Chapter 4

Acceleration And Deceleration In Curvature Induced Phantom Model

4.1 Prelude

In this chapter, cosmology of the late and future universe is obtained from $f(R)$– gravity with non-linear curvature terms $R^2$ and $R^3$ ($R$ being the Ricci scalar curvature). It is different from $f(R)$–dark energy models where non-linear curvature terms are taken as gravitational alternative for dark energy. In the present model, neither linear nor non-linear curvature terms are taken as dark energy. Rather, dark energy terms are induced by curvature terms and appear in the Friedmann equation derived from $f(R)$–gravitational equations. This approach has an advantage over $f(R)$–dark energy models in three ways (i) results are consistent with WMAP observations, (ii) dark matter is produced from the gravitational sector and (iii) the universe expands as $\sim t^{2/3}$ during dominance of the curvature-induced dark matter, which is consistent with the standard cosmology.

Here, curvature terms induce dark energy, dark matter and cosmolog-
ical constant, which appear in the Friedmann equation (FE) for the late universe, derived from $f(R)$—gravitational equations. It is interesting to see that curvature-induced dark energy, obtained here, mimics phantom with the equation of state (EOS) parameter $\omega = -5/4$. Moreover, FE contains phantom DE term as $\rho_{\text{DE}}[1 - \rho_{\text{DE}}/2\lambda]$. The correction term $-\rho_{\text{DE}}^2/2\lambda$, with $\lambda$ being the cosmic tension [140, 141], is analogous to such a term in RS-II model FE (2.2.2) as well as loop quantum gravity correction [142]. Like Refs. [140, 141], here also, this term is obtained from $f(R)$—gravity. Here, cosmic tension $\lambda$ is evaluated to be $5.77\rho_{\text{cr}}^0$ with $\rho_{\text{cr}}^0$ being the present critical density of the universe. Further, it is shown that universe, derived by curvature-induced dark matter, decelerates up to time $0.59t_0$ ($t_0$ being the present age of the universe). At this epoch and small red-shift $z = 0.36$, transition from deceleration to acceleration takes place. Interestingly, it is noted that as phantom energy density increases, effect of the term $-\rho_{\text{DE}}^2/2\lambda$ gradually increases. As a result, it is found that universe will super-accelerate (expansion with high acceleration) during the period $0.59t_0 < t < 2.42t_0$, it will accelerate (expansion with low acceleration) during the period $2.42t_0 < t < 3.44t_0$ and, universe will decelerate even during the phantom phase when $3.44t_0 < t < 3.87t_0$. Phantom-dominance will end when $\rho = 2\lambda = 11.54\rho_{\text{cr}}^0$ at time $t = 3.87t_0$ and dark matter will re-dominate causing decelerated cosmic expansion. It is natural to think that, even during decelerating phase with deceleration (derived by dark matter), phantom DE will grow with expansion. This chapter is based on [144].

Natural units ($k_B = \hbar = c = 1$) are used here.
4.2 Phantom phase of the late and the future universe from $f(R)$-gravity

Here, the action for $f(R)$-gravity is taken as

$$S = \int d^4x \sqrt{-g} \left[ \frac{R}{16\pi G} + \alpha R^2 + \beta R^3 \right], \quad (4.2.1)$$

where $\alpha$ is a dimensionless coupling constant and $\beta$ is a constant having dimension (mass)$^{-2}$ (as $R$ has mass dimension 2).

Action (4.2.1) yields field equations

$$\frac{1}{16\pi G} \left( R_{ij} - \frac{1}{2} g_{ij} R \right) + \alpha \left( 2R_{;ij} - 2g_{ij} \square R - \frac{1}{2} g_{ij} R^2 + 2RR_{ij} \right)$$

$$+ \beta \left( 3R_{;ij}^2 - 3g_{ij} \square R^2 - \frac{1}{2} g_{ij} R^3 + 3R^2 R_{ij} \right) = 0 \quad (4.2.2a)$$

using the condition $\delta S/\delta g^{ij} = 0$. The operator $\square$ is defined as

$$\square = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^i} \left( \sqrt{-g} g^{ij} \frac{\partial}{\partial x^j} \right). \quad (4.2.2b)$$

Taking trace of equation (4.2.2a), it is found that

$$- \frac{R}{16\pi G} - 6(\alpha + 3\beta R) \square R - 18\beta g^{ij} R_{;i} R_{;j} + \beta R^3 = 0. \quad (4.2.3)$$

In equation (4.2.3), $(\alpha + 3\beta R)$ emerges as a coefficient of $\square R$ due to presence of terms $\alpha R^2$ and $\beta R^3$ in the action (4.2.1). If $\alpha = 0$, effect of $R^2$ vanishes and effect of $R^3$ is switched off for $\beta = 0$. So, an effective scalar curvature $\tilde{R}$ is defined as

$$\gamma \tilde{R} = \alpha + 3\beta R, \quad (4.2.4)$$

where $\gamma$ is a constant having dimension (mass)$^{-2}$ being used for the dimensional corrections.
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Connecting equations (4.2.3) and (4.2.4), it is found that

\[-\Box \tilde{R} - \frac{1}{\tilde{R}} g^{ij} \tilde{R}_{;i} \tilde{R}_{;j} = \frac{1}{6\gamma} \left[ \frac{1}{16\pi G} - \frac{\alpha^2}{3\beta} \right] - \frac{1}{6\gamma} \left[ \frac{1}{16\pi G} - \frac{\alpha^2}{9\beta} \right] \frac{\alpha}{\gamma \tilde{R}} \]

\[-\frac{\tilde{R}}{54\beta} \left[ \gamma \tilde{R} - 3\alpha \right]. \tag{4.2.5}\]

In the space-time, given by equation (2.2.1), the above equation (4.2.5) reduces to

\[-\ddot{\tilde{R}} - \frac{3}{a} \dot{\tilde{R}} - \frac{\dot{\tilde{R}}^2}{\tilde{R}} = \frac{1}{6\gamma} \left[ \frac{1}{16\pi G} - \frac{\alpha^2}{3\beta} \right] - \frac{1}{6\gamma} \left[ \frac{1}{16\pi G} - \frac{\alpha^2}{9\beta} \right] \frac{\alpha}{\gamma \tilde{R}} \]

\[-\frac{\tilde{R}}{54\beta} \left[ \gamma \tilde{R} - 3\alpha \right], \tag{4.2.6}\]

due to spatial homogeneous flat model of the universe.

For \(a(t)\), being the power-law function of \(t\), \(\tilde{R} \sim a^{-n}\). For example, \(\tilde{R} \sim a^{-3}\) for matter-dominated model. So, there is no harm in taking

\[\tilde{R} = \frac{A}{a^n}, \tag{4.2.7}\]

where \(n > 0\) is a real number and \(A\) is a constant with mass dimension 2.

Using equation (4.2.7) in (4.2.6), it is found that

\[\frac{d}{dt} \left( \frac{\dot{a}}{a} \right) + (3-2n) \left( \frac{\dot{a}}{a} \right)^2 = Ca^n \left[ 1 - \frac{Da^n}{nA} \right] - \frac{\alpha}{18n\beta}, \tag{4.2.8}\]

In the late universe, \(a(t)\) is large, so this equation reduces to

\[\frac{d}{dt} \left( \frac{\dot{a}}{a} \right) + (3-2n) \left( \frac{\dot{a}}{a} \right)^2 = Ca^n \left[ 1 - \frac{Da^n}{nA} \right] + \frac{\alpha}{18n\beta}, \tag{4.2.9a}\]

where

\[C = \frac{1}{6\gamma} \left[ \frac{1}{16\pi G} - \frac{\alpha^2}{3\beta} \right]. \tag{4.2.9b}\]
and

\[
D = \frac{1}{6\gamma} \left[ \frac{1}{16\pi G} - \frac{\alpha^2}{9\beta} \right].
\]

(4.2.9c)

Equation (4.2.9a) can be re-written as

\[
\ddot{a} + (2 - 2n) \frac{\dot{a}^2}{a} = Ca^{n+1} \left[ 1 - \frac{Da^n}{C} \right] + \frac{\alpha}{18n\beta},
\]

(4.2.10)

which integrates to

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{E}{a^{6-4n}} + \frac{2C}{nA} \left[ \frac{a^n}{(6-3n)} - \frac{Da^{2n}}{C(6-2n)} \right] + \frac{\alpha}{9n\beta(6-4n)},
\]

(4.2.11a)

where \(E\) is an integration constant having dimension \((\text{mass})^2\).

Equation (4.2.11a) is the modified Friedmann equation (MFE) giving cosmic dynamics. Terms, on r.h.s. of this equation, are imprints of curvature. The first term, proportional to \(a^{-(6-4n)}\) emerges spontaneously. It is interesting to see that this term corresponds to matter density if \(n = 3/4\), i.e. for this value of \(n\) it reduces to \(Ea^{-3}\) and yields the density of non-baryonic matter being spontaneously induced by curvature. So, it is identified as dark matter density.

Thus, for \(n = 3/4\), equation (4.2.11a) looks like

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{E}{a^3} + \frac{4\alpha}{81\beta} + \frac{32Ca^{3/4}}{45A} \left[ 1 - \frac{5Da^{3/4}}{6C} \right].
\]

(4.2.11b)

On the r.h.s. of equation (4.2.11b), there are terms proportional to \(a^{3/4}\) and \(a^{3/2}\). If the density term \(\rho_{DE} = 4Ca^{3/4}/15A\pi G\) is put in the conservation equation given by equation (1.3.4), we have

\[
\dot{\rho}_{DE} + 3H(\rho_{DE} + p_{DE}) = 0.
\]

(4.2.11c)

With \(p_{DE} = \omega\rho_{DE}\), we obtain using (4.2.11c)

\[
\omega = -\frac{5}{4} < -1.
\]

(4.2.11d)
This result shows that $\rho_{DE} = 4Ca^{3/4}/15A\pi G$ behaves as phantom dark energy density being induced by $f(R)$—gravity.

Now, equation (4.2.11b) is re-written as

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \left[\frac{8\pi G}{3}\rho_{DM} + \frac{4\alpha}{81\beta}\right] + \frac{32Ca^{3/4}}{45A} \left[1 - \frac{5Da^{3/4}}{6C}\right], \quad (4.2.11e)$$

with

$$\rho_{DM} = \frac{3E}{8\pi Ga^3}. \quad (4.2.11f)$$

Using current value of $\rho_{DM}$ as $0.23\rho^0_{cr}$, (4.2.11f) yields

$$\rho_{DM} = 0.23\rho^0_{cr}\left(\frac{a_0}{a}\right)^3, \quad (4.2.12a)$$

where $a_0 = a(t_0), 3E/8\pi G = 0.23\rho^0_{cr}a^3_0$ and

$$\rho^0_{cr} = \frac{3H^2_0}{8\pi G},$$

with $H_0 = 100\text{km/Mpc/sec} = 2.32 \times 10^{-42}h\text{GeV}$ being the current Hubble’s rate of expansion and $h = 0.68$. The present age of the universe is estimated to be $t_0 \simeq 13.7\text{Gyr} = 6.6 \times 10^{41}\text{GeV}^{-1}$ [145]. So,

$$H_0^{-1} = 0.96 t_0. \quad (4.2.12b)$$

Further, $a_0$ is normalized as

$$a_0 = 1. \quad (4.2.12c)$$

Connecting equations (4.2.12a) and (4.2.12c), it is found that

$$\rho_{DM} = \frac{0.23\rho^0_{cr}}{a^3}. \quad (4.2.12d)$$

WMAP [145] gives decoupling of matter from radiation at red-shift

$$z_d = \frac{1}{a_d} - 1 = 1089. \quad (4.2.13)$$
So, it is supposed that dark matter begins to dominate cosmic dynamics when \( a > a_d \).

Now, equation (4.2.11e) is re-written as
\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \left[ \left\{ \rho_{\text{DM}} + \frac{\alpha}{54\beta \pi G} \right\} + \rho_{\text{DE}} \left\{ 1 - \frac{\rho_{\text{DE}}}{2\Lambda} \right\} \right],
\]
where
\[
\rho_{\text{DE}} = \rho_{\text{DE}}^0 a^{3/4},
\]
with \( \rho_{\text{DE}}^0 = 0.73\rho_{\text{cr}}^0 = 4C/15A\pi G \), using \( a_0 = 1 \) and
\[
\lambda = \frac{3C\rho_{\text{DE}}^0}{5D},
\]
where \( C \) and \( D \) are given by equations (4.2.9b) and (4.2.9c) respectively.

The FE (4.2.14a) is obtained from \( f(R) \)- gravity. Comparing it with GR based FE
\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho + \frac{\Lambda}{3},
\]
it is found that the constant term in equation (4.2.14a) behaves as cosmological constant
\[
\Lambda = \frac{4\alpha}{27\beta}
\]
with vacuum energy density
\[
\rho_\Lambda = \frac{\Lambda}{8\pi G} = \frac{2\alpha}{27\beta \pi G}.
\]

It is interesting to see that equation (4.2.14a) contains a term \(- (\rho_{\text{DE}})^2/2\lambda\) analogous to the brane-gravity correction to the FE for negative brane-tension \([86]\) and modifications in FE due to loop-quantum effects \([57]\). Here \( \lambda \) is called *cosmic tension* \([140, 146]\). Equation (4.2.14c) shows that *cosmic tension* \( \lambda \) depends on coupling constants \( \alpha \) and \( \beta \) in the gravitational action (4.2.1). Moreover, a positive cosmological constant, too, emerges from curvature.
From (4.2.12a) and (4.2.14b), it is found that $\rho_{\text{DM}} \sim a^{-3}$ and $\rho_{\text{DE}} \sim a^{3/4}$. So $\rho_{\text{DM}}$ decreases and $\rho_{\text{DE}}$ increases with expansion of the universe. So, it is natural to think for values of these to come closer and to be equal at a certain time $t_\ast$. At this particular time, we have

$$0.23a_\ast^{-3} = 0.73a_\ast^{3/4},$$

using (4.2.12a), (4.2.12c) and (4.2.14b) as well as $a_\ast = a(t_\ast)$. This equation yields

$$a_\ast = \left(\frac{23}{73}\right)^{4/15}. \tag{4.2.15}$$

It shows that, for $a < a_\ast = (23/73)^{4/15}$, $\rho_{\text{DM}} > \rho_{\text{DE}} > \rho_{\text{DE}}^2$. So, equation (4.2.14a) is approximated as

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \left[\rho_{\text{DM}} + \frac{\alpha}{54\beta\pi G}\right]. \tag{4.2.16}$$

Connecting equations (4.2.12a) and (4.2.16), it is found that

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{0.23H_0^2}{a^3} \left[1 + \frac{a^3}{B}\right], \tag{4.2.17a}$$

with

$$B = \frac{\rho_{\text{DM}}^0}{\rho_\Lambda} = \frac{4.66\beta H_0^2}{\alpha}, \tag{4.2.17b}$$

obtained using (4.2.14e).

Equation (4.2.17a) integrates to

$$a = B^{1/2} \sinh^{2/3} \left[\frac{3H_0\sqrt{0.23}}{2B^{3/2}}(t - t_d) + \sinh^{-1}\left(\frac{a_d}{B^{1/3}}\right)^{3/2}\right]. \tag{4.2.18a}$$

This result is not consistent with the scale factor, obtained in the standard model of cosmology during matter dominance. So, to have a viable cosmology, it needs to be approximated as

$$a = a_d \left[1 + \frac{3H_0\sqrt{0.23}}{2a_d^{3/2}}(t - t_d)\right]^{2/3}, \tag{4.2.18b}$$
which is possible till
\[
\frac{3H_0 \sqrt{0.23}}{2B^{3/2}} (t - t_d) + \sinh^{-1} \left( \frac{a_d}{B} \right)^{3/2} \lesssim 1 \quad (4.2.18c)
\]
as \(\sinh 1 \approx 1.18 \approx 1\). Here \(a_d\) is given by (4.2.13), which is the scale factor at time \(t = t_d = 386\,\text{kyr} = 2.8 \times 10^{-5} t_0\). Connecting equations (4.2.13), (4.2.17b), (4.2.18a) and (4.2.18c), it is found that
\[
\frac{\alpha}{\beta H_0^2} \gtrsim 10^{-6}. \quad (4.2.18d)
\]
Connecting equation (4.2.16) and \(\alpha/\beta H_0^2 \approx 10^{-6}\) from (4.2.18d), it is evaluated that
\[
\rho_\Lambda = 2.19 \times 10^{-8} \rho_{cr}^0 = 5.48 \times 10^{-55}\text{GeV}^4. \quad (4.2.18e)
\]
The approximated form (4.2.17a) is obtained when \(a \ll a_*\), but, as discussed above, \(\rho_{DM} \approx \rho_{DE}\) when \(a \lesssim a_*\). So, in the narrow strip around \(a = a_*\) for \(a < a_*\), equation (4.2.14a) is approximated as
\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \left[ 2\rho_{DM} + \frac{\alpha}{54\beta \pi G} \right], \quad (4.2.19)
\]
using \(\rho_{DM} \approx \rho_{DE} > \rho_{DE}^2\) in (4.2.14a).

Connecting (4.2.12a) and (4.2.19), it is found that
\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{0.46 H_0^2}{a^3} \left[ 1 + \left( \frac{a}{a_*} \right)^3 \right], \quad (4.2.20)
\]
with \(a_*^3 = \alpha/167.67\beta H_0^2\). This is because at \(a = a_*\), matter-dominated phase ends and dark energy dominance begins.

Equation (4.2.20) integrates to
\[
a^{3/2} = a_*^{3/2} \sinh \left[ \frac{3H_0 \sqrt{0.46}}{2a_*^{3/2}} (t - t_d) + \sinh^{-1} \left( \frac{a_d}{a_*} \right)^{3/2} \right],
\]
which is approximated as

\[a = a_d \left[1 + \frac{3H_0 \sqrt{0.46}}{2a_d^{3/2}}(t-t_d)\right]^{2/3}, \quad (4.2.21)\]
as in the above case.

Using equation (4.2.21), it is found that

\[a^\star_{3/2} \simeq a_d^{3/2} + \frac{3H_0 \sqrt{0.46}}{2}(t^\star - t_d).\]

This result yields

\[t^\star \simeq 0.59t_0, \quad (4.2.22)\]
using values of \(a_\star\) [from (4.2.15)], \(H_0^{-1}\) [from (4.2.12b)] and \(t_d\).

It has been discussed above that for \(a > a_\star\), \(\rho_{\text{DM}} < \rho_{\text{DE}}\). In this case, equation (4.2.14a) is approximated to

\[\left(\frac{\dot{a}}{a}\right)^2 \simeq \frac{8\pi G}{3}\rho_{\text{DE}}\left\{1 - \frac{\rho_{\text{DE}}}{2\lambda}\right\}. \quad (4.2.23)\]

Thus, in the late universe, a phantom model is obtained from curvature without using an unknown scalar \(\phi\) as a source of exotic matter. But this model contains a correction term \(-4\pi G \rho_{\text{DE}}^2/3\lambda\) due to curvature-induced cosmic tension \(\lambda\) being evaluated below.

Connecting equations (4.2.14b) and (4.2.23), it is found that

\[H^2 = \left(\frac{\dot{a}}{a}\right)^2 = 0.73H_0^2a^{3/2}\left[a^{-3/4} - \frac{0.73\rho_{\text{cr}}^0}{2\lambda}\right]. \quad (4.2.24)\]

Equation (4.2.24) integrates to

\[a(t) = \left[\frac{0.73\rho_{\text{cr}}^0}{2\lambda} + \left\{\sqrt{1.26 - \frac{0.73\rho_{\text{cr}}^0}{2\lambda}} - \frac{3}{8}H_0\sqrt{0.73}(t-t^\star)\right\}\right]^{-4/3}. \quad (4.2.25a)\]
as $a_{*}^{-3/4} = 1.26$. Equation (4.2.25a) shows that phantom model obtained here is singularity-free.

Also from equation (4.2.25a), it is found that

$$
\ddot{a} = 0.27H_0^2a^{5/2}\left[\frac{1.7\rho_{cr}^0}{\lambda} - \frac{11}{3}a^{-3/4}\right].
$$

This shows $\ddot{a} > 0$, when

$$
\frac{1.7\rho_{cr}^0}{\lambda} > \frac{11}{3}a^{-3/4}.
$$

Further, equation (4.2.25a) yields

$$
1 = a_0 = \left[\frac{0.73\rho_{cr}^0}{2\lambda} + \sqrt{1.26 - \frac{0.73\rho_{cr}^0}{2\lambda} - \frac{3}{8}H_0\sqrt{0.73}(t_0 - t_*)}\right]^{27/43}.
$$

Using (4.2.22) for $t_*$ in (4.2.26), $\lambda$ is evaluated as

$$
\lambda = 5.77\rho_{cr}^0.
$$

Equation (4.2.25a) exhibits accelerating universe when $t > t_*$. Thus, a transition from deceleration to acceleration takes place at

$$
t = t_* = 0.59t_0
$$

and red-shift

$$
z_* = \frac{1}{a_*} - 1 = \left(\frac{73}{23}\right)^{4/15} - 1 = 0.36,
$$

which is within the range $0.33 \leq z_* \leq 0.59$ given by 16 Type supernova observations [41]. Equation (4.2.25a) shows that universe expands till $\rho_{DE}$ becomes equal to $2\lambda$ as it grows with expansion. It happens till $a(t)$ increases to $a_{pe}$, satisfying

$$
a_{pe}^{-3/4} = \frac{2\lambda}{0.73\rho_{cr}^0}.
$$
Thus, expansion (4.2.25a) stops at time

\[ t_{\text{pe}} = t_* + \frac{8}{3H_0\sqrt{0.73}} \sqrt{1.26 - \frac{0.73\rho^0_{\text{cr}}}{2\lambda}}. \] (4.2.31)

### 4.3 Cosmic energy conditions as well as acceleration and deceleration during phantom era

In what follows, like RS-II model of 2nd chapter it is found that correction term \(-4\pi G\rho^2_{\text{DE}}/3\lambda\) effects the behavior of the phantom model drastically. In this section since DE dominates, therefore we take \(\rho = \rho_{\text{DE}}\) and \(p = p_{\text{DE}}\)

From equations (4.2.11c) and (4.2.23), it is seen that

\[ \frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \left[ 3(\rho + p) \left( 1 - \frac{\rho}{\lambda} \right) - 2\rho \left( 1 - \frac{\rho}{2\lambda} \right) \right]. \] (4.3.1)

The correction term, in this equation, is caused due to curvature-induced cosmic tension \(\lambda\) in equation (4.2.23). This type of equation was obtained earlier in 2nd chapter in the context of RS-II model of brane-gravity.

Comparing equation (4.3.1) and analogous equation (2.2.6) of GR based theory, the effective pressure density \(P\) is given by

\[ \rho + 3P = 3(\rho + p) \left[ 1 - \frac{\rho}{\lambda} \right] - 2\rho \left[ 1 - \frac{\rho}{2\lambda} \right]. \] (4.3.2)

Using (4.2.11d), equation (4.3.2) yields the effective pressure density in the curvature induced phantom model as

\[ P = -\frac{5}{4}\rho + \frac{7\rho^2}{12\lambda}. \] (4.3.3)

Equation (4.3.3) yields

\[ \rho + P = -\frac{\rho}{4} + \frac{7\rho^2}{12\lambda}. \] (4.3.4)
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This equation shows that the WEC is violated when \( \rho < \frac{3\lambda}{7} = 2.47\rho_{cr}^{0} \), with \( \lambda \) given by (4.2.27). Moreover, \( \rho + P = 0 \) for \( \rho = 2.47\rho_{cr}^{0} \) and \( \rho + P > 0 \) for \( \rho > \frac{(3/7)\lambda = 2.47\rho_{cr}^{0}}{\rho_{cr}^{0}} \).

From equation (4.2.14b), it is found that energy density \( \rho \) for phantom fluid increases with increasing scale factor \( a(t) \). It is interesting to note from equation (4.3.4) that curvature-induced phantom fluid, obtained here, behaves effectively as phantom violating WEC till \( \rho < 2.47\rho_{cr}^{0} \), but phantom characteristic to violate WEC is suppressed by cosmic tension when \( \rho \) increases and obeys the inequality \( \rho > 2.47\rho_{cr}^{0} \).

Further, using (4.2.11d) in (4.3.2), it is also found that

\[
\rho + 3P = -\frac{11\rho}{4} + \frac{7\rho^{2}}{4\lambda}.
\]  

(4.3.5)

Connecting equations (4.2.27) and (4.3.5), it is seen that SEC is violated when \( \rho < \frac{11\lambda}{7} = 9.07\rho_{cr}^{0} \). Also, it is found that \( \rho + 3P = 0 \) for \( \rho = 9.07\rho_{cr}^{0} \) and \( \rho + 3P > 0 \) for \( \rho > 9.07\rho_{cr}^{0} \).

Thus, it is seen that (i) WEC is violated for \( \rho < 2.47\rho_{cr}^{0} \), (ii) for \( 2.47\rho_{cr}^{0} \leq \rho < 9.07\rho_{cr}^{0} \) WEC is not violated, but SEC is violated and (iii) for \( \rho > 9.07\rho_{cr}^{0} \) neither of the two conditions is violated. Also it is interesting to note that these corrections cause effective phantom divide at

\[
\rho = 2.47\rho_{cr}^{0}.
\]  

(4.3.6)

Moreover, these results suggest that a transition from violation of SEC to non-violation of SEC will take place at

\[
\rho = 9.07\rho_{cr}^{0}.
\]  

(4.3.7)

Also, universe will super-accelerate till \( \rho^{*} < \rho < 2.47\rho_{cr}^{0} \), accelerate when \( 2.47\rho_{cr}^{0} < \rho < 9.07\rho_{cr}^{0} \) and decelerate when \( 9.07\rho_{cr}^{0} < \rho < 11.54\rho_{cr}^{0} \).
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as expansion of phantom phase of the universe will stop at $\rho = 11.54\rho_{cr}^0$.

These results are also supported by equation (4.2.25b), as (4.2.25b) and (4.2.14b) yield

$$\ddot{a} = 0.27H_0^2a^{5/2}\left[0.29 - \frac{2.68\rho_{cr}^0}{\rho}\right]. \quad (4.3.8)$$

Connecting equations (4.2.14b), (4.2.25a), (4.2.27) and (4.2.28a), it is found that

$$\rho = 0.73\rho_{cr}^0[0.06 + \left\{1.094 - 0.32H_0(t - 0.59t_0)\right\}^2]^{-1}. \quad (4.3.9)$$

Equations (4.3.6) and (4.3.9) yield that effective phantom divide is obtained at time

$$t \approx 2.42t_0. \quad (4.3.10)$$

Equations (4.3.8) and (4.3.10) yield that transition time for violation of SEC to non-violation of SEC will take place at

$$t \approx 3.44t_0. \quad (4.3.11)$$

These results imply super-acceleration during the time interval $0.59t_0 < t < 2.42t_0$, acceleration during the time interval $2.42t_0 < t < 3.44t_0$ and deceleration during the time interval $3.44t_0 < t < 3.87t_0$. Expansion, driven by phantom, will stop at time $t = 3.87t_0$ as $\rho_{DE}$ will acquire the value $2\lambda$ by this time.

When $t > 3.87t_0$, deceleration, driven by matter, will resume and Freidmann equation reduces to (4.2.17).
4.4 Re-appearance of matter-dominance and cosmic collapse

As mentioned above, DE terms are switched off in equation (4.2.14a) at \( \rho_{\text{DE}} = 2\lambda = 11.54\rho_{\text{cr}}^0 \) [obtained from (4.2.27)], \( a = a_{\text{pe}} \), given by equation (4.2.30) when \( t = t_{\text{pe}} = 3.87t_0 \), given by equation (4.2.31). So, for \( t > t_{\text{pe}} = 3.87t_0 \), equation (4.2.14a) will look like

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho_{\text{DM}}^{\text{pe}} \left( \frac{a_{\text{pe}}}{a} \right)^3 \left\{ 1 + \tilde{\Lambda}^{-1} \left( \frac{a}{a_{\text{pe}}} \right)^3 \right\},
\]

(4.4.1a)

where

\[
\rho_{\text{DM}}^{\text{pe}} = 3.68 \times 10^{-6} \rho_{\text{cr}}^0,
\]

(4.4.1b)

\[
\tilde{\Lambda} = \frac{\rho_{\text{DM}}^{\text{pe}}}{\rho_{\Lambda}} = \frac{243\pi G\beta\rho_{\text{DM}}^{\text{pe}}}{2\alpha} = 168,
\]

(4.4.1c)

which is evaluated using equations (4.2.18d), (4.2.30), (4.4.1b), and

\[
\rho_{\text{DE}}^{\text{pe}} \left( \frac{a}{a_{\text{pe}}} \right)^{3/4} \left\{ 1 - \left( \frac{a}{a_{\text{pe}}} \right)^{3/4} \right\} = 0
\]

(4.4.1d)

as

\[
\rho_{\text{DE}}^{\text{pe}} = 2\lambda = 11.54\rho_{\text{cr}}^0.
\]

(4.4.1e)

Equation (4.4.1a) integrates to

\[
a(t) = \tilde{\Lambda}^{1/3} a_{\text{pe}} \sinh^{2/3} \left[ \sinh^{-1} \tilde{\Lambda}^{-1/2} + 2.22 \times 10^{-4} H_0(t - t_{\text{pe}}) \right],
\]

(4.4.2a)

yielding

\[
\frac{\ddot{a}}{a} = -\frac{2}{9} [2.22 \times 10^{-4} H_0]^2 [\text{cosech}^2 \theta(t) - 2],
\]

(4.4.2b)

with \( \theta(t) = \sinh^{-1} \tilde{\Lambda}^{-1/2} + 2.22 \times 10^{-4} H_0(t - t_{\text{pe}}) \).
Equation (4.4.2b) shows decelerated expansion caused by matter-dominance till
\[
\sinh \theta(t) < 1/\sqrt{2} = 0.707. \tag{4.4.2c}
\]

The result (4.4.2a) is obtained when expansion is driven by the term
\[
\rho_{DM}^{pe} \left( \frac{a_{pe}}{a} \right)^3 \left\{ 1 + \tilde{\Lambda}^{-1} \left( \frac{a}{a_{pe}} \right)^3 \right\}
\]
in the FE.

Moreover, though at \( a = a_{pe} \),
\[
\rho_{DE}^{pe} \left( \frac{a}{a_{pe}} \right)^{3/4} \left\{ 1 - \left( \frac{a}{a_{pe}} \right)^{3/4} \right\}
\]
vanishes, it will be negative for \( a > a_{pe} \). So for \( a > a_{pe} \), FE (4.2.14a) is obtained as
\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho_{DM}^{pe} \left( \frac{a_{pe}}{a} \right)^3 \left\{ 1 + \tilde{\Lambda}^{-1} \left( \frac{a}{a_{pe}} \right)^3 \right\}
- \rho_{DE}^{pe} \left( \frac{a}{a_{pe}} \right)^{3/4} \left\{ \left( \frac{a}{a_{pe}} \right)^{3/4} - 1 \right\}. \tag{4.4.3}
\]

Obviously, the negative term in equation (4.4.3) will try to stop expansion and, on sufficient growth of \( a(t) \) up to \( a_m \), expansion will reach its maximum such that \( \dot{a}_{a=a_m} = 0 \) and \( a_m \) satisfies the equation
\[
\rho_{DM}^{pe} \left\{ \left( \frac{a_{pe}}{a_m} \right)^3 + \tilde{\Lambda}^{-1} \right\} = 2\lambda \left( \frac{a_m}{a_{pe}} \right)^{3/4} \left\{ \left( \frac{a_m}{a_{pe}} \right)^{3/4} - 1 \right\} \tag{4.4.4a}
\]
being approximated as
\[
\rho_{DM}^{pe} \tilde{\Lambda}^{-1} \approx 2\lambda \left( \frac{a_m}{a_{pe}} \right)^{3/4} \left\{ \left( \frac{a_m}{a_{pe}} \right)^{3/4} - 1 \right\}. \tag{4.4.4b}
\]
Equation (4.4.4b) yields the solution
\[
\left( \frac{a_m}{a_{pe}} \right) = \left\{ \frac{1}{2} \left[ 1 + \sqrt{1 + 2 \rho_{DM}^{pe} / \lambda} \right] \right\}^{4/3} = 1 + 4.25 \times 10^{-7}. \tag{4.4.4c}
\]
The negative sign (−) is ignored here as it yields \(a_m < a_{pe}\), which is not possible. Using equation (4.4.4c), it is found that
\[
\rho_{DM}^m = \rho_{DM}^{pe} \left[ 1 - 1.28 \times 10^{-6} \right]. \tag{4.4.4d}
\]
Using (4.4.4c) in equation (4.4.2a), it is seen that the time \(t = t_m\) corresponding to \(a = a_m\) is given by
\[
\left[ \sinh^{-1} \tilde{\Lambda}^{-1/2} + 2.22 \times 10^{-4} H_0 (t_m - t_{pe}) \right] = \sinh^{-1} \left[ \frac{1}{4 \sqrt{\tilde{\Lambda}}} \left[ 1 + \sqrt{1 + 2 \rho_{DM}^{pe} / \lambda} \right]^2 \right] = \sinh^{-1}(0.18123) = 0.18123.
\tag{4.4.4e}
\]
Equation (4.4.4e) confirms deceleration during time period \(t_{pe} < t < t_m\) as it satisfies the condition (4.4.2c).

So, for \(t_{pe} < t < t_m\), equation (4.4.2a) is obtained as
\[
a(t) = a_{pe} \left[ 1 + 2.22 \times 10^{-4} H_0 \tilde{\Lambda}^{1/3} a_{pe}^{-3/2} (t - t_{pe}) \right]^{2/3}. \tag{4.4.5}
\]

Connecting equations (4.2.12b), (4.4.1c) and (4.4.4e), \(t_m\) is evaluated as
\[
t_m - t_{pe} = 5.32 \times 10^{-4} t_0 = 3.51 \times 10^{38} \text{GeV}^{-1} = 694.4 \text{kyr}. \tag{4.4.6}
\]
Further, it is interesting to note that the curve \(a = a(t)\) will be continuous at \(t = t_m\), but the direction of tangent to this curve (pointed
at $a = a_m$) will change because it will attain its maximum at $t = t_m$, yielding $\dot{a} < 0$ for $t > t_m$, which used to be positive for $t < t_m$. It means that universe will retrace back at $t = t_m$ and will begin to contract. During the contraction phase, term proportional to $a^{-3}$ will dominate over terms proportional to $a^{3/4}$ and (4.4.3) will yield

$$
\frac{\dot{a}}{a} \approx - \left[ \frac{8\pi G}{3} \rho_{DM}^m \left( \frac{a_m}{a} \right)^3 \left\{ 1 + \bar{\Lambda}^{-1} \left( \frac{a}{a_m} \right)^3 \right\} \right]^{1/2}.
$$

(4.4.7)

On integrating equation (4.4.7), it is found that

$$
a(t) = \bar{\Lambda}^{1/3} a_m \{ \sinh[2.22 \times 10^{-4} H_0](t_{col} - t) \}^{2/3},
$$

(4.4.8a)

where

$$
t_{col} = t_m + [2.22 \times 10^{-4} H_0]^{-1} \sinh^{-1} \bar{\Lambda}^{-1/2}
$$

$$
= t_m + 3.33 \times 10^2 t_0.
$$

(4.4.8b)

Equation (4.4.8a) shows that $a(t) = 0$ at $t = t_{col}$ and the energy density of the universe

$$
\rho = \left[ \rho_{DM}^m \left( \frac{a_m}{a} \right)^3 + \rho_{DE}^m \left( \frac{a}{a_{pe}} \right)^{3/4} \right]
$$

$$
\simeq \rho_{DM}^m \left( \frac{a_m}{a} \right)^3
$$

(4.4.9)

will diverge. It means that universe will collapse at $t = t_{col}$.

This result, suggested by the classical mechanics, is unphysical due to divergence of cosmic energy density. So, to have a viable physics around the epoch $t = t_{col}$, the diverging component of the density of the universe, which is proportional to $a^{-3}$, needs to be finite. As this
unphysical situation is predicted by classical mechanics, we have no other alternative other than to resort to quantum gravity. In [144], it is shown that cosmic collapse obtained from the classical mechanics can be avoided on making quantum gravity corrections relevant near collapse time due to extremely high energy density and large curvature analogous to the state of very early universe. This situation is analogous to very early universe, where quantum gravity effects are dominant. Earlier also, quantum gravity corrections were made in the equations of future universe under such circumstances to avoid finite time singularities [97, 98, 147].

4.5 Conclusion

Here, cosmology of the late and future universe is obtained from $f(R)$—gravity, obtained by adding higher-order curvature terms $R^2$ and $R^3$ to the Einstein-Hilbert term linear in scalar curvature $R$. Here, problems of $f(R)$—dark energy models, pointed out in [82], do not appear in $f(R)$—gravity cosmology [84, 140, 141, 143, 146] as well as in the present model. Here, curvature scalar contributes to both geometrical and physical components of the theory. Thus, it plays dual role as a geometrical as well as physical fields, which was obtained earlier in [149].

Here, it is found that, in the late universe for the red-shift $z < 1089$, the dark matter term emerges spontaneously and phantom dark energy emerges as imprints of linear as well as non-linear terms of curvature. It is found that, during $0.36 < z < 1089$, dark matter dominated and universe expanded with deceleration as $t^{2/3}$. A transition from
deceleration to acceleration took place at \( z = 0.36 \) and at time \( t = 0.59t_0 \) (\( t_0 \) being the present age of the universe). This transition is caused by dominance of curvature-induced phantom dark energy over curvature-induced dark matter. Dark energy gives anti-gravity effect and phantom dark energy exhibits this effect more strongly due to violation of WEC. So, phantom energy imposes a high jerk, causing super-acceleration.

The \( f(R) \)- gravity inspired Friedmann equation, obtained here, contains two terms (i) \( 8\pi G \rho_{DM}/3 \) (with \( \rho_{DM} \) being the dark matter density) and (ii) \( (8\pi G \rho_{DE}/3)[1 - \rho_{DE}/2\lambda] \) (with \( \rho_{DE} \) being the phantom dark energy density). Here \( \lambda = 5.777\rho^0_{cr} \) is called the cosmic tension, which is also curvature-induced. It is interesting to see that Friedmann equation, obtained here, contains a correction term \(-4\pi G \rho^2_{DE}/3\lambda\) analogous to such a term in Friedmann equations from RS-II model of brane-gravity [86] and loop quantum cosmology [57]. This correction is not effective in the present universe as \( \rho^0_{DE} << 2\lambda \), as well as till \( \rho_{DE} << 2\lambda \). But, as \( \rho_{DE} \) will increase in future with the growing scale factor \( a(t) \), the effect of this term will increase. It is found that, on sufficient growth of \( \rho_{DE} \), the effective EOS does not violate WEC (which characterizes phantom), but violates SEC. On further increase in \( \rho_{DE} \), even SEC is not violated. Thus, at the beginning of the phantom phase, universe will super-accelerate during the period \( 0.59t_0 < t < 2.42t_0 \); it will accelerate during the period \( 2.42t_0 < t < 3.44t_0 \); and, during the period \( 3.44t_0 < t < 3.87t_0 \), universe will decelerate even during the phantom phase. Phantom-dominance will end when \( \rho = 2\lambda = 11.54\rho^0_{cr} \) at time \( t = 3.87t_0 \). As a consequence, re-dominance of dark matter will begin giving decelerated expansion. But, as universe will expand,
growth of $\rho_{\text{DE}} \sim a^{3/4}$ will also continue giving $\rho_{\text{DE}} > 2\lambda$. It causes the term $(8\pi G \rho_{\text{DE}}/3)[1 - \rho_{\text{DE}}/2\lambda]$ to switch over from positive to negative. The growth of this negative term will try to slow down expansion more rapidly. As a result, universe will reach a stage where the expansion will stop and the scale factor will acquire its maximum value in finite time $t_m = 3.87t_0 + 694.4\text{kyr}$. When $t > t_m$, universe will bounce and contract. Results, obtained here, show that contraction of the universe will continue for sufficiently long period, $333t_0$, and universe will collapse at time $t_{\text{col}} = 336.87t_0 + 694.4\text{kyr}$, where energy density of the universe will diverge and scale factor will vanish. These results are obtained using prescriptions of the classical mechanics. In [144], it is probed further whether quantum gravity corrections can save the universe from the menace of collapse.