[ f_{gl}(v, G) > (p^2 - 1) + (p - 1)(s_1 - 2) \geq p^2 + (p - 1)(s_1 - 2). \]

Suppose \( p^2 + (p - 1)(s_1 - 2) \) pebbles are placed on the vertices of \( G \). Let the target vertex be \( v \) of \( C_k \).

If there is a vertex in some \( C_j \) \((j \neq k)\) with at least \( p \) pebbles then a pebble can be placed on \( v \) using \( p \) pebbles.

If not, then every vertex of \( G - C_k \) will contain either zero or at most \((p - 1)\) pebbles on it. If there is a vertex say \( w \) in some \( C_j \) \((j \neq k)\) with a pebble on it we use \( p \) pebbles from a vertex of \( C_k \) to put a pebble on \( w \) then from the remaining
Ch. 4: The Generalized Pebbling Number of Some Graphs

$p(p - 1) + (p - 1)(s_1 - 2) - 1$ vertices we can put $(p - 1)$ pebbles on $w$ and from $w$ a pebble can be moved to $v$.

Otherwise every vertex of $G - C_k$ will have zero pebbles on it. Then all the $p^2 + (p - 1)(s_1 - 2)$ pebbles are distributed on the vertices of $C_k$. Then using $p^2$ pebbles a pebble can be moved to the vertex $v$ of $C_k$.

Hence $f_{gl}(v, G) \leq (p - 1)(s_1 - 2) + p^2$.

Therefore $f_{gl}(G) \leq p^2 + (p - 1)(s_1 - 2)$, and so the case is complete.

4.3 Computation of generalized $t$-pebbling number

We define the generalized $t$-pebbling number of any connected graph $G$ as follows.

**Definition 4.3.1.** The generalized $t$-pebbling number of a vertex $v$ in a graph $G$ is the smallest number $f_{gl}(v, G)$ with the property that from every placement of $f_{gl}(v, G)$ pebbles on $G$, it is possible to move $t$ pebbles to $v$ by a sequence of pebbling moves, where a pebbling move consists of the removal of $p$ pebbles from a vertex and the placement of one of these pebbles on an adjacent vertex.

The generalized $t$-pebbling number of the graph $G$, denoted by $f_{gl}(G)$, is the maximum $f_{gl}(v, G)$ over all vertices $v$ of $G$.

**Theorem 4.3.2.** Let $P_n : v_0, v_1, v_2, \ldots, v_n$ be a path of length $n$. Then $f_{gl}(P_n) = tp^n$.

**Proof.** Place $t(p^n) - 1$ pebbles at one end of the path, we cannot move $t$ pebbles to the other end of the path. So, $f_{gl}(P_n) \geq t$.

We will use induction on $t$ to prove that $f_{gl}(P_n) \leq tp^n$. When $t = 1$, theorem is true by Theorem 4.2.4. Assume $t > 2$. Then there are at least $2p^n$ pebbles on the
path $P_n$. Using at most $p^n$ pebbles we can move one pebble to the target vertex. With the remaining $(t - 1)p^n$ pebbles we can move $(t - 1)$ pebbles to the target vertex completes the proof.

Theorem 4.3.3. Let $U$ be a rooted tree and let $v$ be the root of $U$. Let $(a_1, a_2, \ldots, a_n)$ be the path-size sequence for some maximum path partition for $U$. Without loss of generality, $a_1$ can be taken to be $h$, where $h$ is the height of the tree. Then

$$f_{glt}(v, U) = tp^h + \sum_{i=2}^{n} p^{ai} - n + 1.$$ 

Proof. Let $(P_1, P_2, \ldots, P_n)$ be a maximum path partition for $U$. Then $a_i$ edges in $P_i$ will touch $a_i + 1$ vertices. Let $Q_i \subseteq V(U)$ contain the $a_i$ of these vertices away from $v$ and let $v_i$ be the vertex in $Q_i$ farthest from $v$. The $Q_i$’s are disjoint and do not contain $v$.

Place $tp^h - 1$ pebbles at $v_1$ and put $p^{ai} - 1$ pebbles on $v_i$ for $i = 2$ to $n$. With this configuration, we cannot move $t$ pebbles to $v$, so $f_{glt}(v, U) \geq tp^h + \sum_{i=2}^{n} p^{ai} - n + 1$.

We will prove the result $f_{glt}(v, U) \leq tp^h + \sum_{i=2}^{n} p^{ai} - n + 1$ by using induction on the number of path partitions.

When $n = 1$, $U$ becomes a path and by Theorem 4.3.2,

$$f_{glt}(v, U) = tp^h = tp^h - 1 + 1.$$ 

Assume the result is true for $1 \leq n' < n$. To prove that the result is true when the number of path partitions is $n$, by placing $tp^h + \sum_{i=2}^{n} p^{ai} - n + 1$ pebbles on $v$, except the root vertex $v$. At least one of the paths $P_i$ receives at least $p^{ai}$ pebbles. Using $p^{ai}$ pebbles, we can bring one pebble to the end of the path $P_i$. Remaining $(n - 1)$ paths receive $tp^h + p^{a_2} + p^{a_3} + \cdots + p^{a_{i-1}} + p^{a_{i+1}} + \cdots + p^{a_n} - (n - 1) + 1$ pebbles.
By induction, we can bring $t$ pebbles to a target vertex $v$.

So $f_{gh}(v, U) \leq tp^n + \sum_{i=2}^{n} p^{n_i} - n + 1$

\[ f_{gh}(C_n) = \begin{cases} 
  tp^{n/2}, & \text{if } n \text{ is even} \\
  1 + (t-1)p^{[n/2]} + 2\left[ \frac{p}{p+1}\right](p^{[n/2]} - 1), & \text{if } n \text{ is odd} 
\end{cases} \]

**Proof.** Let $C_n = (v_0, v_1, v_2, \ldots, v_{n-1})$ be a cycle of length $n$.

**Case i:** $n$ is even. Let $v_0$ be the target vertex. By placing $t(p^{n/2}) - 1$ pebbles at $v_{n/2}$ which is the farthest vertex from $v_0$. We cannot move $t$ pebbles to $v_0$. So $f_{gh}(v_0, C_n) \geq t(p^{n/2})$.

**Case ii:** $n$ is odd. Let $k = \left\lceil \left( \frac{p}{p+1}\right)(p^{[n/2]} - 1) \right\rceil$, $m = (t-1)p^{[n/2]}$ and $N = m + 2k$. Place $m + k$ pebbles at $v_{[n/2]}$, and place $k$ pebbles at $v_{[n/2]+1}$. If we move pebbles from $v_{[n/2]}$ to $v_{[n/2]+1}$ we can move $\left\lfloor \frac{m+k}{p} \right\rfloor$ pebbles to $v_{[n/2]+1}$ and so the vertex $v_{[n/2]+1}$ has a total of $\left\lfloor \frac{m+k}{p} \right\rfloor + k$ pebbles which is equal to $(p+t-1)p^{[n/2]} - 1 - 1$ pebbles. But the vertex $v_{[n/2]+1}$ is at a distance $\lfloor n/2 \rfloor$ from $v_0$ and so we cannot move $t$ pebbles to $v_0$. If we move pebbles from $v_{[n/2]+1}$ to $v_{[n/2]}$ we can move $\left\lfloor \frac{k}{p} \right\rfloor$ pebbles to $v_{[n/2]}$ and so $v_{[n/2]}$ has $tp^{[n/2]} - 1$ pebbles and so we cannot move $t$ pebbles to $v_0$. So $f_{gh}(v_0, C_n) \geq 1 + N$.

We use induction on $t$ to prove these numbers of pebbles are sufficient, where the case $t = 1$ is given by Theorem 4.2.5. Consider the paths

$P_1 = (v_0, v_1, v_2, \ldots, v_{[n/2]})$ and $P_2 = (v_{[n/2]+1}, v_{[n/2]+2}, \ldots, v_{n-1}, v_0)$.

If $t > 1$, (regardless of whether $n$ is even or odd), there are at least $2p^{n/2}$ pebbles on the graph, so we may assume that at least $p^{[n/2]}$ pebbles are on one of the paths $P_1$ and $P_2$. Therefore, we can move one of these pebbles to $v_0$. By induction, the
Ch. 4: The Generalized Pebbling Number of Some Graphs

remaining \( f_{glt}(C_n) + p^{\lfloor n/2 \rfloor} (t - 2) \) pebbles suffice to put \((t-1)\) additional pebbles on \(v_0\).

\[ \text{Theorem 4.3.5.} \] Let \( G \) be a connected graph on \( n \) vertices, where \( n \geq 3 \). Let there be a vertex \( v \) such that \( d(v) = n - 1 \), the generalized \( t \)-pebbling number \( f_{glt}(v, G) = pt + (p - 1)(n - 2) \).

\[ \text{Proof.} \] Place \( pt - 1 \) pebbles at any vertex other than \( v \) and place \((p - 1)\) pebbles at every other vertex except on \( v \), then \( t \) pebbles cannot be moved to \( v \). So, \( f_{glt}(v, G) \geq pt + (p - 1)(n - 2) \).

We use induction on \( t \), to prove that \( f_{glt}(v, G) \leq pt + (p - 1)(n - 2) \). Place zero pebbles on \( v \). For \( t = 1 \), the result is true by Theorem 4.2.2. For \( t > 1 \), there are at least \( 2p + (p - 1)(n - 2) \) pebbles on \( G \). Using \((p - 1)n - (p - 2)\) pebbles, we can put a pebble on \( v \), since \( d(v) = n - 1 \). By induction, we can use the remaining \( p(t - 1) + (p - 1)(n - 2) \) pebbles to put \((t - 1)\) additional pebbles on \( v \). So \( f_{glt}(v, G) \leq pt + (p - 1)(n - 2) \).

\[ \text{Theorem 4.3.6.} \] Let \( K_n \) be the complete graph on \( n \) vertices, where \( n \geq 3 \). Then \( f_{glt}(K_n) = pt + (p - 1)(n - 2) \).

\[ \text{Proof.} \] Follows from Theorem 4.3.5.

\[ \text{Theorem 4.3.7.} \] Let \( K_{1,n} \) be an \( n \)-star where \( n > 1 \), then

\[ f_{glt}(K_{1,n}) = p^2 t + (p - 1)(n - 2), \quad \text{where} \quad p \geq 2. \]

\[ \text{Proof.} \] Let \( V(K_{1,n}) = V_1 \cup V_2 \), where \( V_1 = \{r\} \) and \( V_2 = \{s_1, s_2, \ldots, s_n\} \). By Theorem 4.3.6, \( f_{glt}(r, K_{1,n}) = pt + (p - 1)(n - 1) \).
We place zero pebbles on \( r \) and \( s_1 \). Place \( p^2 t - 1 \) pebbles at \( s_2 \) and \((p - 1)\) pebbles at each of the vertices \( s_3, s_4, \ldots, s_n \). We cannot move \( t \) pebbles to \( s_1 \). So 
\[
\text{fglt}(s_1, K_{1,n}) \geq p^2 t + (p - 1)(n - 2).
\]

We will use induction on \( t \) to prove that \( p^2 t + (p - 1)(n - 2) \) pebbles are sufficient to put \( t \) pebbles on \( s_1 \). Place zero pebbles on \( s_1 \).

For \( t = 1 \), the result is true by Theorem 4.2.6.

For \( t > 1 \), there are at least \( 2p^2 + (n - 2)(p - 1) \) pebbles on \( K_{1,n} \).

**Case i:** If \( r \) has at least \( p \) pebbles, then we can put a pebble on \( s_1 \) using \( p \) pebbles of \( r \). From the remaining \( p^2 t - p + (n - 2)(p - 1) \) pebbles, \( p^2(t - 1) + (p - l)(n - 2) \) pebbles will be sufficient to put \( (t - 1) \) additional pebbles on \( s_1 \).

**Case ii:** If there is a vertex in \( V_2 - \{s_1\} \) with at least \( p^2 \) pebbles or there are \( p \) vertices in \( V_2 - \{s_1\} \) with at least \( p \) pebbles each, then by using \( p^2 \) pebbles, we can put \( p \) pebbles on \( r \) and so we can move a pebble to \( s_1 \). By induction, the remaining \( p^2(t - 1) + (n - 2)(p - 1) \) pebbles will be sufficient to put \( (t - 1) \) additional pebbles on \( s_1 \). These are the only two cases which arise. Otherwise, the total number of pebbles placed will be at most \( 2p^2 + (n - 2)(p - 1) \), which contradicts the fact that there are at least \( 2p^2 + (n - 2)(p - 1) \) pebbles on the graph. So, \( \text{fglt}(s_1, K_{1,n}) \leq p^2 t + (n - 2)(p - 1) \).

So, \( \text{fglt}(s_1, K_{1,n}) = p^2 t + (n - 2)(p - 1) \).

Hence, \( \text{fglt}(K_{1,n}) = p^2 t + (n - 2)(p - 1) \) (since \( s_1 \) is arbitrary).

**Theorem 4.3.8.** Let \( K_1 = \{h\} \). Let \( C_n = \{v_1, v_2, \ldots, v_n\} \) be a cycle of length \( n \). Then the generalized \( t \)-pebbling number of the wheel graph \( W_n \) is
\[
\text{fglt}(W_n) = p^2(t - 1) + (p - 1)n + (p^2 - 2p + 1).
\]
Ch. 4: The Generalized Pebbling Number of Some Graphs

**Proof.** By Theorem 4.3.6, \( f_{glt}(h, W_n) = pt + (p - 1)(n - 1) \). Let us now find the generalized \( t \)-pebbling number of \( v_1 \). Assume that \( v_1 \) has zero pebbles. Let us place \((p^2t - 1)\) pebbles at \( v_{\lceil n/2 \rceil} \), \((p-2)\) pebbles at \( v_n \) and \((p - 1)\) pebbles at each of \( W_n - \{v_1, v_{\lceil n/2 \rceil}, v_n\} \). Then \( t \) pebbles cannot be moved to \( v_1 \).

So \( f_{glt}(v_1, W_n) \geq p^2(t - 1) + (p - 1)n + (p^2 - 2p + 1) \).

Let us use induction on \( t \) to prove that

\[
f_{glt}(v_1, W_n) \geq p^2(t - 1) + (p - 1)n + (p^2 - 2p + 1)\]

For \( t = 1 \), the result is true by Theorem 4.2.15.

By distributing \( p^2(m - 2) + (p - 1)n + (p^2 - 2p + 1) \) pebbles on \( W_n - \{v_1\} \), then we can move \((m - 1)\) pebbles to the target vertex \( v_1 \).

i.e., \( f_{gl}(m - 1)(W_n) = p^2(m - 2) + (p - 1)n + (p^2 - 2p + 1) \). Suppose \( p^2(m - 1) + (p - 1)n + (p^2 - 2p + 1) \) pebbles are distributed onto the vertices of \( W_n - \{v_1\} \). Let the target vertex be \( v_1 \) of \( C_n \).

If there is a vertex in \( C_n \) with at least \( p^2 \) pebbles, then a pebble can be moved to \( v_1 \), using only \( p^2 \) pebbles through \( h \). The remaining \( p^2(m - 2) + (p - 1)n + (p^2 - 2p + 1) \) pebbles are sufficient to put \((m - 1)\) additional pebbles on \( v_1 \) by using induction. Otherwise any one of the vertices of \( W_n - \{v_1\} \) say \( v_{\lceil n/2 \rceil} \) receive at least \( p \) pebbles and each of the vertices \( W_n - \{v_1, v_{\lceil n/2 \rceil}\} \) receive \( p-1 \) pebbles then from \( v_{\lceil n/2 \rceil} \), using a sequence of pebbling moves \( v_{\lceil n/2 \rceil}, v_{\lceil n/2 \rceil - 1}, \ldots, v_1 \), we can move a pebble to \( v_1 \). Remaining \( p^2+(p-1) \) \((n - \lceil n/2 \rceil + 2) + (p^2 - 3p + 1) \) > 0. So by induction, \((m - 1)\) pebbles can be moved to \( v_1 \). Hence in all cases \( f_{glm}(v_1, W_n) \geq p^2(m - 1) + (p - 1)n + (p^2 - 2p + 1) \).

Therefore \( f_{glt}(W_n) = p^2(m - 1) + (p - 1)n + (p^2 - 2p + 1) \). ■
Theorem 4.3.9. For $G = K^s_{s_1 s_2 \ldots s_r}$, the generalized $t$-pebbling number for a complete $r$-partite graph $G$ is given by

$$f_{glt}(G) = \begin{cases} 
pt + (p-1)(n-2), & \text{if } pt < n - s_1 \\
pt + (p-1)(s_1 - 2), & \text{if } pt \geq n - s_1.
\end{cases}$$

Proof. Case 1: Assume $pt < n - s_1$.

Let us place $pt + (p-1)(n-2) - 1$ pebbles on the vertices of $G - \{v\}$ as follows. Let us choose $(t-1)$ vertices and we place $p + (p-1)$ pebbles on each of the $(t-1)$ vertices and we place $(p-1)$ pebbles each on the remaining vertices. Clearly $t$ pebbles cannot be moved to $v$. Hence

$$f_{glt}(v,G) > (t-1)[p + (p-1)] + (p-1)(n-t)$$

$$= pt + (p-1)(n-2) - 1$$

$$\geq pt + (p-1)(n-2).$$

Next we will use induction to show that $pt + (p-1)(n-2)$ pebbles are sufficient to move $t$ pebbles to any desired vertex. For $t = 1$ result is true by Theorem 4.2.17. Suppose $t > s_1$, and $pt + (p-1)(n-2)$ pebbles are placed on the vertices of $G$. Let the target vertex be $v$ of $C_k$ for some $k = 1, 2, \ldots, n$. If there is a vertex $w$ of $C_j$ ($(j \neq k)$ with at least $p$ pebbles then a pebble can be placed on $v$.

The remaining $p(t-1) + (p-1)(n-2)$ pebbles are sufficient to put $(t-1)$ additional pebbles on $v$ by induction. If not then every vertex of $G - C_k$ will have at most $(p-1)$ pebbles on it. Suppose among these $n - s_k$ vertices, $q$ is the number of vertices with at least one pebble. Therefore there will be $pt + (p-1)(n-2) - q$ pebbles on the vertices of $C_k$. We consider the following cases.
Subcase 1.1: \( q \geq t \).

We use pebbling moves from \( s_k - 1 \) vertices of \( C_k - \{v\} \) to put the remaining (at most) \( (p-1) \) pebbles on each of the \( t \) of the \( q \) occupied vertices of \( G - C_k \). Using \( (p-1)t \) pebbles we can pebble \( t \) vertices with \( (p-1) \) pebbles. Then remaining \( (p-1)(n-2) - (q-t) \) pebbles are in \( C_k - \{v\} \). From the \( t \) vertices with \( p \) pebbles we can move \( t \) pebbles to \( v \).

Subcase 1.2: \( q < t \).

As in Subcase (i) first we will put \( (p-1) \) more pebbles on each of these \( q \) vertices by making \( (p-1)q \) moves from the vertices of \( C_k - \{v\} \) in order to put \( q \) pebbles on \( v \). Then we have to place \( (t-q) \) additional pebbles on \( v \). So we use \( p^2(t-q) + (p-1)pq = p^2t - pq \) pebbles among \( pt + (p-1)(n-2) - q \) pebbles in the vertices of \( C_k - \{v\} \).

Hence in all the cases \( f_{gl}(v,G) \leq pt + (p-1)(n-2) \).

Case ii: Assume \( pt \geq n-s_1 \).

Let the vertices of \( C_1 \) be \( v_1, v_2, \ldots, v_{s_1} \) and let \( v_{s_1} \) be the target vertex. Let us place \( p^2t + (p-1)(s_1-2) \) pebbles on the vertices of \( C_1 \) as follows. Let us place \( p^2t - 1 \) pebbles on \( v_1 \) and place \( (p-1) \) pebbles each on \( (s_1-2) \) vertices of \( C_1 \) other than \( v_1 \) and \( v_{s_1} \). In this case \( t \)-pebbles cannot be moved to \( v_{s_1} \).

Hence \( f_{gl}(G) \geq p^2t + (p-1)(s_1-2) \).

Next we will use induction on \( t \) to prove that \( p^2t + (p-1)(s_1-2) \) pebbles are sufficient to put \( t \) pebbles on any desired vertex. Clearly the claim is true for \( pt = n-s_1 \) since by Case (i),
\[
f_{gl}(G) = pt + (p-1)(n-2) = pt + (p-1)(pt+s_1-2) = p^2t + (p-1)(s_1-2).
\]

Suppose \( p(m-1) > n-s_1 \) and
\[
f_{gl(n-1)}(G) = p^2(m-1) + (p-1)(s_1-2) = p^2m + (p-1)s_1 - (p^2 + 2p + 2).\]
Ch. 4: The Generalized Pebbling Number of Some Graphs

We prove the result is true for \( m \) where \( pm > n - s_1 \). Suppose \( p^2 m + (p - 1)(s_1 - 2) \) pebbles are distributed on the vertices of \( G \). Let the target vertex be \( v \) of \( C_k \). If there is a vertex in some \( C_j \) \((j \neq k)\) with at least \( p \) pebbles, then a pebble can be placed on \( v \) using only \( p \) pebbles. The remaining \( p^2 m + (p - 1)s_1 - 3p + 2 \) pebbles are sufficient to put \((m - 1)\) additional pebbles on \( v \), since \( p^2 + 2p - 2 - 3p + 2 > 0 \). If not then every vertex of \( G - C_k \) will contain either zero or at least one pebble on it. If there is a vertex say \( w \) in some \( C_j \) \((j \neq k)\) with at least one pebble on it, we use \((p - 1)p\) pebbles from the vertices of \( C_k \) to put \((p - 1)\) pebbles on \( w \) and hence a pebble can be placed on \( v \). Since \( p^2 + 2p - 2 - (p - 1)(p + 3) > 0 \), the remaining \( f_{glt}(m - 1)(G) \) pebbles would suffice to put \((m - 1)\) additional pebbles on \( v \). Otherwise, every vertex of \( G - C_k \) will have zero pebbles, using \( p^2 \) pebbles we can place a pebble on \( v \) in this case the remaining \( p^2(m - 1) + (p - 1)(s_1 - 2) \) pebbles would suffice to put \((m - 1)\) additional pebbles on \( v \). Thus \( f_{glt}(v, G) \leq p^2 m + (p - 1)(s_1 - 2) \). Therefore by induction \( f_{glt}(v, G) \leq p^2 t + (p - 1)(s_1 - 2) \) for all \( pt > n - s_1 \).

Thus \( f_{glt}(G) < p^2 t + (p - 1)(s_1 - 2) \) for all \( pt \geq n - 1 \) and so the proof is over.

**Theorem 4.3.10.** The generalized \( t \)-pebbling number of a cube \( Q_n \) is \( f_{glt}(Q_n) = tp^n \).

**Proof.** Suppose we place \( t(p^n) - 1 \) pebbles at a vertex which is at a distance \( n \) from any target vertex, then \( t \) pebbles cannot be moved to \( v \), so \( f_{glt}(Q_n) \geq tp^n \).
Let us use induction on \( t \) to prove that \( f_{gl}(Q_n) \leq tp^n \). For \( t = 1 \), the result is true by Theorem 4.2.13.

For \( t > 1 \), there are at least \( 2p^n \) pebbles on \( Q_n \). Using \( p^n \) pebbles by Theorem 4.2.13 we can move a pebble on any target vertex. Then by induction, the remaining \( (t-1)p^n \) pebbles will be sufficient to put \( (t-1) \) additional pebbles on the target vertex. So \( f_{gl}(Q_n) \leq tp^n \). ■

4.4 The generalized \( p \)-pebbling property

Chung [2] defined the two pebbling property of a graph and Wang [28] extended Chung’s definition to the odd two-pebbling property. We define the generalized \( p \) pebbling property as follows:

**Definition 4.4.1 (Generalized \( p \)-pebbling property).** Suppose \( a \) pebbles are distributed on the vertices of \( G \) in such a way that \( b \) vertices of \( G \) are occupied. i.e., there are exactly \( b \) vertices which have at least one pebble. We say the graph \( G \) satisfies the generalized \( p \) pebbling property, if we can put \( p \) pebbles on any specified vertex of \( G \) starting from every configuration in which \( a \geq pf_{gl}(G) - b + 1 \) or equivalently \( \frac{a+b}{p} > f_{gl}(G) \).

We show that path \( P_n \), complete graph \( K_n \), cycle \( C_n \), star \( K_{1,n} \) satisfy the generalized \( p \) pebbling property.

**Theorem 4.4.2.** Path \( P_2 \) satisfies the generalized \( p \) pebbling property.
Proof. Let \( V(P_2) = \{u_1, u_2\} \). Suppose \( pf_{gl}(P_2) - b + 1 \) which is \( p^2 - b + 1 \) pebbles are placed on the vertices of \( P_2 \). Without loss of generality assume that \( u_1 \) is our target vertex.

Case 1: If \( b = 1 \), and \( p^2 \) pebbles are placed on \( u_1 \), then we are done. Otherwise we can move \( p \) pebbles to \( u_1 \) from \( u_2 \).

Case 2: If \( b = 2 \), and \( u_1 \) receives \( x < p \) pebbles, then \( u_2 \) has at least \( p^2 - 1 - x \) pebbles. Using \( p(p-x) \) pebbles, \( (p-x) \) pebbles can be moved to \( u_1 \) while leaving \( ((p-1)x-1) \) pebbles on \( u_2 \). Hence we are done.

Notation 4.4.3. Let the vertices of \( P_n \) be \( \{v_1, v_2, \ldots, v_n\} \). Given a distribution of pebbles on \( P_n \) we let \( a \) and \( b \) denote the number of pebbles and number of occupied vertices in \( P_n \) respectively. We define the vertex sets \( A \) and \( B \) by \( A = \{v_1, v_2, \ldots, v_{n-1}\}, B = \{v_2, v_3, \ldots, v_n\} \). We let \( a_A, a_B, b_A \) and \( b_B \) denote the number of pebbles on \( A \) and \( B \) and number of occupied vertices in \( A \) and \( B \) respectively.

We let \( a_n \), the number of pebbles on vertex \( v_n \) and let

\[
b_n = \begin{cases} 
1, & \text{if } v_n \text{ is occupied} \\
0, & \text{otherwise}
\end{cases}
\]

For \( n \geq 3 \), we call \( v_1 \) and \( v_n \) as end vertices and each \( v_i \) \((2 \leq i \leq n-1)\) as an internal vertex.

Theorem 4.4.4. Any Path \( P_n \) satisfies the generalized \( p \) pebbling property.

Proof. The proof is by induction on \( n \), the number of vertices. For \( n = 2 \), the theorem is true by Theorem 4.4.2. Suppose as the induction hypothesis, that \( n > 2 \) and that whenever \( pf_{gl}(P_{n-1}) - b + 1 \) pebbles are distributed onto the vertices of path on \( n-1 \) vertices, \( p \) pebbles can be moved to our desired vertex.
Let $p_{fg}(P_n) - b + 1 = p^n - b + 1$ pebbles are distributed on the vertices of $P_n$.

**Case 1:** Let $v_i$ $(2 \leq i \leq n-1)$ be the target vertex. In the given distribution, assume without loss of generality there are at least $p_{fg}(P_n)$ pebbles on $A$. Otherwise there are at least $p_{fg}(P_n)$ pebbles on $B$. So, by induction we can put $p$ pebbles on the target.

**Case 2:** Now let the end vertex $v_1$ be our target vertex. If $a_A > p_{fg}(P_{n-1}) - b_A$ then we are through.

If $f_{gl}(P_{n-1}) < a_A < p_{fg}(P_{n-1}) - b_A$, then one pebble can be moved to $v_1$.

Now $a_n \geq p_{fg}(P_n) - b - p_{fg}(P_{n-1}) + b_A + 1. = (p - 1)f_{gl}(P_n) - b_n + 1$.

Hence using $(p - 1)f_{gl}(P_n)$ we may move $(p - 1)$ pebbles to our target vertex $v_1$ and we are done.

Finally if $a_A < f_{gl}(P_{n-1})$, then let $a_A = f_{gl}(P_{n-1}) - x$ for some integer $x$. Now

$$a_n = a - a_A$$

$$= p_{fg}(P_n) - f_{gl}(P_{n-1}) - x$$

$$= (p - 1)f_{gl}(P_n) + px + (p - 1)p^{n-2} - (p - 1)x - b + 1.$$  

With the help of $px$ pebbles in $v_n$, $x$ pebbles can be moved to $A$ from $v_n$ and hence with the help of $f_{gl}(P_{n-1})$ pebbles in $A$ one pebble can be moved to our target vertex. Again with the help of $(p - 1)f_{gl}(P_n)$ pebbles in $v_n$, $(p - 1)$ pebbles can be moved to our target vertex. This leaves us with at least $(p - 1)p^{n-2} - (p - 1)x - b + 1$ pebbles on $v_n$.

By symmetry we can pebble $v_n$, using any configuration of $p_{fg}(G) - b + 1$ pebbles on $P_n$.  

\[\blacksquare\]
Notation 4.4.5. Let $K_n$ be a complete graph on $n$ vertices $v_1, v_2, \ldots, v_n$ where $n \geq 2$. Let $K_{n-1}$ be the complete sub graph of $K_n$ induced by vertices $v_1, v_2, \ldots, v_{n-1}$. Let $a', a, a_n$ be the number of pebbles in $K_{n-1}, K_n$ and vertex $v_n$ respectively. Let $b', b$ be the number of occupied vertices in $K_{n-1}, K_n$ respectively.

Let $b_n = \begin{cases} 1, & \text{if } v_n \text{ is occupied}, \\ 0, & \text{otherwise}. \end{cases}$

Theorem 4.4.6. Any complete graph $K_n$ on $n$ vertices satisfies the generalized $p$ pebbling property.

Proof. The proof is by induction on $n$. For $n = 2$, the result is true by Theorem 4.4.2. Suppose generalized $p$ pebbling property is true in complete graph on $(n - 1)$ vertices say $K_{n-1}$.

Assume $pf_{gl}(K_n) - a + 1$ pebbles are distributed onto the vertices of $K_n$ where $n \geq 3$. Let $v_1$ be our target vertex. In the distribution suppose $a' > pf_{gl}(K_{n-1}) - b'$ then we are through.

If $f_{gl}(K_{n-1}) < a' < pf_{gl}(K_{n-1}) - b'$ then we can move a pebble to our target vertex $v_1$.

Now $a_n \geq a - a' = pf_{gl}(K_n) - b - pf_{gl}(K_{n-1}) + b' + 1 = p(p - 1) - (b - b') + 1 = p(p - 1) - b_n + 1$.

Hence $(p - 1)$ pebbles can be moved to our target vertex $v_1$, from $v_n$ using $p(p - 1)$ pebbles.
Finally if $a' > f_{gl}(K_{n-1})$, then let $x = f_{gl}(K_{n-1}) - a'$.

Now $a_n = a - a'$

\[
\geq (pf_{gl}(K_n) - b + 1) - (f_{gl}(K_{n-1}) - x)
\]

\[
= (p - 1)f_{gl}(K_n) + px + (p - 1)(1 - x) - b + 1.
\]

Using $px$ pebbles in vertex $v_n$, $x$ pebbles can be moved to $K_{n-1}$ and hence one pebble can be moved to $v_1$. Again using $(p - 1)f_{gl}(K_n)$ pebbles in $V_n$, we can move $(p - 1)$ pebbles to our target vertex $v_1$ while keeping $(p - 1)(1 - x) - b + 1$ pebbles in $v_n$, hence we are done.

**Notation 4.4.7.** Let the vertices of $C_m$ be $\{x_0, x_1, \ldots, x_{m-1}\}$ in order. Without loss of generality assume $x_0$ is the target vertex in $C_m$. Given a distribution of pebbles on $C_m$, we let $a_i$ represent the number of pebbles on $x_i$, and we let $b_i$ be 1 if $x_i$ is occupied, and 0 otherwise. If $m$ is even, we suppose $m = 2k$, and if $m$ is odd, we let $m = 2k + 1$. In either case we define the vertex sets $A$ and $B$ by

\[
A = \{x_1, x_2, \ldots, x_{k-1}\};
\]

\[
B = \{x_{m-1}, x_{m-2}, \ldots, x_{m-k+1}\}.
\]

We let $a_A$, $a_B$, $b_A$ and $b_B$ denote the number of pebbles on $A$ and $B$ and number of occupied vertices in $A$ and $B$ respectively.

**Theorem 4.4.8.** All cycles satisfy the generalized $p$ pebbling property.

**Proof.** Let $x_0 \in V(C_n)$ be our target vertex.

**Case 1:** Let $n = 2k$. Let us assume $a \geq pf_{gl}(C_n) - b + 1$ pebbles are distributed on the vertices of $C_n$ with $b$ occupied vertices. If $x_0$ is occupied with $x$ pebbles, then
using \((p-x)f_{gl}(C_n)\) pebbles we can move \((p-x)\) pebbles to \(x_0\) while keeping
\((x-1)(f_{gl}(C_n) - 1)\) pebbles in \(C_n - \{x_0\}\) since
\(a \geq (p-x)f_{gl}(C_n) + (x-1)(f_{gl}(C_n) - 1)\). Suppose \(x_0\) is not occupied. Then

\[a_A + b_A + a_B + b_B + a_k + b_k = a + b > p^{k+1} + 1.\]

Either \(a_A + b_A > p^k\) or \(a_B + b_B > p^k\). Without loss of generality assume
\(a_A + b_A > p^k\). If \(a_A + b_A > p^k\), then pebbles on \(A\) are sufficient to put \(p\) pebbles on \(x_0\), since \(A \cup \{x_0\}\) is isomorphic to path on \(k\) vertices \(P_k\), which satisfies the
generalized \(p\) pebbling property by Theorem 4.4.4. Thus we assume \(a_A + b_A \leq p^k\)
and similarly \(a_B + b_B < (p-1)p^k\) and we show that we can simultaneously put
\(p\) pebbles on \(x_1\) and \(p(p-1)\) pebbles on \(x_{2k-1}\). Suppose \(a_A + b_A = p^k\). Then we
can put \(p\) pebbles on \(x_1\) since path \(P_A = P_{k-1}\) and \(f_{gl}(P_A) = p^k\). Again suppose
\(a_B + b_B = (p-1)p^k\). Then we can put \((p-1)p\) pebbles on \(x_{2k-1}\), since path
\(P_B = P_{k-1}\) and \(f_{gl}(P_B) = p^{k-2}\) and hence \(p\) pebbles can be moved to our target \(x_0\).

If \(a_A + b_A < p^k\) and \(a_B + b_B < (p-1)p^k\), then let

\[a'_k = p^k - a_A - b_A\]
\[a''_k = (p-1)p^k - a_B - b_B\]

But \(a_k \geq (p^k - a_A - b_A) + ((p-1)p^k - a_B - b_B) + (1 - b_k) \geq a'_k + a''_k\).

So, there are enough pebbles on \(x_k\), we let \(b'_k = 1\) and \(b''_k = 1\). Hence \(a_A + b_A +
\[a'_k + b'_k = p^k + 1\] and we can put \(p\) pebbles on \(x_1\). Similarly \(a_B + b_B + a''_k + b''_k =
(p-1)p^k + 1\) and we can put \((p-1)p\) pebbles on \(x_{2k-1}\), so we can put \((p-1)\)
pebbles on \(x_0\) from \(x_{2k-1}\) and one pebble from \(x_1\) and we are done.
Case 2: \( n = 2k + 1 \) \((k \geq 2)\).

We know that \( f_{\text{glt}}(C_n) = 1 + (t - 1)p^{\lfloor n/2 \rfloor} + 2 \left\lceil \left( \frac{p}{p+1} \right) \left( p^{\lfloor n/2 \rfloor} - 1 \right) \right\rceil \)

Let us place \( p \left(1 + 2 \left\lceil \left( \frac{p}{p+1} \right) \left( p^{\lfloor n/2 \rfloor} - 1 \right) \right\rceil \) - \( b + 1 \) pebbles on the vertices of \( C_n \). It is enough to show that

\[
p \left(1 + 2 \left\lceil \left( \frac{p}{p+1} \right) \left( p^{\lfloor n/2 \rfloor} - 1 \right) \right\rceil \right) - \( b + 1 \)
\geq 1 + 2 \left\lceil \left( \frac{p}{p+1} \right) \left( p^{\lfloor n/2 \rfloor} - 1 \right) \right\rceil + (p - 1)p^{\lfloor n/2 \rfloor}
\]

For this it is enough to show that \( 1 + 2 \left\lceil \left( \frac{p}{p+1} \right) \left( p^{\lfloor n/2 \rfloor} - 1 \right) \right\rceil > p^{\lfloor n/2 \rfloor} \).

Consider \( 1 + 2 \left\lceil \left( \frac{p}{p+1} \right) \left( p^{\lfloor n/2 \rfloor} - 1 \right) \right\rceil - p^{\lfloor n/2 \rfloor} \).

\[
1 + 2 \left\lceil \left( \frac{p}{p+1} \right) \left( p^{\lfloor \frac{n}{2} \rfloor} - 1 \right) \right\rceil - p^{\lfloor \frac{n}{2} \rfloor} = 1 + 2 \left[ \frac{p \cdot p^{\lfloor n/2 \rfloor} - p}{p + 1} \right] - \frac{p \cdot p^{\lfloor n/2 \rfloor} - p^{\lfloor n/2 \rfloor}}{p + 1}
= 1 + \frac{2p \cdot p^{\lfloor n/2 \rfloor} - 2p - p \cdot p^{\lfloor n/2 \rfloor} - p^{\lfloor n/2 \rfloor}}{p + 1}
= 1 + \frac{p \cdot p^{\lfloor n/2 \rfloor} - 2p - p^{\lfloor n/2 \rfloor}}{p + 1}
= 1 + \frac{p^{\lfloor n/2 \rfloor}(p - 1) - 2p}{p + 1}
> 1
\]

Since \( p \geq 2 \), \( p - 1 \geq 1 \) and \( p^{\lfloor n/2 \rfloor}(p - 1) \geq p^2 \geq 4 \) as \( n \geq 5 \).

Again \( 2p \geq 4 \).

**Theorem 4.4.9.** Any star \( K_{1, n} \) \((n \geq 2)\) satisfies generalized pebbling property.

**Proof.** Let \( V(K_{1, n}) = U \cup W \) where \( U = \{u\} \) and \( W = \{w_1, w_2, \ldots, w_n\} \). Without loss of generality, let \( V(K_{1, n-1}) = \{u, w_1, w_2, \ldots, w_{n-1}\} \). We will prove the result by induction on \( n \). For \( n = 2 \), \( K_{1, 2} \cong P_3 \) and from Theorem 4.3.4, result follows.
Assume \( n \geq 3 \). Let \( a', a, a_n \) be number of pebbles in \( K_{1,n-1}, K_{1,n} \) and \( v_n \) respectively. Let \( b', b, b_n \) be the number of occupied vertices in \( K_{1,n-1}, K_{1,n} \) and vertex \( v_n \) respectively. Suppose \( a' > pf_{gl}(K_{1,n-1}) - b' \) then we are done by induction. If \( f_{gl}(K_{1,n-1}) < a' < pf_{gl}(K_{1,n-1}) - b' \), then we can move a pebble to our target vertex.

Then \( a - f_{gl}(K_{1,n-1}) = pf_{gl}(K_{1,n}) - b + 1 - f_{gl}(K_{1,n-1}) \)

\[ = (p - 1)f_{gl}(K_{1,n}) - b + p. \]

Hence using \( (p - 1)f_{gl}(K_{1,n}) \) pebbles in \( K_{1,n} \) we can move \( (p - 1) \) pebbles to our target vertex while leaving \( p - b \) pebbles in \( K_{1,n} \). Finally if \( a' < f_{gl}(K_{1,n-1}) \) then let \( a' = f_{gl}(K_{1,n-1}) - x \). Number of pebbles in \( v_n \) is

\[ a_n = a - a' \geq pf_{gl}(K_{1,n}) - b + 1 - f_{gl}(K_{1,n-1}) \]

\[ = (p - 1)f_{gl}(K_{1,n}) + px + (p - 1)(1 - x) - b + 1. \]

Using \( px \) pebbles we can move \( x \) pebbles to \( K_{1,n-1} \) from \( v_n \) and hence one pebble can be moved to our target vertex. Again using \( (p - 1) f_{gl}(K_{1,n}) \) pebbles \( (p - 1) \) pebbles can be moved to our target vertex while keeping \( (p - 1)(1 - x) - b + 1 \) pebbles in \( v_n \). Hence we are done.

\[ \square \]

### 4.5 The generalized pebbling conjecture on the products of graphs

We show that for any connected graphs \( G \) and \( H \), and if \( H \) satisfies the generalized \( p \)-pebbling property, then the pebbling number of \( G \times H \) satisfies \( f_{gl}(G \times H) \leq f_{gl}(G)f_{gl}(H) \).
Theorem 4.5.1. Let $P_2$ be the path on two vertices say $u_1$ and $u_2$. Let $G$ be a graph with generalized $p$ pebbling property. Then $f_{gl}(P_2 \times G) \leq f_{gl}(P_2)f_{gl}(G) = pf_{gl}(G)$. Furthermore, if $f_{gl}(P_2 \times G) = f_{gl}(P_2)f_{gl}(G)$ then $P_2 \times G$ has the generalized $p$ pebbling property.

Proof. Let $V(P_2) = \{u_1, u_2\}$ and let $y \in G$. Without loss of generality we assume the target vertex on $P_2 \times G$ is $r = (u_1, y)$ and then choose $r' \in V(G_2) \cap N(r)$ where $N(v)$ is the neighborhood of a vertex $v$. We denote the two copies of $G$ in $P_2 \times G$, i.e., $\{u_i\} \times G$, $i = 1, 2$ respectively by $G_i$. Let $a_i$ denote the number of pebbles on $G_i$ with $b_i$ occupied vertices and let $a = a_1 + a_2$ and $b = b_1 + b_2$. Suppose we start with a configuration of $pf_{gl}(G)$ pebbles.

Suppose $a_1 \geq f_{gl}(G_1)$, then a pebble can be moved to $(u_1, y)$. Assume $a_1 < f_{gl}(G_1)$. Then for some integer $x$ we have

$$a_1 = f_{gl}(G_1) - x \text{ and } a_2 = (p - 1)f_{gl}(G_2) + x.$$ 

Since $G \cong G_2$ has the generalized $p$ pebbling property we may assume

$$x \leq f_{gl}(G_2) - b_2. \quad (4.4)$$

Otherwise we could move $p$ pebbles to $r'$ and then one to $r$. From Eq. (4.4), it follows that $b_2 \leq f_{gl}(G_2) - x$.

Now we will move as many pebbles as possible from $G_2$ to $G_1$. We can move at least

$$\frac{(p - 1)f_{gl}(G_2) + x - b_2}{p} \geq \frac{(p - 1)f_{gl}(G_2) + x - (p - 1)(f_{gl}(G_2) - x)}{p} = x$$
pebbles to \( G_1 \) yielding \( f_{gl}(G_1) \) pebbles on \( G_1 \) and then we can move a pebble to \( r \). Therefore \( f_{gl}(P_2 \times G) \leq p f_{gl}(G) \). Now to prove the second part of the theorem assume that \( f_{gl}(P_2 \times G) < f_{gl}(P_2) f_{gl}(G) = p f_{gl}(G) \) and that \( a = p f_{gl}(G) - b + 1 \).

If \( a_1 > p f_{gl}(G_1) - b_1 \), then since \( G_1 \) has the generalized \( p \) pebbling property we are done. Otherwise \( f_{gl}(G_1) < a_1 \leq p f_{gl}(G_1) - b_1 \). We can move one pebble to \( r \).

And since \( a_2 \geq p f_{gl}(P_2) f_{gl}(G) - b - (p f_{gl}(G_1) - b_1) + 1 \)
\[ = (p - 1) p f_{gl}(G_2) - b_2 + 1. \]

So we can move \((p - 1)p\) pebbles to \( r' \) and then \((p - 1)\) to \( r \). Finally, if \( a_1 < f_{gl}(G_1) \) then \( a_1 = f_{gl}(G_1) - x \) for some \( x \).

Then \( a_2 = a - a_1 \)
\[ = (p^2 f_{gl}(G) - b + 1) - (f_{gl}(G_1) - x) \]
\[ = (p^2 - 1) f_{gl}(G) - b_1 - b_2 + x + 1 \]
\[ \geq p(p - 1) f_{gl}(G) - b_2 + px + 1 \]

Since \( b_1 \leq f_{gl}(G_1) - x \leq (p - 1) f_{gl}(G_1) - (p - 1)x \)

Thus \( a_2 \geq p(p - 1) f_{gl}(G) - b_2 + px + 1 \)
\[ \geq (p^2 - p - 1) b_2 + px + 1 \] since \( b_2 \leq f_{gl}(G_2) \)
\[ \geq b_2 + px \text{ as } p \geq 2. \]

Thus we can move \( x \) pebbles to \( G_1 \) yielding \( f_{gl}(G_1) \) pebbles on \( G_1 \). Therefore we can move one pebble to \( r \). Also \( a_2 - px \geq p(p - 1) f_{gl}(G) - b_2 + 1 \) pebbles remain on \( G_2 \). Hence \( p(p - 1) \) pebbles can be moved to \( r' \) and then \((p - 1)\) pebbles can be moved to \( r \) and hence we are done.
Theorem 4.5.2. Let $P_m$ be a path on $m$ vertices. If $G$ satisfies the generalized $p$ pebbling property then $f_{gl}(P_m \times G) \leq p^{m-1} f_{gl}(G)$. If $f_{gl}(P_m \times G) = f_{gl}(P_m)f_{gl}(G)$, then $P_m \times G$ has the generalized $p$ pebbling property.

Proof. Let $V(P_m) = \{v_1, v_2, \ldots, v_m\}$ where $m \geq 2$. The proof is by induction on $m$. For $m = 2$, the theorem is true by Theorem 4.5.1. Let $y \in V(G)$. For $m \geq 3$ we call $(v_1, y)$ and $(v_m, y)$ as end vertices, we call $(v_i, y)$ as an internal vertex for $i = 2$ to $(m - 1)$.

Let us take an internal vertex to be the target vertex. Then without loss of generality there are at least $p^{m-2} f_{gl}(G)$ pebbles on $\{v_1, v_2, \ldots, v_{m-1}\} \times G$ (otherwise there are at least $p^{m-2} f_{gl}(G)$ pebbles on $\{v_2, v_3, \ldots, v_m\} \times G$). So by induction we can pebble the target vertex.

Now let $(v_1, y)$ be the target vertex. Let $a'$ denote the number of pebbles in $\{v_1, v_2, \ldots, v_{m-1}\} \times G$ and let $b'$ denote the number of vertices with at least one pebble in $\{v_1, v_2, \ldots, v_{m-1}\} \times G$ and let $a_n$ denote the number of pebbles in $\{v_n\} \times G$ and let $b_n$ denote the number of occupied vertices in $\{v_n\} \times G$ and let $a = a' + a_n$ and $b = b' + b_n$. Suppose $a' > p^{m-2} f_{gl}(G)$. Then we are done. Assume $a' < p^{m-2} f_{gl}(G)$. Then for some integer $x$ we have $a' = p^{m-2} f_{gl}(G) - x$ and $a_n = p^{m-2}(p - 1) f_{gl}(G) + x$. Since $G$ has the generalized $p$ pebbling property we may assume

$$x \leq p^{m-2} f_{gl}(G) - b_n \quad (4.5)$$

Otherwise we could move $p^{m-1}$ pebbles to a vertex in $\{v_n\} \times G$ and hence a pebble can be moved to our target vertex.

From Eq. 4.5 it follows that $b_n \leq p^{m-2} f_{gl}(G) - x$.

Now we will move as many pebbles as possible from
\{v_n\} \times G \text{ to } \{v_1, v_2, \ldots, v_{n-1}\} \times G. \text{ We can move at least}
\[ p^{m-2}(p-1)f_{gl}(G) + x - b_n \]
\[ \geq \frac{p^{m-2}(p-1)f_{gl}(G) + x - (p-1)p^{m-2}f_{gl}(G) + (p-1)x}{p} \]
\[ = x \]

pebbles to \(\{v_1, v_2, \ldots, v_{n-1}\} \times G\) yielding \(p^{m-2}f_{gl}(G)\) pebbles and then we can
move a pebble to \((v_1, y)\). Therefore \(f_{gl}(P_m \times G) \leq p^{m-1}f_{gl}(G)\).

Without loss of generality assume that \((v_1, y)\) is our target vertex. We will
prove the theorem by induction on \(m\). When \(m = 2\) and \(f_{gl}(P_2 \times G) = pf_{gl}(G)\)
then \(P_2 \times G\) satisfies the generalized \(p\) pebbling property by Theorem 4.5.1. Assume the result is true for \(m-1\). i.e., whenever \(f_{gl}(P_{m-1} \times G) = p^{m-2}f_{gl}(G)\) then \(P_{m-1} \times G\) satisfies the generalized \(p\) pebbling property. Let \(a_i\) denote the number
of pebbles on \(\{v_i\} \times G\) with \(b_i\) occupied vertices where \(i = 1, 2, \ldots, m\).

And let \(a = a_1 + a_2 + \cdots + a_m\),
and let \(b = b_1 + b_2 + \cdots + b_m\).

Assume that \(f_{gl}(P_m \times G) = p^{m-1}f_{gl}(G)\) and that \(a = p^m f_{gl}(G) - b + 1\).

If \(a_1 > pf_{gl}(G) - b_1\), then we are done. Otherwise \(f_{gl}(G) < a_1 \leq pf_{gl}(G) - b_1\),
we can move one pebble to our target vertex \((v_1, y)\).

Again \(a_2 + a_3 + \cdots + a_m \geq p^m f_{gl}(G) - b - (pf_{gl}(G) - b_1) + 1 \]
\[ = (p^m - p)f_{gl}(G) - (b - b_1) + 1 \]
\[ = (p^m - p^{m-1})f_{gl}(G) + (p^{m-1} - p)f_{gl}(G) - (b_2 + b_3 + \cdots + b_m) + 1 \]
We can move \( p(p-1) \) pebbles to a vertex in \( \{v_2\} \times G \) by induction. Hence \( (p-1) \) pebbles can be moved to \((v_1, y)\).

Finally if \( a_1 < f_{gl}(G) \) and let \( a_1 = f_{gl}(G) - x \) for some integer \( x \). Now

\[
a_2 + a_3 + \cdots + a_m
\]

\[
= a - a_1
\]

\[
= (p^m f_{gl}(G) - b + 1) - (f_{gl}(G) - x)
\]

\[
= p^m f_{gl}(G) - b_1 - b_2 - \cdots - b_m + 1 - f_{gl}(G) + x
\]

\[
= p^m f_{gl}(G) - (p-1)f_{gl}(G) + (p-1)x - b_2 - b_3 - \cdots - b_m + 1 - f_{gl}(G) + x,
\]

since

\[
b_1 \leq f_{gl}(G) - x
\]

\[
\leq (p-1)(f_{gl}(G) - x)
\]

\[
= (p^m - p^{m-1}) f_{gl}(G) + (p^{m-1} - p)f_{gl}(G) + px - (b_2 + b_3 + \cdots + b_m) + 1
\]

Using \( px \) pebbles, \( x \) pebbles can be moved to \( \{v_1\} \times G \) and hence one pebble can be moved to \((v_1, y)\). Again using \( p(p-1)p^{m-2}f_{gl}(G) \), pebbles we can move \( p(p-1) \) pebbles to a vertex in \( \{v_2\} \times G \) namely \((v_2, y)\) and hence \( (p-1) \) pebbles can be moved to \((v_1, y)\) and we are done.

**Corollary 4.5.3.** Let \( P_m \) be a path on \( m \) vertices and \( K_n \) be a complete graph on \( n \) vertices. Then \( f_{gl}(P_m \times K_n) \leq f_{gl}(P_m)f_{gl}(K_n) \).

**Proof.** The corollary follows from Theorem 4.4.6 and Theorem 4.5.2.

\[\]
Corollary 4.5.4. Let $P_m$ be a path on $m$ vertices and $K_{1,n}$ be a star. Then $f_{gl}(P_m \times K_{1,n}) \leq f_{gl}(P_m)f_{gl}(K_{1,n})$.

**Proof.** The corollary follows from Theorem 4.4.8 and Theorem 4.5.2. □

Corollary 4.5.5. Let $P_m$ be a path on $m$ vertices and $C_n$ ($n \geq 3$) be a Cycle on vertices. Then $f_{gl}(P_m \times C_n) \leq f_{gl}(P_m)f_{gl}(C_n)$.

**Proof.** The corollary follows from Theorem 4.4.4 and Theorem 4.5.2. □

Theorem 4.5.6. Let $K_n$ be a complete graph on $n$ vertices where $n \geq 2$ and let $G$ be a graph with generalized $p$ pebbling property. Then $f_{gl}(K_n \times G) \leq f_{gl}(K_n)f_{gl}(G)$.

**Proof.** Let $v_1, v_2, \ldots, v_n$ be the vertices of $K_n$. The proof is by induction on $n$.

For $n = 2$, $K_2 = P_2$ and hence the theorem is true by Theorem 4.5.1. Let us assume the result is true when $n' < n$ and $n \geq 3$. Without loss of generality assume the target is $(v_1,y)$. Assume that in $K_n \times G$, we have $a_i$ pebbles occupying $b_i$ vertices of $\{v_i\} \times G$ for $i = 1, 2, \ldots, n$. Let $a = a_1 + a_2 + \cdots + a_n$. If $a_1 \geq f_{gl}(G)$, then we can move a pebble to $(v_1,y)$ since $\{v_1\} \times G$ is isomorphic to $G$. Hence assume $a_1 < f_{gl}(G)$. There are two cases which arise.

**Case 1:** Assume $a_1 + a_2 + \cdots + a_{n-1} < b_n$. Then

\[
a_n = a - (a_1 + a_2 + \cdots + a_{n-1}) \\
\geq (p + (p-1)(n-2))f_{gl}(G) - b_n \\
> pf_{gl}(G) - b_n \quad \text{as } p \geq 2 \text{ and } n \geq 3
\]

Hence we can move $p$ pebbles to $(v_n,y)$ and hence a pebble can be moved to $(v_1,y)$ and we are done.
Case 2: Assume \( b_n \leq a_1 + a_2 + \cdots + a_{n-1} \).

We can transfer \( \frac{a_n - (p-1)b_n}{p} \) (by Lemma 4.2.12) pebbles to \( (K_n - \{v_n\}) \times G \) which is \( K_{n-1} \times G \), from \( \{v_n\} \times G \).

Therefore, if\( a_1 + a_2 + \cdots + a_{n-1} + \frac{a_n - (p-1)b_n}{p} \geq ((p-1)(n-1) \cdot f_{gl}(G), \)
then a pebble can be placed on \( (v_1,y) \) by induction. Hence the only distribution from which we cannot pebble the target satisfies the inequalities

\[
\frac{a_n + b_n}{p} < f_{gl}(G)
\]

i.e.,

\[
\frac{(p-1)a_n + (p-1)b_n}{p} < (p-1)f_{gl}(G)
\] (4.6)

\[
a_1 + a_2 + \cdots + a_{n-1} + \frac{a_n - (p-1)b_n}{p} \leq ((p-1)(n-1) \cdot f_{gl}(G) \) (4.7)

But adding (4.6) and (4.7) gives

\[
a_1 + a_2 + \cdots + a_{n-1} + a_n \leq ((p-1)n - (p-2)) f_{gl}(G).
\]

Thus any distribution of pebbles from which we may not put a pebble on \( (v_1,y) \) must begin with less than \( ((p-1)n - (p-2)) f_{gl}(G) \) pebbles.

Corollary 4.5.7. Let \( K_m \) be a complete graph on \( m \) vertices. Let \( P_n \) be a path on \( n \) vertices. Then \( f_{gl}(K_m \times P_n) \leq f_{gl}(K_m) f_{gl}(P_n) \).

Proof. The corollary follows from Theorem 4.4.4 and Theorem 4.5.6.

Corollary 4.5.8. Let \( K_m \) be a complete graph on \( m \) vertices. Then \( f_{gl}(K_m \times K_n) \leq f_{gl}(K_m) f_{gl}(K_n) \).

Proof. The corollary follows from Theorem 4.4.6 and Theorem 4.5.6.
Corollary 4.5.9. Let $K_m$ be a complete graph on $m$ vertices. Let $C_n$ ($n \geq 3$) be a cycle on $n$ vertices. Then $f_{gl}(K_m \times C_n) \leq f_{gl}(K_m)f_{gl}(C_n)$.

**Proof.** The corollary follows from Theorem 4.4.8 and Theorem 4.5.6. ■

Theorem 4.5.10. Let $K_{1,n}$ ($n > 1$) be a star. If $G$ satisfies the generalized $p$ pebbling property then $f_{gl}(K_{1,n} \times G) \leq f_{gl}(K_{1,n})f_{gl}(G)$. Moreover if $f_{gl}(K_{1,n} \times G) = f_{gl}(K_{1,n})f_{gl}(G)$, then $K_{1,n} \times G$ has the generalizing $p$ pebbling property.

**Proof.** By Theorem 4.2.6, the generalized pebbling number of $K_{1,n}$ is $f_{gl}(K_{1,n}) = (p - 1)n + (p^2 - 2p + 2)$ if $n > 1$ and $p \geq 2$. Let $V(K_{1,n}) = V_1 \cup V_2$ where $V_1 = \{u\}$ and $V_2 = \{w_1, w_2, \ldots, w_n\}$. We use induction on $n$ to prove $f_{gl}(K_{1,n} \times G) \leq f_{gl}(K_{1,n})f_{gl}(G)$. For $n = 1$, $K_{1,2} \cong P_3$, a path on three vertices namely $w_1, u$ and $w_2$. Therefore by Theorem 4.5.2,

$$f_{gl}(P_3 \times G) \leq f_{gl}(P_3)f_{gl}(G) = ((p - 1)n + (p^2 - 2p + 2))f_{gl}(G).$$

Let us assume $n > 2$. Let $a_i, a_j$ be the number of pebbles and $b_i, b_j$ be the number of occupied vertices in $\{u\} \times G$ and $\{w_i\} \times G$ for $i = 1, 2, \ldots, n$ respectively. Let $y \in G$.

**Case 1:** Let the target vertex be $(u, y)$. Let us fix some $w_i \in V_2$.

**Subcase 1.1:** Assume $a_1 + a_2 + \cdots + a_{i-1} + a_{i+1} + \cdots + a_n + a < b_i$. Now

$$a_i = f_{gl}(K_{1,n})f_{gl}(G) - (a_1 + a_2 + \cdots + a_{i-1} + a_{i+1} + \cdots + a_n + a)$$

$$= f_{gl}(K_{1,n})f_{gl}(G) - b_i$$

$$> pf_{gl}(G) - b_i$$

So we can move $p$ pebbles to $(w_i, y)$ and hence one pebble can be moved to $(u, y)$.
Subcase 1.2: Assume $a + a_1 + a_2 + \cdots + a_{i-1} + a_{i+1} + \cdots + a_n \geq b_i$.

By Lemma 4.2.12, we can transfer $\frac{a_i - (p-1)b_i}{p}$ pebbles to $\{u\} \times G$.

If $a_1 + a_2 + \cdots + a_{i-1} + a_{i+1} + \cdots + a_n + \frac{a_i - (p-1)b_i}{p} + a \geq f_{gl}(K_{1,n-1})$, then by induction a pebble can be moved to $(u, y)$. The only distribution from which we cannot pebble the target satisfies the inequalities, $\frac{a_i - h_i}{p} \leq f_{gl}(G)$ and hence

\[
\frac{(p-1)a_i + (p-1)b_i}{p} < (p-1)f_{gl}(G) \tag{4.8}
\]

\[
a + a_1 + a_2 + \cdots + a_{i-1} + a_{i+1} + \cdots + a_n + \frac{a_i - (p-1)b_i}{p} < ((p-1)(n-1) + (p^2 - 2p + 2)) f_{gl}(G) \tag{4.9}
\]

But adding (4.8) and (4.9) we get

\[
a + a_1 + a_2 + \cdots + a_{i-1} + a_{i+1} + \cdots + a_n + a_i < ((p-1)n + (p^2 - 2p + 2)) f_{gl}(G).
\]

Thus any distribution of pebbles from which we may not place a pebble on $(u, y)$ must begin with fewer than $((p-1)n + (p^2 - 2p + 2)) f_{gl}(G)$ pebbles.

Case 2: Let $(w_j, y)$ be the target vertex for some $j = 1, 2, \ldots, n$. Without loss of generality, we assume that $(w_n, y)$ is the target vertex. Let us name the subgraph obtained by deleting the vertices $w_1, w_2, \ldots, w_{n-1}$ as $K'$ and hence $V(K') = \{u, w_n\}$. Again $K' \times G$ is isomorphic to $P_2 \times G$.

If $a + a_n \geq p f_{gl}(G)$, then by Theorem 4.5.2, one pebble can be moved to $(w_n, y)$ Let us assume $a + a_n < p f_{gl}(G)$. Consider the following two cases.

Subcase 2.1: Assume $a + a_n < b_1 + b_2 + \cdots + b_{n-1}$. Consider

\[
a_1 + a_2 + \cdots + a_{n-1} = ((p-1)n + (p^2 - 2p + 2)) f_{gl}(G) - (a + a_n)
\]

\[
\geq ((p-1)n + (p^2 - 2p + 2)) f_{gl}(G) - (b_1 + b_2 + \cdots + b_n).
\]
Hence
\[
\sum_{k=1}^{n-1} (a_k + b_k) \geq ((p - 1)n + (p^2 - 2p + 2))\ f_gl(G)
\]
\[
\geq p^2\ f_gl(G) \quad \text{as } p \geq 2 \text{ and } n \geq 2.
\]

Then we could move at least \( p\ f_gl(G) \) pebbles to the vertices of \( K' \times G \) and we are done.

**Subcase 2.2:** Assume \( b_1 + b_2 + \cdots + b_{n-1} \leq a + a_n \).

We apply sequence of pebbling moves \( \left\{ \bigcup_{i=1}^{n-1} w_i \right\} \times G \), we could move at least
\[
\left\lfloor \frac{(a_1 + a_2 + \cdots + a_{n-1}) - (p-1)(b_1 + b_2 + \cdots + b_{n-1})}{p} \right\rfloor
\]
pebbles to the vertices of \( K' \times G \) then after pebbling, number of pebbles on \( K' \times G \) will be
\[
a + a_n + \left\lfloor \frac{\sum_{k=1}^{n-1} a_k - (p-1)\sum_{k=1}^{n-1} b_k}{p} \right\rfloor
\]
\[
= a + a_n + \frac{\sum_{k=1}^{n-1} a_k - (p-1)(a + a_n)}{p}
\]
\[
= a + a_1 + a_2 + \cdots + a_n
\]
\[
= \frac{(p-1)n + (p^2 - 2p + 2)}{p}\ f_gl(G)
\]
\[
\geq \frac{p^2}{p}\ f_gl(G).
\]

Hence a pebble can be moved to \((w_n, y)\) by Theorem 4.5.2. Hence the only distribution from which we cannot pebble the target satisfies the inequalities,
\[
\sum_{k=1}^{n-1} \frac{(a_k + b_k)}{p} < p\ f_gl(G)
\]
and hence
\[
\sum_{k=1}^{n-1} \frac{(p-1)(a_k + b_k)}{p} < p(p-1)\ f_gl(G) \quad (4.10)
\]
\[ a + a_1 + a_2 + \cdots + a_n < p^2 f_{gl}(G) = ((p-1)n + (p^2 - 2p + 2)) f_{gl}(G). \]

Hence any distribution from which \((w_n, y)\) cannot be pebbled must have fewer than \(f_{gl}(K_{1,n}) f_{gl}(G)\) pebbles.

Now to prove the second part of the theorem we assume that \(f_{gl}(K_{1,n} \times G) = f_{gl}(K_{1,n}) f_{gl}(G)\). We want to prove that \(K_{1,n} \times G\) satisfies the generalized \(p\) pebbling property. Suppose \(n = 2\), \(K_{1,n} \cong P_3\). Then by Theorem 4.5.2 we are done. Assume the result is true when \(n' < n\). Suppose \(p f_{gl}(K_{1,n} \times G) - b + 1\) pebbles are distributed on the vertices of \(K_{1,n} \times G\).

**Case 1:** Let the target vertex be \((u, y)\). Let \(V(K_{1,n-1}) = \{u, w_1, w_2, \ldots, w_{n-1}\}\). Let \(a', a_n\) be the number of pebbles and \(b', b_n\) be number of occupied vertices in \(K_{1,n-1} \times G\) and \(\{w_n\} \times G\) respectively. Consider the following three cases.

**Subcase 1.1:** Suppose \(a' > p f_{gl}(K_{1,n-1} \times G) - b'\) by induction \(p\) pebbles can be moved to our target vertex.

**Subcase 1.2:** Assume \(f_{gl}(K_{1,n-1} \times G) \leq a' \leq p f_{gl}(K_{1,n} \times G) - b'\).

Since \(a' \geq f_{gl}(K_{1,n-1} \times G)\), by induction in \(K_{1,n-1} \times G\) one pebble can be moved to our target vertex. Since

\[ a_n = a - a' \]

\[ > p f_{gl}(K_{1,n} \times G) - b - a' \]

\[ = p f_{gl}(K_{1,n} \times G) - b' - b_n - a' \]

\[ = p f_{gl}(K_{1,n} \times G) - b' - b_n - p f_{gl}(K_{1,n-1} \times G) + b'. \]
Since $a' \leq p f_{gl}(K_{1,n-1} \times G) - b'$

$$= p(p-1)f_{gl}(G) - b_n.$$  

Hence $p(p-1)$ pebbles can be moved to a vertex which is adjacent to $(u,y)$ and hence $(p-1)$ pebbles can be moved to $(u,y)$ and we are done.

**Subcase 1.3:** Assume $a' < f_{gl}(K_{1,n-1} \times G)$.

Then $a_n > p((p-1)n + (p^2 - 2p + 2))f_{gl}(G) - b - a'$

$$= p((p-1)(n-1) + (p^2 - 2p + 2))f_{gl}(G) + p(p-1)f_{gl}(G) - b' - b_n - a'$$

$$= (p^2 - p - 1)b_n + p((p-1)(n-1) + (p^2 - 2p + 2))f_{gl}(G) - b' - a',$$

since $b_n < f_{gl}(G)$.

Hence $f_{gl}(K_{1,n-1})f_{gl}(G) - \left\lceil \frac{b' + a'}{p} \right\rceil$ pebbles can be moved to $K_{1,n-1} \times G$ leaving more than $p(p-1)f_{gl}(G) - b_n$ pebbles in $\{w_n\} \times G$.

In $K_{1,n-1} \times G$, there are

$$a' + f_{gl}(K_{1,n-1})f_{gl}(G) - \left\lceil \frac{b' + a'}{p} \right\rceil = f_{gl}(K_{1,n-1})f_{gl}(G) + \left\lceil \frac{a' (p-1) - b'}{p} \right\rceil$$

$$\geq f_{gl}(K_{1,n-1})f_{gl}(G)$$

pebbles.

We can then move one pebble to a target vertex $(u,y)$ in $K_{1,n-1} \times G$ and at the same time we can move $p(p-1)$ pebbles to a vertex which is adjacent to $(u,y)$ in $\{w_n\} \times G$, which will result in $(p-1)$ additional pebbles to a target vertex.

**Case 2:** Let the target vertex be $(w_i,y)$ for some $i = 1,2,\ldots,n$. Without loss of generality, let the target vertex be $(w_n,y)$. Let $a, a_i (i = 1,2,\ldots,n)$ be the number of pebbles and $b, b_i (i = 1,2,\ldots,n)$ be the number of occupied vertices in $\{u\} \times G$ and $\{w_i\} \times G (i = 1,2,\ldots,n)$ respectively and

$$b_0 = b + b_1 + b_2 + \cdots + b_n.$$
Suppose \( a + a_n > p f_{gl}(G) - (b + b_n) \). Then we are done by Theorem 4.5.2.

Assume \( a + a_n \leq p f_{gl}(G) - (b + b_n) \). Then

\[
\sum_{k=1}^{n-1} a_k > p((p - 1)n + (p^2 - 2p + 2)) f_{gl}(G) - b_0 - (a + a_n)
\]
\[
= p((p - 1)n + (p^2 - 2p + 2)) f_{gl}(G) - b - b_1 - b_2 - \cdots - b_n - (a + a_n)
\]
\[
= p((p - 1)n + (p^2 - 2p + 2)) f_{gl}(G) - (b + a) - (b_n + a_n) - \sum_{k=1}^{n-1} b_k
\]
\[
\sum_{k=1}^{n-1} (a_k + b_k) > p((p - 1)n + (p^2 - 2p + 2)) f_{gl}(G) - (b + a) - (b_n + a_n).
\]

Hence \((p - 1)n + (p^2 - 2p + 2)) f_{gl}(G) - \left\lceil \frac{b + a}{p} \right\rceil - \left\lceil \frac{b_n + a_n}{p} \right\rceil\) pebbles can be moved to \( P_2 \times G \) where \( V(P_2) = \{u, w_n\} \).

Hence in \( \{u, w_n\} \times G \), there are

\[
a + a_n + ((p - 1)n + (p^2 - 2p + 2)) f_{gl}(G) - \left\lceil \frac{a + b}{p} \right\rceil - \left\lceil \frac{b_n + a_n}{p} \right\rceil
\]
\[
= ((p - 1)n + (p^2 - 2p + 2)) f_{gl}(G) + \left\lfloor \frac{(p - 1)a_0 - b_0}{p} \right\rfloor + \left\lfloor \frac{(p - 1)a_n - b_n}{p} \right\rfloor
\]
\[
\geq p f_{gl}(G)\) pebbles since \( p \geq 2, n \geq 3 \).

We can then move \( p \) pebbles to \((w_n, y)\) and we are done.

**Corollary 4.5.11.** Let \( K_{1,n} (n > 1) \) be a star and let \( K_m \) be a complete graph. Then \( f_{gl}(K_{1,n} \times K_m) \leq f_{gl}(K_{1,n}) f_{gl}(K_m) \).

**Proof.** The Corollary follows from Theorem 4.4.6 and Theorem 4.5.10.

**Corollary 4.5.12.** Let \( K_{1,n} (n > 1) \) and let \( K_{1,m} (m > 1) \) be any two stars. Then \( f_{gl}(K_{1,n} \times K_{1,m}) \leq f_{gl}(K_{1,n}) f_{gl}(K_{1,m}) \).

**Proof.** The Corollary follows from Theorem 4.4.9 and Theorem 4.5.10.
Corollary 4.5.13. Let $K_{1,n}$ ($n > 1$) and $C_m$ ($m \geq 3$) be a cycle on $m$ vertices. Then 
$f_{gl}(K_{1,n} \times C_m) \leq f_{gl}(K_{1,n})f_{gl}(C_m)$.

Proof. The Corollary follows from Theorem 4.4.8 and Theorem 4.5.10.

Theorem 4.5.14. Let $K_{2,2}$ be a bipartite graph and $G$ be a graph with the generalized $p$ pebbling property. Then 
$f_{gl}(K_{2,2} \times G) \leq f_{gl}(K_{2,2})f_{gl}(G)$.

i.e., $f_{gl}(K_{2,2} \times G) \leq p^2f_{gl}(G)$ since $f_{gl}(K_{2,2}) = p^2$.

Proof. Let $V(K_{2,2}) = V_1 \cup V_2$ where $V_1 = \{u_1, u_2\}$ and $V_2 = \{u_3, u_4\}$. Let $y \in G$.

Without loss of generality we assume the target vertex on $K_{2,2} \times G$ is $(u_1, y)$. Let $a_i$ denote the number of pebbles on $\{u_i\} \times G$ with $b_i$ occupied vertices. Suppose we start with a configuration of $p^2f_{gl}(G)$ pebbles.

Suppose $a_1 + a_3 > pf_{gl}(G)$. Then we are done by Theorem 4.5.2. Assume $a_1 + a_3 \leq pf_{gl}(G)$. Let $a_1 + a_3 = pf_{gl}(G) - x$ for some integer $x$. Then $a_2 + a_4 = (p^2 - p)f_{gl}(G) + x$. If $f_{gl}(P_2 \times G) = pf_{gl}(G)$ and $G$ has the generalized $p$ pebbling property then $P_2 \times G$ has the generalized $p$ pebbling property. So we may assume

$$x \leq pf_{gl}(G) - (b_2 + b_4). \quad (4.12)$$

Otherwise we could move $p$ pebbles to a vertex in $\{u_2, u_4\} \times G$ which is adjacent to $(u_1, y)$. And a pebble can be moved to $(u_1, y)$. From Eq. (4.5), it follows that

$$b_2 + b_4 \leq pf_{gl}(G) - x \quad (4.13)$$

Now we will move as many pebbles as possible from $\{u_2, u_4\} \times G$ to $\{u_1, u_3\} \times G$.

We can move at least

$$\frac{(p^2 - p)f_{gl}(G) + x - b_2 - b_4}{p}$$
\[ \geq \frac{(p^2 - p)f_{gl}(G) + x - (p - 1)pf_{gl}(G) + (p - 1)x}{p} \] (from (4.13))

\[ = \frac{px}{p} = x \] pebbles to \( \{u_1, u_3\} \times G \),

yielding \( pf_{gl}(G) \) pebbles on \( \{u_1, u_3\} \times G \). Then we can move a pebble to \((u_1, y)\) and we are done.

\[ \text{Theorem 4.5.15.} \] Let \( K_{s_1, 2} \) be a bipartite graph with \( s_1 \geq 2 \) and \( G \) be a graph with the generalized \( p \) pebbling property. Then

\[ f_{gl}(K_{s_1, 2} \times G) \leq f_{gl}(K_{s_1, 2}) f_{gl}(G), \]

i.e., \( f_{gl}(K_{s_1, 2} \times G) \leq (p^2 + (p - 1)(s_1 - 2))f_{gl}(G) \) as \( p \geq n - s_1 \),

\[ f_{gl}(K_{s_1, 2} \times G) \leq (p + (p - 1)(n - 2))f_{gl}(G) \] as \( p \leq n - s_1 \).

**Proof.** We use induction on \( s_1 \) to prove the result. The result is true for \( s_1 = 2 \) by Theorem 4.5.14. We assume \( s_1 > 2 \). Let \( V(K_{s_1, 2}) = V_1 \cup V_2 \) where \( V_1 = \{v_1, v_2, \ldots, v_{s_1}\} \) and \( V_2 = \{u_1, u_2\} \). Let \( a_{1j} \) be the number of pebbles on \( \{v_j\} \times G \) with \( b_{1j} \) occupied vertices and \( a_{2i} \) be the number of pebbles on \( \{u_i\} \times G \) with \( b_{2i} \) occupied vertices. Let \( y \in G \). Suppose we start with a configuration of \( (p^2 + (p - 1)(s_1 - 2))f_{gl}(G) \) pebbles on \( (K_{s_1, 2} \times G) \).

**Case 1:** Suppose the target vertex is \((u_i, y)\) for some \( i = 1, 2 \).

Without loss of generality, we assume the target vertex is \((u_1, y)\). Let us choose \( v_j \in V_1 \) for some \( j = 1, 2, \ldots, s_1 \). Since \( G \) satisfies the generalized \( p \) pebbling property if \( \frac{a_{1j} + b_{1j}}{p} > f_{gl}(G) \) then \( p \) pebbles can be placed on \((v_j, y)\) and hence a pebble can be moved to \((u_1, y)\). Otherwise by Lemma 4.2.12 we transfer \( \frac{a_{1j} - (p - 1)b_{1j}}{p} \) pebbles to the vertices of \( \{u_1\} \times G \).
If \( \left( \sum_{k 
eq j} a_{1k} \right) + a_{21} + a_{22} + \frac{a_j - (p-1)b_{1j}}{p} \geq (p^2 + (p-1)(s_1 - 3)) f_{gl}(G) \) then by induction the result follows, since \( \{K_{s_1,2} - v_j\} \times G \) is isomorphic to \( K_{s_1-1,2} \times G \).

The only distribution from which we cannot pebble the target satisfies the inequalities

\[
\frac{(p-1)a_{1j} + (p-1)b_{1j}}{p} \leq (p-1)f_{gl}(G)
\]

(4.14)

\[
\left( \sum_{k 
eq j} a_{1k} \right) + a_{21} + a_{22} + \frac{a_j - (p-1)b_{1j}}{p} < (p^2 + (p-1)(s_1 - 3)) f_{gl}(G)
\]

(4.15)

Adding these together gives \( \sum_{k=1}^{s_1} a_{1k} + a_{21} + a_{22} < (p^2 + (p-1)(s_1 - 2)) f_{gl}(G) \).

Thus the initial distribution has lesser than \( (p^2 + (p-1)(s_1 - 2)) f_{gl}(G) \) pebbles.

**Case 2:** Suppose the target vertex is \((v_{s_1}, y)\). Suppose \( a_{1s_1} + a_{21} \geq p f_{gl}(G) \), then a pebble can be moved to \((v_{s_1}, y)\) by Theorem 4.5.2.

Let us assume \( a_{1s_1} + a_{21} < p f_{gl}(G) \). Then for some integer \( x \), \( a_{1s_1} + a_{21} = p f_{gl}(G) - x \) and \( \sum_{k=1}^{s_1-1} a_k + a_{22} = ((p-1)(s_1 - 2) + (p^2 - 3p + 2)) f_{gl}(G) + x \).

Assume \( x \leq p f_{gl}(G) - \left( \sum_{k=1}^{s_1-1} b_{1k} + b_{22} \right) \). Otherwise

\[
\sum_{k=1}^{s_1-1} a_{1k} + a_{22} > p^2 f_{gl}(G) - \sum_{k=1}^{s_1-1} (b_{1k} + b_{22})
\]

then by Theorem 4.5.10, \( p f_{gl}(G) \) pebbles can be moved to \( \{v_{s_1}, u_2\} \times G \) and we are done.

Hence \( \sum_{k=1}^{s_1-1} b_{1k} + b_{22} \leq p f_{gl}(G) - x \).

Hence we could move at least \( \frac{(p-1)(s_1-2)+(p^2-3p+2)) f_{gl}(G)+x-(\sum_{k=1}^{s_1-1} (b_{1k}+b_{22}))}{p} \)

pebbles to \( \{v_{s_1}, v_{21}\} \times G \) from \( K_{s_1-1,1} \times G \) which is isomorphic to \( \{v_1, v_2, \ldots, v_{s_1-1}, u_1\} \times G \).
i.e., we could move at least
\[
\frac{((p-1)(s_1-2) + (p^2 - 3p + 2)) f_{gl}(G) + x - p(p-1)f_{gl}(G) + (p-1)x}{p}
\]

\[
\geq x + \frac{(p-1)(s_1-4)}{p} f_{gl}(G),
\]
x pebbles to \(v_{s_1}, u_2\) while leaving \(\frac{(p-1)(s_1-4)}{p} f_{gl}(G)\) pebbles on \(\{v_1, v_2, \ldots, v_{s_1-1}, u_1\} \times G\). Hence we are done.

For \(p \leq n - s_1\), the proof is similar to the above cases.

\[\square\]

**Theorem 4.5.16.** Let \(G\) be a graph with the generalized \(p\) pebbling property. Then
\[
f_{gl}(K_{s_1,s_2,\ldots,s_r}^*\times G) \leq f_{gl}(K_{s_1,s_2,\ldots,s_r}) f_{gl}(G).
\]

**Proof.** We prove the theorem by induction on \(n\). Suppose we start with a configuration of \((p^2 + (p-1)(s_1-2)) f_{gl}(G)\) pebbles if \(p \geq n - s_1\) and \((p + (p-1)(n-2)) f_{gl}(G)\) pebbles if \(p \leq n - s_1\) on \(K_{s_1,s_2,\ldots,s_r}^*\times G\). By Theorem 4.5.15 the result is true when \(r = 2\) and \(s_2 = 2\). Therefore we assume \(r \geq 2\) and \(s_2 > 2\) if \(r = 2\).

Let \(\{v_{i_1}, v_{i_2}, \ldots, v_{i_{s_1}}\}\) be the vertices of \(C_i\) for \(i = 1, 2, \ldots, r\). Let \(a_{i_j}\) denote the number of pebbles on \(\{v_{i_j}\} \times G\) with \(b_{i_k}\) occupied vertices. Let \(y \in G\) and \((v_{i_j}, y)\) be the target vertex.

**Case 1:** Assume \(2 \leq p \leq n - s_1\). Suppose \(a_{i_j} \geq f_{gl}(G)\). Then one pebble can be moved to \((v_{i_j}, y)\). Let us assume \(a_{i_j} < f_{gl}(G)\).

Suppose there exists some \(v_{lm} \in K_{s_1,s_2,\ldots,s_r}^*\) such that \(v_{lm} \neq v_{i_j}\) and
\[
\left( \sum_{w=1}^{r} \left( \sum_{k=1}^{s_w} a_{wk} \right) - a_{lm} \right) < b_{lm}.
\]

Then \(a_{lm} = \sum_{w=1}^{r} \left( \sum_{k=1}^{s_w} a_{wk} \right) - \left( \sum_{w=1}^{r} \left( \sum_{k=1}^{s_w} a_{wk} \right) - a_{lm} \right)\)

\[
= (p + (p-1)(n-2)) f_{gl}(G) - b_{lm}
\]
Since \( \{v_{lm}\} \times G \) is isomorphic to \( G \) and it satisfies the generalized \( p \) pebbling property then we can move \( p \) pebbles to \((v_{lm}, y)\) and hence a pebble can be moved to \((v_{ij}, y)\). Suppose \( b_{lm} \leq \left( \sum_{w=1}^{r} \left( \sum_{k=1}^{s_w} a_{wk} \right) \right) - a_{lm} \).

By Lemma 4.2.12 we can transfer \( a_{lm} \) pebbles to \((K^*_{s_1, s_2, \ldots, s_r} - \{v_{lm}\}) \times G\) which is \((K^*_{s_1, s_2, \ldots, s_r-1, \ldots, s_r}) \times G\).

Therefore if \( \sum_{w=1}^{r} \left( \sum_{k=1}^{s_w} a_{wk} \right) - a_{lm} + \frac{a_{lm}-(p-1)b_{lm}}{p} \geq (p + (p-1)(n-3)) f_{gl}(G) \) pebbles, a pebble can be moved to \((v_{ij}, y)\) by induction. If we cannot pebble the target, the following inequalities are satisfied.

\[
\frac{a_{lm} + b_{lm}}{p} < f_{gl}(G) \tag{4.16}
\]

\[
\frac{(p-1)a_{lm} + (p-1)b_{lm}}{p} < (p-1)f_{gl}(G) \tag{4.17}
\]

But adding (4.16) and (4.17) we have

\[
\sum_{w=1}^{r} \left( \sum_{k=1}^{s_w} a_{wk} \right) - a_{lm} + \frac{a_{lm}-(p-1)b_{lm}}{p} \geq (p + (p-1)(n-3)) f_{gl}(G)
\]

Thus the original configuration has fewer than \((p + (p-1)(n-2)) f_{gl}(G)\) pebbles.

**Case 2:** Assume \( p \geq n - s_1 \).

The proof is similar to Case 1 and so the proof is omitted. \(\blacksquare\)
Corollary 4.5.17. Let $P_m$ be a path on $m$ vertices. Then

$$f_{gl}(K_{s_1,s_2,\ldots,s_r} \times P_m) \leq f_{gl}(K_{s_1,s_2,\ldots,s_r}) f_{gl}(P_m).$$

Proof. The Corollary follows from Theorem 4.4.4 and Theorem 4.5.16.

Corollary 4.5.18. Let $K_n$ be a complete graph on $n$ vertices. Then

$$f_{gl}(K_{s_1,s_2,\ldots,s_r} \times K_n) \leq f_{gl}(K_{s_1,s_2,\ldots,s_r}) f_{gl}(K_n).$$

Proof. The Corollary follows from Theorem 4.4.6 and Theorem 4.5.16.

Corollary 4.5.19. Let $K_{1,n} (n > 1)$ be a star. Then

$$f_{gl}(K_{s_1,s_2,\ldots,s_r} \times K_{1,n}) \leq f_{gl}(K_{s_1,s_2,\ldots,s_r}) f_{gl}(K_{1,n}).$$

Proof. The Corollary follows from Theorem 4.4.9 and Theorem 4.5.16.

Corollary 4.5.20. Let $C_n (n \geq 3)$ be a cycle on $n$ vertices. Then

$$f_{gl}(K_{s_1,s_2,\ldots,s_r} \times C_n) \leq f_{gl}(K_{s_1,s_2,\ldots,s_r}) f_{gl}(C_n).$$

Proof. The Corollary follows from Theorem 4.4.8 and Theorem 4.5.16.