Chapter 6

Maximum Independent Set Cover

Pebbling Number of a Complete Binary Tree and a Complete $m$-ary Tree

6.1 Introduction

Trees are useful in describing any structure which involves hierarchy. Familiar examples of such structures are family trees, the decimal classification of books in a library etc. Every tree can be uniquely represented by a binary tree, so that for the computer representation of a tree it is possible to consider the representation of its corresponding binary tree. More particularly, our discussion will be limited to complete binary trees because their representation and manipulation are relatively simple when we compare with general trees. Binary trees are useful in several
applications. Binary trees permit storage of data for quick access. We store each item at a leaf and access it by following the path from the root.

**Notation 6.1.1.** $p(a)$ denotes the number of pebbles placed at the vertex. Also $p(G)$ denotes the number of pebbles on the graph $G$.

### 6.2 Maximum independent set cover pebbling number of a complete binary Tree

**Definition 6.2.1.** A complete binary tree denoted by $B_n$ is a tree of height $n$ with $2^i$ vertices at distances $i$ from the root. Each vertex of $B_n$ has two “children” except for the set of $2^n$ vertices that are at distance $n$ away from the root, none of which have children. The root will be denoted by $R = R_n$.

Let us compute the maximum independent set cover pebbling number of $B_0$ in the following theorem.

**Theorem 6.2.2.** For a binary tree $B_0$, $\rho(B_0) = 1$.

**Proof.** $\rho(B_0) = 1$ is obvious. ■

**Theorem 6.2.3.** For a binary tree $B_1$, $\rho(B_1) = 6$.

**Proof.** By Theorem 5.2.2, $\rho(B_1) = \rho(P_3) = 6$. ■
Theorem 6.2.4. For a binary tree $B_2$, $\rho(B_2) = 41$.

Proof. Binary tree $B_2$ consists of left subtree $B_{12}$ induced by the vertices $v_1, v_2, v_3$ in which $v_2$ is the root vertex of $B_{12}$ and right subtree $B_{22}$ induced by the vertices $u_1, u_2, u_3$ in which $u_2$ is the root vertex of $B_{22}$.

If we place forty pebbles on $u_3$ we can pebble $u_1, v_1, v_3$ and root vertex $R_2$ of $B_2$. But $u_3$ remains unpebbled.

Hence $\rho(B_2) > 40$.

Consider the distribution of forty one pebbles on $B_2$. Based on the number of pebbles on each of the two subtrees, following three cases arise.

Case 1: Suppose there are at least six pebbles on each of the two subtrees. If the root vertex $R_2$ has a pebble, then we are done. Otherwise one subtree say $B_{22}$ contains at least fifteen of the twenty nine pebbles. There are two paths leading from the root to the bottom of the subtree. By pigeonhole principle one of the paths contains at least eight pebbles then we are done since $\rho(P_3) = 6$.

Case 2: If there are fewer than six pebbles on each of the two disjoint subtrees in $B_2$, then we can use twenty five of the thirty one pebbles on the root vertex to cover the maximum independent set of the sub trees and the root vertex $R_2$.

Case 3: Suppose that right subtree $B_{22}$ contains at least six pebbles and the left subtree has less than six pebbles. If there are nine pebbles on the root vertex $R_2$, then we are done since $\rho(B_{12} \cup R_2) = \rho(K_{1,3}) = 9$. Suppose $1 \leq \rho(R_2) \leq 8$. Then using at most $4(9 - \rho(R_2))$ pebbles we can place $9 - \rho(R_2)$ pebbles on the root vertex $R_2$. Then $B_{22}$ has $41 - \rho(R_2) - 36 + 4\rho(R_2) = 5 + 3\rho(R_2) \geq 8$.

Using at least six pebbles on $B_{22}$ we can cover maximum independent set of $B_{22}$ and hence we are done. Suppose $\rho(R_2) = 0$. We claim and prove that we can
send a pebble to $R_2$ as long as $B_{22}$ has a total of nine or more pebble on it. Suppose this is impossible. Let us distribute nine pebbles on the vertices of $B_{22}$. One path say $u_2 - u_3$ contains at least five pebbles. If there are at least six pebbles then we are done since $\rho(P_3) = 6$. If there are exactly five pebbles, on the path $u_2 - u_3$ vertex $u_1$ has at least four pebbles on it. Using exactly four pebbles from $u_1$ we can place a pebble on the root $R_2$. Then $p(u_3) \geq 5$ and hence we are done. If we use at most three pebbles to pebble the root, then we are done since $\rho(P_3) = 6$ and $u_1u_2u_3$ contains at least six pebbles. If there are at most four pebbles on the path $u_2 - u_3$, then there are at least five pebbles on $u_1$. If $p(u_2) = 2$ then we move a pebble to $R_2$ and we are done. If $p(u_2) = 1$ or $3$ then consider the following sequence of pebbling moves. $u_1 \overset{1}{\rightarrow} u_2 \overset{1 \text{ or } 2}{\rightarrow} R_2$ and we are done. Thus we had started with $35 + 6$ pebbles on $B_{22}$. Using thirty two pebbles, eight pebbles can be sent to $R_2$ with as few as nine pebbles left on $B_{22}$. With these nine pebbles we can pebble the maximum independent set of $B_{22} \cup R_2$. Hence $\rho(B_2) < 41$.

**Theorem 6.2.5.** For a binary tree $B_3$, $\rho(B_3) = 313$.

**Proof.** Binary tree $B_3$ consists of left subtree $B_{13}$ induced by the vertices, a the root vertex, $b_1$, $b_2$-the middle vertices, $C_1$, $C_2$, $C_3$, $C_4$-the bottom vertices of $B_{13}$ and right subtree $B_{23}$ induced by the vertices $d'$-the root vertex $b'_1$, $b'_2$-the middle vertices, $C'_1$, $C'_2$, $C'_3$, $C'_4$-the bottom vertices of $B_{23}$ respectively. Based upon the number of pebbles on these two subtrees following three possible cases arise.

**Case 1:** Suppose there are at least forty one pebbles on each of the two disjoint subtrees. If the root vertex $R_3$ has no pebbles, then we are done. If not, one of the subtrees say $B_{23}$ contains at least one hundred and sixteen pebbles. So one of the
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four paths leading from $\rho_3$ to the bottom of the subtree contains at least twenty nine pebbles then we are done since $\rho(P_3) = 6$ and using at most eight pebbles a pebble can be sent to $\rho_3$.

**Case 2:** If there are less than forty one pebbles on each of the two disjoint subtrees in $B_3$, then we can use one hundred and sixty four of the two hundred and thirty three pebbles on the root vertex $\rho_3$ to cover maximum independent set of $B_3$ and we are done.

**Case 3:** Assume that one of the subtree say $B_{23}$ contains at least forty one pebbles and the other subtree say $B_{13}$ contains less than forty one pebbles. Consider the root vertex $R_3$. If $R_3$ has $6 \leq p(R_3) \leq 34$, then we are done. Otherwise $0 \leq p(R_3) \leq 5$. Then using at most $8(34 - p(R_3))$ pebbles we can place $34 - p(R_3)$ pebbles on $R_3$. So number of pebbles in $B_{23}$ is $313 - p(R_3) - 8(34 - p(R_3)) \geq 41 + 8p(R_3) \geq 41$. Using these pebbles we can pebble the maximum independent set of $B_{23}$. Assume root vertex $\rho_3$ has zero pebbles. By pigeonhole principle at least one of the four paths leading from $\rho_3$ to the bottom of the subtree has at least twelve pebbles. If the path say $a'b_2'c_4'$ has at least thirteen pebbles, using at most eight pebbles we can send a pebble to the root $\rho_3$ of $B_3$. If we use exactly eight pebbles to pebble $\rho_3$, then $p(c_4') \geq 5$. From $c_4'$ a pebble can be moved to $c_3'$. If we use at most seven pebbles to send a pebble to the root $\rho_3$ then we can pebble $c_3', c_4'$ since $\rho(P_3) = \rho(c_3'b_2'c_4') = 6.$

Now $p(a') = 0$. At least one of the vertices $b_1', c_1', c_2'$ receives at least twelve pebbles. Using at most four pebbles we can move a pebble to $a'$. Then the path induced by the vertices $c_1'b_1'c_2'$ contains at least thirty two pebbles hence we are done since $\rho(P_3) = 6$. By our assumption $P_3$ has no pebbles. Also by pigeonhole principle then there exists at least two paths with twelve pebbles each since no path
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from $\rho_3$ to the bottom can have thirteen pebbles. Assume root vertex $a'$ of $B_{23}$ has no pebbles. If $p(a') = 2$, then from $a'$ we can move a pebble to $R_3$. If $p(a') = 1$, then using at most four pebbles a pebble can be moved to $a'$ from $V(B_{23}) - \{a'\}$.

Assume $p(a') = 0$. Let us assume the two paths overlap at the vertex $a'$. Let the two paths be $P_1$ induced by the vertices $a'b'_1$ and $c'_1$ and $P_2$ induced by the vertices $a', b'_2$ and $c'_4$. Using nine pebbles along $P_1$ we pebble $a', c'_1, c'_2$. Again using at least nine pebbles along $P_2$, we pebble $a' c'_2 c'_4$. Hence $a'$ receives two pebbles, then from $a'$, one pebble can be moved to $R_3$ ($49 - 24 = 25$ pebbles left over). Then among the two vertices $c'_2, c'_3$ one of the vertices say $c'_2$ receives at least thirteen pebbles. From $c'_2$ using exactly four pebbles a pebble can be sent to $a'$ and hence we are done.

So let us assume the two paths overlap at the vertices $a', b'_1$ they may be of the form $a' - b'_1 - c'_1$ and $a' - b'_1 - c'_2$. Assume $p(b'_1) = 1$, again by our assumption $p(a') = 0$. Then $p(c'_1) = p(c'_2) = 11$. Consider the following sequence of pebbling moves. $c'_1 \xrightarrow{1} b'_1 \xrightarrow{1} a' \xrightarrow{1} R_3$.

After pebbling $p(c'_1) = 5$ and $p(c'_2) = 7$

Again remaining twenty five pebbles are in the subtree induced by the vertices $b'_2, c'_3, c'_4$. Using at least six pebbles we can pebble the maximum independent set since $\rho(P_3) = 6$. Hence using $34 \times 8 = 272$ pebbles we can move thirty four pebbles to $R_3$ again using forty one pebbles we can pebble the maximum independent set of the subtree $B_{23}$. Hence we are done.
Theorem 6.2.6. For a binary tree $B_4$, $\rho(B_4) = 2505$.

Proof. Binary tree $B_4$ consists of left subtree $B_{14}$ induced by the vertices $a$-the root vertex, $b_1$, $b_2$-vertices in the row third from the bottom, $c_1$, $c_2$, $c_3$, $c_4$ vertices in the row second from the bottom, $d_1$, $d_2$, $d_3$, $d_4$, $d_5$, $d_6$, $d_7$, $d_8$ the bottom vertices and right subtree $B_{24}$ induced by the vertices $a'$-the root vertex, $b'_1$, $b'_2$ vertices in the row third from the bottom row, $c'_1$, $c'_2$, $c'_3$, $c'_4$ vertices in the row second from the bottom row, $d'_1$, $d'_2$, $d'_3$, $d'_4$, $d'_5$, $d'_6$, $d'_7$, $d'_8$-bottom vertices of $B_{24}$. Based on the distribution of pebbles on the vertices of the subtrees $B_{14}$ and $B_{24}$, following three cases arise.

Case 1: Suppose there are at least three hundred and thirteen pebbles on each of the two disjoint subtrees. If the root vertex $R_4$ has pebbles then we are done. Otherwise one subtree contains at least one thousand eight hundred and seventy nine pebbles, so one of the eight paths leading from $R_4$ to the bottom of the subtree contains at least two hundred and thirty five pebbles, then we are done since $\rho(P_5) = 21$.

Case 2: If there are fewer than three hundred and thirteen pebbles on each of the two subtrees then we can use one thousand two hundred and fifty three of the one thousand eight hundred and eighty one pebbles, we cover maximum independent set of each of the subtrees and root vertex $R_4$.

Case 3: Suppose subtree $B_{24}$ has $\rho(B_{3})$ pebbles and $B_{14}$ has no pebbles.

Suppose $p(R_4) \geq 137$. Then we are done. Suppose $\rho_4$ has $1 \leq p(R_4) \leq 136$ pebbles then using $16(137 - p(R_4))$ pebbles, we can move one hundred and thirty seven pebbles to $R_4$. 

Suppose $p(R_4) \geq 137$. Then we are done. Suppose $\rho_4$ has $1 \leq p(R_4) \leq 136$ pebbles then using $16(137 - p(R_4))$ pebbles, we can move one hundred and thirty seven pebbles to $R_4$. 

Suppose $p(R_4) \geq 137$. Then we are done. Suppose $\rho_4$
Then \( 2505 - p(R_4) = 16(137 - p(R_4)) \geq 313 + 15p(R_4) \), using three hundred and thirteen pebbles we can pebble the maximum independent set of \( B_{24} \). Suppose \( p(R_4) = 0 \), \( p(a') = 0 \). We claim and prove that using three hundred and twenty nine pebbles we can move one pebble to \( R_4 \) and also pebble the maximum independent set of \( B_{24} \). If we distribute three hundred and twenty nine pebbles on the vertices of \( B_{24} \), then one of the eight paths leading from \( R_4 \) to the bottom of the subtree contains at least forty one pebbles. Let the path \( b'_2c'_4d'_8 \) contains at least forty one pebbles and hence we can pebble the maximum independent set of the right subtree \( B^*_{24} \) of \( B_{24} \). Using the remaining two hundred and eighty eight pebbles we can pebble the left subtree \( B'_{24} \) of \( B_{24} \) and the root vertex \( R_4 \). If there are less than forty one pebbles on the path induced by the vertices \( b'_2c'_4d'_8 \), from the subtree \( B'_{24} \) of \( B_{24} \) we can move a pebble to \( R_4 \) and \( 15 \leq x \leq 18 \) pebbles can be moved to the root vertex of \( B^*_{24} \). Using \( 41 + 8 + 240 = 289 \) pebbles in \( B'_{24} \), we can cover maximum independent set of \( B^*_{24} \) and a pebble can be moved to \( R_4 \) and also pebbles can be moved to \( B^*_{24} \).

Hence using two thousand one hundred and ninety two pebbles we can bring one hundred and thirty seven pebbles to \( R_4 \) and using three hundred and thirteen pebbles we can pebble the maximum independent set of subtree \( B_{24} \). Hence \( \rho(B_4) \leq 2505 \).

**Theorem 6.2.7.** For a binary tree \( B_n \ (n \geq 3) \),

\[
\rho(B_n) = \sum_{k=0}^{\left\lfloor \frac{n-4}{2} \right\rfloor} 2^{n-2k-1}2^{2n-2k} + \sum_{i=0}^{\left\lfloor \frac{n-4}{2} \right\rfloor} \left( 2^{2i} \sum_{j=1}^{n-2i-1} 2^{j-1}2^{2i+2j} \right) + \gamma_n
\]

\[
= S_{1,n} + S_{2,n} + S_{3,n} \quad \text{(say)}
\]
where \( S_{i,n} \) denotes the \( i \)th term of the above sum and \( \gamma = \gamma_n = 2^{2\left[ \frac{n-2}{2} \right]} + 2 \) if \( n \) is even and \( \gamma = 0 \) otherwise.

**Proof.** There are \( 2^{n-1} \) parents in the row second from the bottom row and each parent has a pair of children. Choose the child from the last pair and place all of the \( \rho(B_n) - 1 \) pebbles on it. Name this vertex as \( v \). In this distribution in order to cover the maximum independent set of \( B_n \) our aim is to place one pebble on every vertex in the bottom row and place one pebble on every vertex in the row third, fifth, seventh etc from the bottom row. Consider \( B_n \). There are two \( B_{n-1} \) subtrees connected to the root \( R_n \) of \( B_n \). Name the subtree which does not contain \( v \) as \( B_{n-1}^{(2)} \) and name the subtree which contain \( v \) as \( B_{n-1}^{(1)} \).

Consider the subtree \( B_{n-1}^{(1)} \). We need \( \left\lfloor \frac{n-1}{2} \right\rfloor \sum_{i=0}^{n-2i-1} (2^{2i} + \sum_{j=1}^{n-2i-1} 2^{j-1}2^{2i+2j}) \) pebbles to cover the maximum independent set of \( B_{n-1}^{(1)} \). Consider the subtree \( B_{n-1}^{(2)} \). In the bottom row, there are \( 2^{n-1} \) vertices to be pebbled. For each vertex it costs \( 2^{2n} \) pebbles from \( v \), hence to cover the bottom row, we need \( 2^{3n-1} = 2^n - 1 \) \( 2^{n-0} \) pebbles that correspond to \( k = 0 \) term of the sum \( \left\lfloor \frac{n-1}{2} \right\rfloor \sum_{k=0}^{n-2k-1} (2^{n-2k-1}2^{2n-2k}) \). So we need \( \left\lfloor \frac{n-1}{2} \right\rfloor \sum_{k=0}^{n-2k-1} (2^{n-2k-1}2^{2n-2k}) \) pebbles to cover the maximum independent set of vertices in \( B_{n-1}^{(2)} \).

Hence the maximum independent set of \( B_n \) is covered if \( n \) is odd.

If \( n \) is even, root vertex \( R_n \) of \( B_n \) is also included in the maximum independent set of \( B_n \). Hence \( S_{2,n} + S_{3,n} \) pebbles are needed to cover the maximum independent set of \( B_{n-1}^{(1)} \) and root vertex \( R_n \).

If we place \( \rho(B_n) - 1 \) pebbles on \( v \), \( v \) is left unpebbled.
Hence $\rho(B_n) > S_{1,n} + S_{2,n} + S_{3,n}$. Next we prove by induction that $\rho(B_n)$ pebbles are sufficient to cover the maximum independent set of vertices in $B_n$.

By Theorem 6.2.5, $\rho(B_3) = 313$ and by Theorem 6.2.6, $\rho(B_4) = 2505$. Hence result is true for $n = 3, 4$. Assume the result is true till $B_{n-1}$. We will consider the three possible cases based on the number of pebbles on these two $B'_{n-1}$s.

**Case i:** Suppose each of the subtree $B_{n-1}$ contains at least $\rho(B_{n-1})$ pebbles. In this case each of the two subtrees $B_{n-1}$ needs at most $\rho(B_{n-1})$ pebbles for its maximum independent set of vertices to be covered and hence maximum independent set of vertices in $B_n$ is also covered if $n$ is odd.

Consider $\frac{\rho(B_n)}{2} - \rho(B_{n-1}) > 2^{3n-3}$. One of the subtrees $B^{(1)}_{n-1}$ contains at least $2^{3n-3}$ extra pebbles, so at least one of the $2^{n-1}$ paths leading to the root vertex from the bottom of the subtree has at least $\frac{2^n}{2}$ pebbles.

For $n \geq 3$, this number exceeds $2^n$, so we can pebble the root vertex $R_n$ if $n$ is even. Hence maximum independent set of vertices of $B_n$ is covered in both the cases.

**Case ii:** Suppose there are less than $\rho(B_{n-1})$ pebbles on each of the two $B_{n-1}$’s. In this case there are at least $\rho(B_n) - 2\rho(B_{n-1}) + 2$ pebbles on the root. We claim that this number is at least as large as $4\rho(B_{n-1}) + 1$, a number that would allow us to move $\rho(B_{n-1})$ pebbles onto the root of each subtree which leaving one pebble on the root if $n$ is even thus covering the maximum independent set of $B_n$. It is enough to show that $\rho(B_n) \geq 6\rho(B_{n-1})$. We have $\rho(B_n) \geq 2^{3n-1}$ by considering only the $k = 0$ term of $S_{1,n}$. 
Also \( 6S_1,n-1 = 6 \left( \sum_{k=0}^{\lfloor n/2 \rfloor} 2^{3n-4-4k} \right) \)
\[ \leq 6 \left( \frac{2^{3n}}{16} \right) \]

Again
\[
6(S_2,n-1) = 6 \sum_{i=0}^{\lfloor n/2 \rfloor} 2^{2i} + 6 \sum_{i=0}^{\lfloor n/2 \rfloor} 2^{2i} \sum_{j=1}^{n-2i} 2^{i-1} 2^{2i+j}
\]
\[ \leq 6 \left( \frac{2^n - 1}{3} \right) + 6 \left( \sum_{i=0}^{\lfloor n/2 \rfloor} 2^{2i} \left( 2^{2i+2} + 2^{2i+5} + 2^{2i+8} + 2^{3n-4i-7} \right) \right) \]
\[ = 6 \left( \frac{2^n - 1}{3} \right) + 6 \left( \sum_{i=0}^{\lfloor n/2 \rfloor} 2^{2i} 2^{2i}(2^2 + 2^5 + 2^8 + \ldots + 2^{3n-6i-7}) \right) \]
\[ = 6 \left( \frac{2^n - 1}{3} \right) + 24 \left( \sum_{i=0}^{\lfloor n/2 \rfloor} 2^{4i} \left( 1 + 8 + 2^2 + \ldots + 8^{n-2i-3} \right) \right) \]
\[ = 6 \left( \frac{2^n - 1}{3} \right) + 24 \sum_{i=0}^{\lfloor n/2 \rfloor} 2^{4i} \left( \frac{8^{n-2i-2} - 1}{7} \right) \]
\[ = 6 \left( \frac{2^n - 1}{3} \right) + \frac{24}{7} \sum_{i=0}^{\lfloor n/2 \rfloor} (2^{3n-2i-6} - 2^{4i}) \]
\[ = 2^{n+1} - 2 + \frac{24}{7} \left( \frac{2^{3n-4}}{3} \left( \frac{2^{n+4} - 1}{2^{n+4}} - \frac{1}{15} (4^n - 1) \right) \right) \]
\[ = 2^{n+1} - 2 + \frac{24}{7} \left( \frac{2^{2n}}{3 \cdot 256} (2^{n+4} - 1) - \frac{1}{15} (4^n - 1) \right) \]
\[ = 2^{n+1} - 2 + \left( \frac{2^{3n+4} - 2^{2n}}{7 \cdot 32} \right) - \frac{8}{7 \cdot 5} (4^n - 1) \]
\[ = 2^{n+1} - 2 + \frac{1}{7} (5 \cdot 2^{3n+4} - 5 \cdot 2^{2n} - 256(4^n - 1)) \]
\[ = 2^{n+1} - 2 + \frac{1}{7} \left( \frac{5 \cdot 2^{3n}}{16} - 5 \cdot 2^{2n} - 256(4^n - 1) \right) \]
Hence

\[ 6(S_{1,n-1}) + 6(S_{2,n-1}) \leq 6 \left( \frac{2^3n}{16} \right) + 2^{n+1} - 2 + \frac{1}{7} \left( \frac{5 \cdot 2^{3n}}{16} - 5 \cdot 2^{2n} - 256(4^n - 1) \right) \]

The above equation holds if

\[ 2^{3n-1} \geq 6 \left( \frac{2^{3n}}{16} \right) + 2^{n+1} - 2 + \frac{1}{7} \left( \frac{5 \cdot 2^{3n}}{16} - 5 \cdot 2^{2n} - 256(4^n - 1) \right) \]

which holds for \( n \geq 3 \).

**Case iii:** Suppose there are at least \( \rho(B_{n-1}) \) pebbles on one of the \( B_{n-1} \) call it \( B_{n-1}^{(1)} \) and less than \( \rho(B_{n-1}) \) on \( B_{n-1}^{(2)} \).

**Case 3a:** If \( n \) is odd, \( \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} (2^{2i} + \sum_{j=1}^{\lfloor \frac{n-2i-1}{2} \rfloor} 2^{j-1} \cdot 2^{i+2j}) \) pebbles are required to cover the maximum independent set of vertices in \( B_{n-1}^{(1)} \). Also we can supply at least one pebble to the root vertex of \( B_n \), for every \( 2^n \) extra pebbles on \( B_{n-1}^{(1)} \). Hence \( 2^n \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (2^{n-2k-1} \cdot 2^{2n-2k}) \) pebbles can be brought to the root vertex \( R_n \) using \( 2^n \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (2^{n-2k-1} \cdot 2^{2n-2k}) \) pebbles we can pebble the maximum independent set of vertices in \( B_{n-1}^{(2)} \).

**Case 3b:** If \( n \) is even, \( \gamma_n - \gamma_{n-1} = 2^n \) using \( 2^n \) pebbles we can move a pebble to the root vertex \( R_n \) of \( B_n \). Again using \( 2^n + \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} (2^{2i} + \sum_{j=1}^{\lfloor \frac{n-2i-1}{2} \rfloor} 2^{j-1} \cdot 2^{i+2j}) \) pebbles we can pebble the root vertex \( R_n \) and maximum independent set of \( B_{n-1}^{(1)} \).

Again for every \( 2^n \) extra pebbles on \( B_{n-1}^{(1)} \) we can bring \( 2^n + \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} 2^{n-2k-1} \cdot 2^{2n-2k} \) pebbles to the root vertex \( R_n \) of \( B_n \). Again using \( \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} 2^{n-2k-1} \cdot 2^{2n-2k} \) pebbles we can pebble the maximum independent set of \( B_{n-1}^{(2)} \). Hence in both cases,

\[ \rho(B_n) \leq \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} 2^{n-2k-1} \cdot 2^{2n-2k} + \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} (2^{2i} + \sum_{j=1}^{\lfloor \frac{n-2i-1}{2} \rfloor} 2^{j-1} \cdot 2^{i+2j}) + \gamma_n \]
where
\[ \gamma_n = \begin{cases} 2^{2\lfloor \frac{n-2}{2} \rfloor + 2}, & \text{if } n \text{ is even} \\ 0, & \text{otherwise}. \end{cases} \]

We have determined the maximum independent set cover pebbling number of a binary tree.

Next, we compute the maximum independent set cover pebbling number of a complete \( m \)-ary tree.

### 6.3 The Maximum independent set cover pebbling number of a complete \( m \)-ary tree

**Definition 6.3.1.** A complete \( m \)-ary tree, denoted by \( M_n \), is a tree of height \( n \) with \( m^i \) vertices at distances \( i \) from the root. Each vertex of \( M_n \) has \( m \) children except for the set of \( m^n \) vertices that are at distance \( n \) away from the root, none of which have children. The root is denoted by \( R_n \). Or simply a complete \( m \)-ary tree with height \( n \), denoted by \( M_n \), is an \( m \)-ary tree satisfying that each vertex \( v \) not in the \( n \)th level.

**Theorem 6.3.2.** (i) \( \rho(M_0) = 1 \) (obvious).

(ii) \( \rho(M_1) = 4m - 3 \) (\( m \geq 3 \)) and if \( m = 2 \) then \( \rho(M_1) = 6 \). Since, for \( m \geq 3 \),
\[ M_1 \equiv K_{1,m} \text{ and for } m = 2, M_1 \equiv P_3, \text{ the path of length two}. \]

(iii) \( \rho(M_2) = 16m^2 - 12m + 1 \).

(iv) \( \rho(M_3) = 64m^3 - 48m^2 + 4m - 15 \) (\( m \geq 3 \)).

(v) \( \rho(M_4) = 256m^4 - 192m^3 + 16m^2 - 60m + 1 \).
Proof of (iii): Note that $M_2$ contains $m - M_1$’s as subtrees which are all connected to the root $R_2$ of $M_2$. Let $R_{11}, R_{12}, \ldots, R_{1m}$ be the root of the $m - M_1$’s (say $M_{11}, M_{12}, \ldots, M_{1m}$). In general, $M_n$ contains $m - M_{n-1}$’s as subtrees which are all connected to the root $R_n$ of $M_n$. Let $R_{(n-1)1}, R_{(n-1)2}, \ldots, R_{(n-1)m}$ be the root of the $m - M_{(n-1)}$’s. Choose the rightmost vertex of this subtree, label it by $v$. Put $16m^2 - 12m$ pebbles on this vertex. Then we cannot cover the maximum independent set of $M_2$. Thus $\rho(M_2) \geq 16m^2 - 12m + 1$.

Now consider the distribution of $16m^2 - 12m + 1$ pebbles on the vertices of $M_2$. According to the distribution of these amounts of pebbles, we find the following cases:

Case 1: $p(M_{1i}) \geq 4m - 3$, where $1 \leq i \leq m$.

Clearly we are done if $p(R_2) \geq 1$. So assume that, $p(R_2) = 0$. This implies that $\sum_{i=1}^{m} p(M_{1i}) = 16m^2 - 12m + 1$. Any one of the $m^2$ paths (of length two) leading from the root $R_2$ to the bottom of $M_2$ must contain at least four pebbles and hence we are done, since any one of the subtree contains at least $\left\lceil \frac{16m^2 - 12m + 1}{m} \right\rceil \geq 16m - 12 + 1$ pebbles.

Case 2: $p(M_{1i}) \leq 4m - 4$, for all $i (1 \leq i \leq m)$.

This implies that $p(R_2) \geq 16m^2 - 12m + 1 - m(4m - 4) = 12m^2 - 8m + 1$. We need $2m(4m - 3) + 1$ pebbles at $R_2$. But $p(R_2) - 2m(4m - 3) - 1 > 0$.

Case 3: $p(M_{1i}) \geq 4m - 3$ for some $i (1 \leq i \leq m)$.

Let $t \geq 1$ subtrees of $M_2$ contains at least $4m - 3$ pebbles. Note that, for every subtree (except one subtree that contains $4m-3$ or more pebbles), we have $16m$ pebbles to cover its maximum independent set.

Let $p(M'_{1j}) = a_j$ where $a_j \leq 4m - 4$. Thus, to cover the maximum independent set of the subtree $M'_{1j}$, we have another $16m - a_j$ pebbles somewhere on the graph.
So, we can send \( \left\lfloor \frac{16m - a_j}{4} \right\rfloor \geq 4m - \frac{a_j}{4} \) pebbles to the root \( R_2 \) and then we move \( 2m - \frac{a_j}{8} \) pebbles to the root \( R'_{1j} \) of \( M'_{1j} \). Thus \( M'_{1j} \) contains \( a_j + 2m - \frac{a_j}{8} = 2m + \frac{7}{8}a_j \). But these numbers of pebbles are enough to cover the maximum independent set of \( M'_{1j} \), or the value of \( 2m + \frac{7}{8}a_j \geq 4m - 3 \), and hence we are done. So using \( (m - t)(16m - a_j) - \sum_{i=1}^{t} a_j \) pebbles, we cover the maximum independent set of the \( (m - t) \) subtrees that contains \( a_j \) pebbles. So we have at least \( (t - 1)16m + 4m + 1 \) pebbles on the \( t \)-subtrees plus \( R_2 \) that contains \( 4m - 3 \) or more pebbles. If \( p(R_2) \geq 1 \) then we are done. Otherwise we can always move a pebble to \( R_2 \) using at most four pebbles from the remaining pebbles on the \( t \)-subtrees.

**Proof of (iv):** Clearly, \( M_3 \) contains \( m - M_2 \)'s as subtrees which are all connected to the root \( R_3 \) of \( M_3 \). Consider the rightmost bottom vertex, say \( v \), of \( M_3 \) and put \( 64m^3 - 48m^2 + 4m - 16 \) pebbles on the vertex \( v \). Then we cannot cover the maximum independent set of \( M_3 \). Thus \( \rho(M_3) \geq 64m^3 - 48m^2 + 4m - 15 \).

Now consider the distribution of \( 64m^3 - 48m^2 + 4m - 15 \) pebbles on the vertices of \( M_3 \). According to the distribution of these amounts of pebbles, we find the following cases:

**Case 1:** \( p(M_{2i}) \geq \rho(M_2) \) where \( 1 \leq i \leq m \).

Clearly we are done if \( p(R_2) = 0 \), or 2 or \( p(R_2) \geq 4 \). So assume that \( p(R_2) = 1 \) or 3. This implies that, \( \sum_{i=1}^{m} p(M_{2i}) \geq 64m^3 - 48m^2 + 4m - 18 \). So, any one of the path (of length three) leading from the root \( R_3 \) to the bottom row of \( M_3 \) must contain at least eight pebbles. Thus we move a pebble to \( R_3 \) and hence we are done.

**Case 2:** \( p(M_{2i}) < \rho(M_2) \) where \( 1 \leq i \leq m \).

We need \( 2mp(M_2) + 5 \) pebbles on the root vertex \( R_3 \) of \( M_3 \). We have \( \rho(M_3) - mp(M_2) + m \) pebbles on the root vertex \( R_3 \). But, \( \rho(M_3) - mp(M_2) + m - (2mp(M_2) + \frac{15}{8}a_j) \geq 16m - a_j \geq 4m - 3 \), and hence we are done.
5) \geq 0. Since, \( \rho(M_3) = 64m^3 - 48m^2 + 4m - 15, \rho(M_2) = 16m^2 - 12m + 1 \) and \( m \geq 3 \).

**Case 3:** \( \rho(M_{2i}) \geq \rho(M_2) \) for some \( i \) (\( 1 \leq i \leq m \)).

Let \( t \geq 1 \) subtrees contains \( \rho(M_2) \) or more pebbles. Label those subtrees by \( M_{2i} \) (\( 1 \leq i \leq t \)) and label the other subtrees by \( M'_{2j} \) (\( 1 \leq i \leq m - t \)). Also, let \( p(M'_{2j}) = a_j \) where \( a_j < \rho(M_2) \). Note that, we have usually \( (64m^2 + 16)(m - 1) \) pebbles each to cover the maximum independent set of \( M_{2i} \)'s and \( M'_{2j} \)'s, except one subtree \( M_{2k} \) (\( 1 \leq k \leq t \)) that contains \( \rho(M_2) \) or more pebbles.

Since \( a_j < \rho(M_2) \), we have another \( 64m^2 + 16 - a_j \) pebble that are somewhere in the graph \( M_3 \) to cover the maximum independent set of \( M'_{2j} \). So we can send \( \left\lfloor \frac{64m^2 + 16 - a_j}{8} \right\rfloor \geq 8m^2 + 2 - \frac{a_j}{8} \) pebbles to the root \( R_3 \) and then we move \( 4m^2 + 1 - \frac{a_j}{16} \) pebbles to the root \( R'_{2j} \) of \( M'_{2j} \). Thus, \( M'_{2j} \) contains \( 4m^2 + 1 + \frac{15}{16}a_j \) pebbles. But these number of pebbles are at least \( \rho(M_2) \) or it is enough to cover the maximum independent set of \( M'_{2j} \) using the pebbles at \( R'_{2j} \) plus \( a_j \) pebbles. Thus the \( t \)-subtrees \( M_{2i} \) plus \( R_3 \) contains \( (64m^2 + 16)(t - 1) + 16m^2 - 12m + 1 \) or more pebbles. We know that \( p(M_{2i}) \geq \rho(M_2) \) where \( 1 \leq i \leq t \). Let \( p(R_3) = 1 \) or 3 (Otherwise, we are done). We can move a pebble to \( R_3 \), using at most eight pebbles from the subtree that contains \( 16m^2 - 12m + 9 \) pebbles or more. And hence we are done.

**Proof of (v):** Consider the rightmost bottom vertex, say \( v \), of \( M_4 \), and put \( 256m^4 - 192m^3 + 16m^2 - 60m \) pebbles. Then we cannot cover the maximum independent set of \( M_4 \). Thus, \( \rho(M_4) \geq 256m^4 - 192m^3 + 16m^2 - 60m + 1 \).

Now consider the distribution of \( 256m^4 - 192m^3 + 16m^2 - 60m + 1 \) pebbles on the vertices of \( M_4 \). According to the distribution of these amounts of pebbles, we find the following cases:
Case 1: \( p(M_{3i}) \geq \rho(M_3) \) for all \( i \) (1 \( \leq i \leq m \)).

Clearly we are done if \( p(R_4) \geq 1 \). So assume that \( p(R_3) = 0 \). This implies that 
\[ \sum_{i=1}^{m} p(M_{3i}) = \rho(M_4) = 256m^4 - 192m^3 + 16m^2 - 60m + 1. \]
So any one of the \( m^4 \) paths (of length four) leading from the root \( R_4 \) to the bottom row of \( M_4 \) contains at least sixteen ‘extra’ pebbles. Thus we can move a pebble to \( R_4 \) and hence we are done.

Case 2: \( p(M_{3i}) < \rho(M_3) \) for all \( i \) (1 \( \leq i \leq m \)).

We need \( 2mp(M_3) + 1 \) pebbles on the root vertex \( R_4 \) of \( M_4 \). We have \( \rho(M_4) - m[\rho(M_3) - 1] \) pebbles on the root vertex \( R_4 \). Since, \( \rho(M_4) = 256m^4 - 192m^3 + 16m^2 - 60m + 1 \), \( \rho(M_3) = 64m^3 - 48m^2 + 4m - 15 \) and \( m \geq 3 \), we get \( p(R_3) \geq 2mp(M_3) + 1 \) and hence we are done.

Case 3: \( p(M_{3i}) \geq \rho(M_3) \) for some \( i \).

Similar to Case (iii) of previous theorems; using the hints, from that \( 256m^3 + 64m \) pebbles, we can send \( \left\lfloor \frac{256m^3 + 64m - a_j}{16} \right\rfloor \geq 16m^3 + 4m - \frac{a_j}{16} \) to the root \( R_4 \) of \( M_4 \).

Theorem 6.3.3. For a complete \( m \)-ary tree \( M_n \) (\( n \geq 3 \)), the maximum independent set cover pebbling number is given by,

\[ \rho(M_n) = (m - 1)P + Q + \gamma_n = S_{1,n} + S_{2,n} + S_{3,n} \text{ where} \]

\[ P = \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} m^{n-2^k-1}2^{2n-2k}, \]

\[ Q = \sum_{i=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \left( 2^{2i} + (m - 1) \sum_{j=1}^{n-2^i-1} m^{j-1}2^{2i+2j} \right) \text{ and} \]

\[ \gamma_n = \begin{cases} 2^n, & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd}. \end{cases} \]
Proof. Consider the rightmost vertex of $M_n$, say $v$, and put $\rho(M_n) - 1$ pebbles on the vertex $v$. Then we cannot cover a maximum independent set of $M_n$. Thus the lower bound follows.

We prove the upper bound of $\rho(M_n)$ by induction on $n$. For $n = 3$ and $n = 4$, this theorem is true by previous theorem (iv) and (v). So assume the result is true for the complete $m$-ary tree $M_{n-1}$ ($n \geq 5$).

Consider the distribution of $\rho(M_n)$ pebbles on the vertices of $M_n$. According to the distribution of these amounts of pebbles, we find the following cases:

**Case 1:** $\rho(M_{n-1}\vert_i) < \rho(M_{n-1})$ for all $i$ ($1 \leq i \leq m$).

We need, $2m\rho(M_{n-1}) + 5$ pebbles on the root $R_n$, to cover the maximum independent set of $M_n$. We have to prove that $\rho(M_n) - m\rho M_{n-1} + m \geq 2m\rho(M_{n-1}) + 5$.

It is enough to prove that, $\rho(M_n) \geq 3m\rho M_{n-1} + 2$ (for $m \geq 3$).

\[
\rho(M_n) \geq 3m\rho M_{n-1} + 2. \quad (6.1)
\]

From the 1st term, by considering $k = 0$ we get,

\[
\rho(M_n) \geq (m - 1)m^{n-1}2^{2n}. \quad (6.2)
\]

\[
S_{1,n-1} = (m - 1) \sum_{k=0}^{\left\lfloor \frac{n-2}{2} \right\rfloor} m^{n-2k-2}2^{2n-2k-2}
= (m - 1)(m^{n-2}2^{2n-2}) \sum_{k=0}^{\left\lfloor \frac{n-2}{2} \right\rfloor} \frac{1}{m^k2^k}
\leq \frac{8}{7}(m - 1)(m^{n-2})(2^{2n-2}) \quad (6.3)
\]

\[
S_{2,n-1} = \sum_{i=0}^{\left\lfloor \frac{n-2}{2} \right\rfloor} \left[ 2^{2i} + (m - 1) \sum_{j=1}^{n-2i-2} m^{j-1}2^{2i+2j} \right]
= \sum_{i=0}^{\left\lfloor \frac{n-2}{2} \right\rfloor} 2^{2i} + (m - 1) \sum_{i=0}^{\left\lfloor \frac{n-2}{2} \right\rfloor} 2^{2i} \sum_{j=1}^{n-2i-2} m^{j-1}2^j
\]

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$\leq \left( \frac{2^n}{3} \right)^{\frac{n-2}{2}+1} - \left( \frac{m-2}{m-1} \right) \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} 2^i \left( \frac{m^{n-2i-2}}{m-1} \right) \left( \frac{4^{n-2i-2}}{3} \right)$

$\leq \frac{2^n}{3} + \frac{4(m-1)(m^{n-2})(4^{n-2})}{3(m-1)} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} 2^{2i} m^{-2i-2} 4^i$

$\leq \frac{2^n}{3} + \frac{(m^{n-2})(4^{n-2})}{3} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{m^{2i}2^i}$

$S_{2,n-1} \leq \frac{2^n}{3} + \frac{4(m^{n-2})(4^{n-2})}{11}$ (6.4)

and

$S_{3,n-1} = \begin{cases} 0, & \text{if } n \text{ is even} \\ 2^{n-1}, & \text{if } n \text{ is odd.} \end{cases}$ (6.5)

Eqs. (6.2) through (6.5) show that (6.1) holds if,

$(m-1)(m^{n-1}2^{2n}) \geq 3m \left[ \frac{8}{7}(m-1)(m^{n-2}2^{2n-2}) \right.

+ \frac{2^n}{3} + \frac{4(m^{n-2})(4^{n-1})}{11} + 2^{n-1} \big] + 2$

$(m-1)(m^{n-1}2^{2n}) \geq \frac{24}{7}(m-1)(\lfloor m^{n-1}4^{n-1} \rfloor) + m2^n

+ \frac{12(m^{n-1})(4^{n-1})}{11} + 3(m^{n-1}) + 2$

$(m-1) \geq \frac{24(m-1)}{7(4)} + \frac{5(2^{n-1})}{m^{n-2}(4^n)} + \frac{12}{44} + \frac{1}{m^{n-1}(4^n)}$

$(m-1) - \frac{24(m-1)}{7(4)} - \frac{12}{44} \geq \frac{5(2^{n-1})}{m^{n-2}(4^n)} + \frac{2}{m^{n-1}(4^n)}$

which holds for $m \geq 3$ and $n \geq 5$. Also, $\rho(M_n) \geq 3m\rho(M_{n-1}) + 2$ for $n = 3$ and $n = 4$.

Case 2: $\rho(M_{(n-1)i}) \geq \rho(M_{n-1})$ for all $i$ ($1 \leq i \leq m$).
Subcase 2.1: \( n \) is odd.

If \( p(R_n) = 0, 2 \) or \( p(R_n) \geq 4 \) then clearly we are done. So assume that \( p(R_n) = 1 \) or \( 3 \). Then, \( \rho(M_n) - 3 \) or more pebbles on the \( m(M_{n-1}) \)'s. We know that, \( \rho(M_n) \geq 3m\rho(M_{n-1}) + 2 \) and \( \rho(M_{n-1}) \geq (m-1)(m^{n-2})(2^{2n-2}) \). We have, \( \rho(M_n) - m\rho(M_{n-1}) \) extra pebbles on the vertices of \( V(M_n) - \{R_n\} \). Thus at least one sub-tree \( M_{(n-1)i} \) contains \( 2\rho(M_{n-1}) \geq 2(m-1)(m^{n-2})(2^{2n}) \) extra pebbles, so at least one of the \( m^{n-1} \) paths leading to the root \( R_n \) from the bottom of the subtree has at least \( 2^n \) pebbles and hence we are done.

Subcase 2.2: \( n \) is even.

If \( p(R_n) \geq 1 \) then we are done. So assume that \( p(R_n) = 0 \). Like, Subcase 2.1, at least one of the \( m^{n-1} \) paths has \( 2^n \) or more pebbles and hence we are done.

Case 3: \( \rho(M_{(n-1)i}) \geq \rho(M_{n-1}) \) for some \( i \).

Let \( t \geq 1 \) subtrees contain \( \rho(M_{n-1}) \) or more pebbles. Label those subtrees by \( M_{(n-1)i} \) \( (1 \leq i \leq t) \) and label the other subtrees by \( M'_{(n-1)j} \) \( (1 \leq j \leq m-t) \). Also let \( \rho(M'_{(n-1)j}) = a_j \) where \( a_j < \rho(M_{n-1}) \) and \( 1 \leq j \leq m-t \). Clearly we can supply at least one pebble to the root \( R_n \) of \( M_n \) for every \( 2^n \) extra pebbles on \( M_{(n-1)i} \) \( (1 \leq i \leq t) \). Also, having one additional pebble in \( M'_{(n-1)j} \) \( (1 \leq j \leq m-t) \) is equivalent to have at least one pebble on the root vertex of \( R_n \) of \( M_n \).

Note that, we have usually used \( P \) pebbles each to cover the maximum independent set of \( M_{(n-1)i} \) \( (1 \leq i \leq t) \) and \( M'_{(n-1)j} \) \( (1 \leq j \leq m-t) \), except one subtree, say \( M_{(n-1)k} \), that contains \( \rho(M_{n-1}) \) or more pebbles. Since \( a_j < \rho(M_{n-1}) \), we have \( P - a_j \) extra pebbles, that are in somewhere of the graph \( M_n \), to cover the maximum independent set of \( M'_{(n-1)j} \). So we can send \( \frac{a_j}{2^{n+1}} \) pebbles to the root vertex of \( R'_{(n-1)j} \) of \( M'_{(n-1)j} \). Thus \( M'_{(n-1)j} \) contains
a_j + \left\lceil \frac{n-2}{2} \right\rceil \sum_{k=0}^{4} m^{n-2k-1} 2^{n-2k-1} - \frac{a_j}{2^{n+1}} \right \rceil \text{ pebbles. But these amounts of pebbles are at least } \rho(M_{(n-1)}) \text{ or it is enough to cover the maximum independent set of } M'_{(n-1)} \text{ using the pebbles at } R'_{(n-1)} \text{ plus } a_j \text{ pebbles. Thus the } t \text{-subtrees } M_{2i} \text{ (} 1 \leq i \leq t \text{) plus } R_3 \text{ contains } (t-1) \left\lceil \frac{n-2}{2} \right\rceil \sum_{k=0}^{4} m^{n-2k-1} 2^{n-2k-1} + Q + \gamma_n \text{ or more pebbles. We know that } p(M_{(n-1)}) \geq \rho(M_{n-1}) \text{ where } 1 \leq i \leq t.

Subcase 3.1: } n \text{ is odd.}

Let } p(R_n) = 1 \text{ or } 3 \text{ (otherwise we are done easily). Then we can move a pebble to } R_n, \text{ using at most } 2^n \text{ pebbles from the subtree that contains at least } \rho(M_{n-1}) + 2^n \text{ pebbles and hence we are done (since } \rho(M_{n-1}) \geq (m-1)(m^{n-2})(2^{n-2}) \text{).}

Subcase 3.2: } n \text{ is even.}

Let } p(R_n) = 0 \text{ (otherwise we are done). Like the Subcase 3.1, we can move a pebble to } R_n, \text{ using at most } 2^n \text{ pebbles (from the subtree that contains } \rho(M_{n-1}) + 2^n \text{ pebbles or more).}

We have determined the maximum independent set cover pebbling number of a complete binary tree and a complete } m \text{-ary tree.}