Appendix B

Friedmann equation in the braneworld

The standard model of cosmology is based on Friedmann-Robertson-Walker space-time. So, it is natural to choose the bulk space-time in such a way that FRW model is recovered from it. In 2000, P.Binétry, C.Deffayet, U.Ellwanger and D.Langlois considered a 5-dimensional bulk space-time enveloping warped FRW type space-time as a 3-brane with ansatz

$$ds^2_5 = n^2(\tau, y)d\tau^2 - S^2(\tau, y)\gamma_{ab}dx^a dx^b - R^2(\tau, y)dy^2,$$

(B.1)

where $\tau$ is the time and $\gamma_{ab}(a, b = 1, 2, 3)$ are components of the metric tensor of maximally symmetric 3-space parameterized by $k$. Here metric tensor components are obtained as

$$a_{00} = n^2(\tau, y), a_{ab} = -S^2(\tau, y)\gamma_{ab}, a_{44} = -R^2(\tau, y)$$  

(B.2a)

and

$$\gamma_{ab}dx^a dx^b = \frac{dr^2}{(1 - k r^2)} + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$  

(B.2b)
In this case, the 5-dimensional Einstein’s field equations (A.13) are obtained as

$$\hat{G}_0^0 = 3\left\{ \frac{\dot{S}}{n^2 S} \left( \frac{\dot{S}}{S} + \frac{\hat{R}}{R} \right) + \frac{1}{R^2} \left[ S'' + \frac{S'}{S} \left( \frac{S'}{S} - \frac{R'}{R} \right) \right] + \frac{k}{R^2} \right\} = -8\pi G_5 \hat{T}_0^0,$$

(B.3)

$$\hat{G}_b^a = \frac{1}{R^2} \delta_b^a \left\{ \frac{S'}{S} \left( \frac{S'}{S} + 2\frac{n'}{n} \right) - \frac{R'}{R} \left( \frac{2S'}{S} + \frac{n'}{n} \right) + 2\frac{S''}{S} + \frac{n''}{n} \right\}$$

$$+ \frac{1}{n^2} \delta_b^a \left\{ \frac{\dot{S}}{S} \left( -\frac{\dot{S}}{S} + \frac{n}{n} \right) - 2\frac{\dot{S}}{S} + \frac{\hat{R}}{R} \left( -2\frac{\dot{S}}{S} + \frac{n}{n} \right) - 2\frac{\ddot{R}}{R} \right\} - k \delta_b^a = -8\pi G_5 \hat{T}_b^a,$$

(B.4)

$$\hat{G}_0^0 = \frac{3}{n^2} \left( \frac{n' S}{n S} - \frac{S' \hat{R}}{S R} - \frac{S'}{S} \right) = -8\pi G_5 \hat{T}_0^0,$$

(B.5)

$$\hat{G}_4^4 = 3\left\{ \frac{S'}{R^2 S} \left( \frac{S'}{S} + \frac{n'}{n} \right) - \frac{1}{n^2} \left[ \frac{\dot{S}}{S} \left( \frac{\dot{S}}{S} - \frac{n}{n} \right) + \frac{\dot{S}}{S} \right] - \frac{k}{R^2} \right\} = -8\pi G_5 \hat{T}_4^4,$$

(B.6)

where dot denotes derivative with respect to \(\tau\) and prime denotes derivative with respect to \(y\). Here

$$\hat{T}_{\alpha \beta} = T_{\alpha \beta}^{(bulk)} + \delta(y) R^{-1} S_{\alpha \beta},$$

(B.7)

where \(N^\alpha S_{\alpha \beta} = 0\). It means that non-zero components of \(S_{\alpha \beta}\) are given by

$$S_{ij} = \lambda g_{ij} + T_{ij}$$

(B.8)

being energy-momentum tensor in the 3-brane. Moreover, \(T_{ij}\) are given by

$$T_{ij} = (\rho + p) u_i u_j - p g_{ij}.$$  

(B.9a)

with

$$p = \omega \rho.$$  

(B.9b)

In the space-time (B.1), extrinsic curvature components in 3-brane,
are given by

\[
[\Omega_{00}] = -[N_{\alpha;\beta}] y_0^\alpha y_0^\beta \\
= -[N_{\alpha;\beta}] y_0^\alpha y_0^\beta + \Gamma_{\alpha\beta}^\gamma N_\gamma y_0^\alpha y_0^\beta \\
= \Gamma_{00}^\gamma N_\gamma \\
= \frac{1}{2} a^{\alpha\delta} \left( a_{00,0} + a_{0\delta,0} - a_{00,\delta} \right) N_\gamma \\
= \frac{n(\tau, 0)n'(\tau, 0)}{R^2(\tau, 0)} N_4 \\
= \frac{n(\tau, 0)n'(\tau, 0)}{R(\tau, 0)} \tag{B.10a}
\]

as \( N^\alpha N_\alpha = -1 \) and \( N^\alpha = (0, 0, 0, 0, 1/R) \). Similarly

\[
[\Omega_{ab}] = -\frac{S(\tau, 0)S'(\tau, 0)}{R(\tau, 0)} \tag{B.10b}
\]

The \([A] = A^+ - A^-\) is the jump function across the brane at \( y = 0 \). So, using equation (A.43), \textit{junction conditions} are given by

\[
[\Omega_{00}] = 8\pi G_5 [S_{00} - \frac{1}{3} g_{00} S] \tag{B.11a}
\]

and

\[
[\Omega_{ab}] = 8\pi G_5 [S_{ab} - \frac{1}{3} g_{ab} S] \tag{B.11b}
\]

From equation (B.8), trace of \( S_{ij} \) is obtained as

\[
S = \rho + 4\lambda - 3p. \tag{B.11c}
\]

Connecting equations (B.10a), (B.11a) and (B.11c), it is obtained that

\[
\frac{n'(\tau, 0)}{n(\tau, 0)R(\tau, 0)} = \frac{8\pi G_5}{3} [3p + 2\rho - \lambda] \tag{B.12}
\]
as $u_0^2 = n^2$ and $g_{00} = n^2$. Similarly, using $g_{ab} = -S^2(\tau, 0)$ and connecting equations (B.10b), (B.11b) and (B.11c), it follows that

$$\frac{S'(\tau, 0)}{S(\tau, 0) R(\tau, 0)} = -\frac{8\pi G_5}{3} [\rho + \lambda]$$  

(B.13)

As $\hat{T}_4^0 = 0$, equation (B.5) yields

$$\frac{n' S'}{n S} + \frac{S' R}{S R} - \frac{\dot{S}'}{S} = 0.$$  

(B.14)

From the condition (B.13),

$$\frac{\dot{S}'}{S} = \dot{S}rac{S'}{S} - \frac{8\pi G_5}{3} \rho R(\tau, 0) - \frac{8\pi G_5}{3} (\rho + \lambda) \dot{R}(\tau, 0).$$  

(B.15)

Using equations (B.12), (B.13), (B.14) and (B.15), we get

$$\dot{\rho} + 3 \frac{\dot{S}(\tau, 0)}{S(\tau, 0)} (\rho + p) = 0,$$

(B.16)

which is the usual conservation equation remaining unchanged for brane-world also.

To recover FRW type space-time at $y = 0$, we should have the synchronized time (cosmic clock synchronized at the beginning of the universe). In this case, we can redefine time as

$$dt = n(\tau, 0) d\tau.$$  

(B.17)

Using the synchronized time, equation (B.17) yields at $y = 0$

$$\frac{1}{R^2(t, 0)} \left( \frac{S'(t, 0)}{S(t, 0)} \right)^2 - \left[ \left( \frac{\dot{S}(t, 0)}{S(t, 0)} \right)^2 + \frac{\ddot{S}(t, 0)}{S(t, 0)} \right] - \frac{k}{R^2(t, 0)} = -\frac{8\pi G_5}{3} \Lambda_5$$

(B.18)

as $T_4^{(bulk)} = \Lambda_5, S_4^4 = 0$ and, in equation (B.17), $n(t, 0) = 1$.

Integrating equation (B.18) with respect to time $t$

$$\left( \frac{\dot{S}_0}{S_0} \right)^2 = \frac{1}{2R_0^2} \left( \frac{S_0'}{S_0} \right)^2 + \frac{4\pi G_5}{3} \Lambda_5 + \frac{C}{S_0^4} - \frac{k}{S_0^2},$$

(B.19)
Here  \( C \) is an integration constant.

Using the condition (B.13), equation (B.19) can be rewritten as

\[
\left( \frac{\dot{S}_0}{S_0} \right)^2 = \frac{1}{2} \left( \frac{8\pi G_5}{3} \right)^2 (\rho + \lambda)^2 + \frac{4\pi G_5}{3} \Lambda_5 + \frac{C}{S_0^4} - \frac{k}{S_0^2}
\]

Now Friedmann equation can be recovered from equation (B.19) by recognizing

\[
\frac{8\pi G}{3} = \left( \frac{8\pi G_5}{3} \right)^2 \lambda
\]

and

\[
\frac{\Lambda_4}{3} = \frac{1}{2} \left( \frac{8\pi G_5 \lambda}{3} \right)^2 + \frac{4\pi G_5}{3} \Lambda_5,
\]

where \( \Lambda_4 \) is the 4-dimensional cosmological constant in 3-brane.

Incorporating equations (B.21a) and (B.21b) in equation (B.20), we obtain

\[
\left( \frac{\dot{S}_0}{S_0} \right)^2 = \frac{8\pi G}{3} \rho \left( 1 + \frac{\rho}{2\lambda} \right) + \frac{\Lambda_4}{3} + \frac{C}{S_0^4} - \frac{k}{S_0^2}. \tag{B.22}
\]

For further simplification on the modified Friedmann equation (B.22), we take \( S_0 = a(t) \). As a result, equation (B.22) looks like

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho \left( 1 + \frac{\rho}{2\lambda} \right) + \frac{\Lambda_4}{3} + \frac{C}{a^4} - \frac{k}{a^2}. \tag{B.23}
\]

**Some cosmological consequences**

For \( p = \omega \rho \), equation (B.16) yields

\[
\rho = \rho(0) \left[ \frac{a(0)}{a} \right]^q \tag{B.24a}
\]
with

\[ q = 3(1 + \omega). \]  \hfill (B.24b)

**Case 1. When** \( \Lambda_4 \neq 0, C = 0, k = 0 \)

In this case, equation (B.23) integrates to

\[
a(t) = a(0) \left[ -\frac{\kappa \rho(0)}{2\Lambda_4} + \rho(0) \sqrt{\frac{\kappa}{2\Lambda_4} \left( \frac{1}{\lambda} - \frac{\kappa}{2\Lambda_4} \right)} \{ \sinh(q\sqrt{\Lambda/3t}) \cosh A + \sinh A \cosh(q\sqrt{\Lambda/3t}) \} \right]^{1/q}, \tag{B.25a}
\]

where

\[
\sinh A = \frac{1 + \frac{\kappa}{2\Lambda_4} \rho(0)}{\rho(0) \sqrt{\frac{\kappa}{2\Lambda_4} \left[ \frac{1}{\lambda} - \frac{\kappa}{2\Lambda_4} \right]}} \tag{B.25b}
\]

and

\[
\kappa = 8\pi G. \tag{B.26c}
\]

When \( \omega < -1, \ q < 0 \). In this case the solution (B.25a) exhibits the *big-rip* singularity at time \( t = t_{br} \) given by

\[
cosh(q\sqrt{\Lambda/3t_{br}}) = \sqrt{\frac{\kappa}{2\Lambda_4} \left( \frac{1}{\lambda} - \frac{\kappa N}{2\Lambda_4} \right)} \cosech A + \sinh(q\sqrt{\Lambda/3t_{br}}) \coth A \tag{B.26}
\]

as \( t \to t_{br}, \ a(t) \to \infty, \ \rho \to \infty \) and \( p \to \infty \).

When brane-tension \( \lambda \) is given by \( \lambda = \frac{2\Lambda_4}{\kappa} \),

\[
a(t) = a(0) \left[ -\frac{\rho(0)}{\lambda} + (1 + \rho(0)/\lambda)e^{q\sqrt{\Lambda/3t}} \right]^{1/q} \nonumber
\]

\[
\simeq a(0)(1 + \rho(0)/\lambda)^{1/q}e^{\sqrt{\Lambda_4/3t}}, \tag{B.27}
\]

which is de Sitter expansion.

**Case 2. When** \( \Lambda_4 = 0, C = 0, k = 0 \)
In this case, equation (B.23) integrates to

\[ a(t) = a(0) \left[ -\frac{\rho(0)}{2\lambda} + \left(1 + \frac{\rho(0)}{\lambda}\right)^{1/2} + \frac{q}{2}\sqrt{\kappa N \rho(0)/3t}\right]^{2/q}. \]  

(B.28)

It exhibits big-rip singularity at

\[ t_{br} = \left[ -\frac{\rho(0)}{2\lambda} + \left(1 + \frac{\rho(0)}{\lambda}\right)^{1/2} \right]. \]  

(B.29)