CHAPTER IV

ON THE DEGREE OF APPROXIMATION TO A FUNCTION BELONGING TO WEIGHTED \((L^p, \psi(t))\) CLASS

4.1. Let \(f(x)\) be a \(2\pi\)-periodic function integrable \(L^p\) \((p > 1)\) and let
\[
\int f(x) = \frac{1}{2} a_0 + \sum_{\nu=1}^{\infty} \frac{a_{\nu}}{\omega} \cos \nu x + \frac{b_{\nu}}{\omega} \sin \nu x = \sum_{\nu=0}^{\infty} A_{\nu}(x) \quad (4.1.1)
\]
be its Fourier series.

We define the norm \(\|f\|_p\) by
\[
\|f\|_p = \left\{ \int_0^{2\pi} |f(x)|^p \, dx \right\}^{1/p}, \quad p > 1
\]
and the degree of approximation \(E_n^*(f)\) by
\[
E_n^*(f) = \min_{T_n} \|f - T_n\|_p,
\]
where \(T_n(x)\) is a trigonometric polynomial of degree \(n\).

Given a positive increasing function \(\psi(t)\) and an integer \(p > 1\), we say† that \(f(x)\) belongs to \(\text{Lip} (\psi(t), p)\) class (written \(f(x) \in \text{Lip} (\psi(t), p)\)) if

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† Siddiqi, A. H. [19]
and that $f(x)$ belongs to weighted $(L^p, \psi(t))$ class if
\[
\{ \int_0^{2\pi} |f(x+t) - f(x)|^p \sin^{\beta} p x \, dx \}^{1/p} = O(\psi(t)) \quad (\beta > 0). \quad (4.1.3)
\]
In case $\beta = 0$, we notice that our newly defined class $W(L^p, \psi(t))$ coincides with the class $\text{Lip}(\psi(t), p)$.

4.2. Let $\{p_n\}, \{q_n\}$ be non-negative, non-increasing generating sequences for the $(N, p_n, q_n)$ method such that
\[
P_n = p_0 + p_1 + \cdots + p_n \to \infty \quad \text{as} \quad n \to \infty, \quad (4.2.1)
\]
\[
q_n = q_0 + q_1 + \cdots + q_n, \quad (4.2.2)
\]
\[
R_n = p_0 q_n + p_1 q_{n-1} + \cdots + p_n q_0 \to \infty \quad \text{as} \quad n \to \infty, \quad (4.2.3)
\]
and
\[
\frac{p_n q_n}{R(n)} = \sum_{k=0}^{n} r_k S_{n-k} \quad (4.2.4)
\]
where $S_k$ is the $k$-th partial sum of the series (4.1.1) and
\[
r_k = p_k q_{n-k}.
\]
4.3. Our main result of this chapter is the following theorem on necessary and sufficient condition for the degree of approximation to a function via the generalized Nörlund means $t_n^{p.q}$ of its Fourier series belonging to our new class $W(L^p, \psi(t))$.

Theorem (4.3.1): If $f(x)$ is a periodic function, then

$$f(x) \in W(L^p, \psi(t)) \text{ if and only if}$$

$$E_n(f) = \min_{t_n} \|f - t_n^{p.q}|| = O((1/n)^{\beta + 1/p}), \quad (4.3.1)$$

provided $\psi(t)$ satisfies the following:

(i) $$\left\{ \int_0^{x/n} \left( \frac{t! \phi(t)!}{\psi(t)} \right)^p \sin^p t \ dt \right\}^{1/p} = O(1/n),$$

(ii) $$\left\{ \int_0^{x/n} \left( \frac{t^\delta \phi(t)!}{\psi(t)} \right)^p \ dt \right\}^{1/p} = O(n^\delta),$$

where $\delta$ is an arbitrary positive number such that $\delta(1-\delta)-1 > 0$.

(iii) $$\left\{ \int_0^{x/n} \left( \frac{\psi(t)}{t^{1+\beta}} \right)^q \ dt \right\}^{1/q} = O((1/n)^{\beta + 1/p}),$$

(iv) $$\int_0^{x^n} \frac{\psi(1/t)}{t^{1+\beta - 1/p}} \ dt = O((1/n)^{\beta + 1/p + 1}),$$

(v) $$\int_0^{x^n} \frac{\psi(1/t)}{t^{\beta - 1 + 1/p}} \ dt = O((1/n)^{\beta + 1/p}).$$
(vi) \( R(n) \not\leq (1/n) \) is non-decreasing.

**Proof:** To prove the theorem we need the following lemmas:

**Lemma (4.3.2):** If \( \{p_n\} \) and \( \{q_n\} \) are non-negative and non-decreasing then for \( 0 \leq a \leq b \leq \infty, 0 \leq t \leq \pi \) and any \( n \), we have

\[
\sum_{k=a}^{b} \frac{i(n-k)t}{p_k q_{n-k}} e^{-ikt} \leq R(1/t) \quad \text{for any } a.
\]

**Proof:** Let \( \Gamma = [t^{-1}] \), then

\[
\sum_{k=a}^{b} \frac{i(n-k)t}{p_k q_{n-k}} e^{-ikt} = \int_{a}^{b} \sum_{k=a}^{\Gamma} p_k q_{n-k} e^{-ikt}.
\]

\[
\leq \int_{a}^{\Gamma - 1} p_k q_{n-k} e^{-ikt} + \int_{\Gamma}^{b} p_k q_{n-k} e^{-ikt},
\]

\[
= T_1 + T_2 \quad \text{(say)}.
\]

Now

\[
T_1 \leq \sum_{k=a}^{\Gamma - 1} p_k q_{n-k} e^{-ikt} \leq \sum_{k=a}^{\Gamma - 1} p_k q_{\Gamma - k} \leq R(\Gamma) \leq R(t^{-1}),
\]

and by Abel's Lemma,

\[
T_2 \leq 2p_{\Gamma} q_{n-b} \quad \max_{\Gamma + 1 \leq k \leq b} \left| 1 - \frac{e^{-i(k+1)t}}{1 - e^{-kt}} \right|,
\]
Since
\[ R_n = \sum_{k=0}^{\infty} p_k q_{p-k} \geq p N q_n, \]
we finally get
\[ T_2 \leq A R_n \leq A R(1/t), \]
which completes the proof of our lemma.

**Lemma (4.3.3):** If for each \( n \geq 1 \), the function \( f(x) \) can be approximated by a trigonometric polynomial \( t_n(x) \) of degree \( n \) (at most) such that
\[ \| f - t_n(x) \| \phi \leq \frac{\Omega(n)}{n^r}, \]
r being a positive integer or zero, then \( f(x) \) is equivalent to an absolutely continuous function having a derivative \( f^r(x) \). 

*Quade, E.S. [17]
of order \( r \), for which

\[
    w_{\phi}(\delta, f^r) \leq \delta^r \int_a^{\infty} \frac{\phi(x)}{x} \, dx + \int_{1/\delta}^{\infty} \frac{\phi(x)}{x} \, dx,
\]

where \( a \) and \( \Delta \) are constants which may depend on \( f(x) \) but not on \( \delta \) and \( \phi(x) \) is a function not identically zero which satisfies the following:

(i) \( \phi(x) \geq 0 \) and, at least for \( x \) greater than some \( x_0 \), decreases monotonically to zero as \( x \to \infty \).

(ii) \[
    \int_{x_0}^{\infty} \frac{\phi(x)}{x} \, dx \text{ exists},
\]

and

\[
    w_{\phi}(\delta, f) = \sup_{0 < |h| < \delta} |f(x+h) - f(x)|,
\]

if \( f(x) \in L_p^* \) and \( \delta > 0 \), for \( p > 1 \), \( L_p \) is class \( L_p^* \).

**Proof of theorem:** First we prove that the condition (4.3.1) is necessary. As in Zygmund, we write

\[
    f(x) - t_n^{p_1 q_1} = 2 \int_{\Lambda} \phi(t) \sum_{k=0}^{n} p_k q_{n-k} \sin(n-k)t \, dt + o(1),
\]

+ Zygmund A [28]
Applying Hölder's inequality and the fact that
\( \phi(t) \in W(L^p, \psi(t)) \), we get

\[
I_1 = \frac{2}{\kappa R(n)} \int_0^{\kappa/n} \phi(t) \frac{1}{t} \sum_{k=0}^{n} p_k q_{n-k} \sin(n-k)t \ dt + o(1),
\]

\[
= I_1 + I_2 + o(1), \text{ say.}
\]
Similarly
\[
I_2 \leq O \left( \frac{1}{R(n)} \right) \left\{ \int_{\kappa/n}^{\tau} t^{-\delta} \frac{\phi(t) \sin^{\beta} t}{\varphi(t)} \, dt \right\}^{1/p}.
\]

Let
\[
\sum_{k=0}^{n} p_k q_{n-k} \sin(n-k)t \varphi(t) q^{1/q}
\]
and
\[
\int_{\kappa/n}^{\tau} \left\{ \frac{R \left( \frac{1}{t} \right) \varphi(t)}{\sin^{\beta} t \cdot t^{-\delta}} \right\}^q \, dt
\]

Then
\[
= O \left( \frac{1}{R(n)} \right) \left\{ \int_{\kappa/n}^{\tau} \left[ \frac{1}{\sin^{\beta} (\kappa/n)} \right] \, dt \right\}^{1/q}
\]

Finally
\[
= O \left( \frac{1}{R(n)} \right) O(n^\delta) \left( \frac{1}{\sin^{\beta} (\kappa/n)} \right) \left\{ \int_{\kappa/n}^{\tau} \frac{R \left( \frac{1}{t} \right) \varphi(t)}{t^{-\delta}} \, dt \right\}^{1/q}
\]

And
\[
= O \left( \frac{1}{R(n)} \right) O(n^\delta) \left( \frac{1}{\sin^{\beta} (\kappa/n)} \right) \int_{\kappa/n}^{\tau} \left[ \frac{R(y) \varphi(y)}{y^{\delta-1}} \right]^{q} \, dy
\]

\[
\int_{\kappa/n}^{\tau} \left[ \frac{R(y) \varphi(y)}{y^{\delta-1}} \right]^{q} \, dy
\]

In order to prove that the condition (4.3.1) is also sufficient, we put \( r = 0, \gamma(t) = \frac{\psi(t)}{t^{-\beta} + \frac{1}{p}} \). M, (M being a positive constant) in Lemma (4.3.3) and get

\[
\begin{align*}
\psi_p\left(\frac{1}{n}, f\right) &= \circ\left(\frac{1}{n}\right) \cdot \int_{a}^{n} \frac{\psi\left(\frac{1}{t}\right)}{t^{-\beta} - \frac{1}{p}} \, dt + \int_{n}^{\infty} \frac{\psi\left(\frac{1}{t}\right)}{t^{-\beta} - \frac{1}{p} + 1} \, dt \\
&= \circ\left(\frac{1}{n}\right) \circ \left( \psi\left(\frac{1}{n}\right)^{\beta + \frac{1}{p} + 1} \right) \\
&\quad + \circ\left( \psi\left(\frac{1}{n}\right)^{\beta + \frac{1}{p}} \right).
\end{align*}
\]
This completes the proof of our theorem.

An immediate corollary of the above theorem is the following result:

Corollary (4.3.4): If \( f(x) \) is a periodic function belonging to the class Lip \( (\alpha, p) \), \( 0 < \alpha < 1 \), and if the sequences \( \{p_n\} \) and \( \{q_n\} \) are defined as (4.2.1) and (4.2.2) such that

\[
\frac{R(y)}{\int f^\alpha} \text{ is non-decreasing} \quad (4.3.2)
\]

then

\[
E_n(f) = \min \left\{ \left\| f - t_n p^\alpha q_n \right\| \right\} = \Theta \left( \frac{1}{n^{\alpha - \frac{1}{p}}} \right).
\]

Remark: It may be noted that the proof as given by B. N. Sahney and Gopal Rao in their result is wrong. The incorrect steps in their proof being

\[
\frac{1}{n} \int_0^{1/n} \frac{\phi(t)}{t^\alpha} p^{1/p} dt \right\} = \Theta(1),
\]

and

** Khan, Huzoor H. [9]
+ Sahney and Gopal Rao [18]
\[ \left\{ \int_{\infty}^{\frac{\pi}{n}} \left( \frac{\phi(t)}{t^{k}} \right)^{p} \, dt \right\}^{1/p} = O(1). \]

For, in this case, \( \phi(t) \notin \text{Lip} (\kappa - \frac{1}{p}) \) such that \( \kappa p > 1 \). We have given the correct proof of the problem.

4.4. In this section we prove the following theorem on the degree of approximation of the partial sums of a Fourier series for the class \( W(L^{p}, \psi(t)) \).

**Theorem** *(4.4.1):* If \( f \in W(L^{p}, \psi(t)) \) class such that

\[ \left\{ \int_{0}^{\frac{\pi}{n}} \left( \frac{u! \phi(u)!}{\psi(u)} \right)^{p} \, du \right\}^{1/p} = O\left( \frac{1}{n} \right), \]

then

\[ f(x) - S_{n}(x) = O\left( n^{\beta} + \frac{1}{p} \psi\left( \frac{1}{n} \right) \right), \]

uniformly almost everywhere.

**Proof:** We see that

\[ \int_{0}^{\frac{\pi}{n}} \phi(x) \, dx = \int_{0}^{\frac{\pi}{n}} \left( \frac{u! \phi(x)! \sin^{\beta} u!}{\psi(u)} \right) \psi(u) \, du, \]

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* Khan Huzoor H. [14]
\[
\left\{ \int_0^{\infty} \left( \frac{\phi_x(u) \log \sin \theta u}{\psi(u)} \right)^p \frac{du}{u} \right\}^{1/p} \times \left\{ \int_0^{\infty} \left( \frac{\psi(u)}{u \sin^\beta u} \right)^q du \right\}^{1/q} \\
\leq \left\{ \int_0^{\infty} \left( \frac{1}{\psi(u)} \right)^p \frac{1}{u^q \sin^\beta u} du \right\} \left\{ \int_0^{\infty} \left( \frac{1}{\psi(u)} \right)^q \frac{du}{u} \right\}^{1/q} \\
= \bigcirc \left( \frac{1}{n} \right) \bigcirc \left( \psi \left( \frac{1}{n} \right) \right) \left\{ \int_0^{\infty} \left( \frac{1}{u} \right) \frac{1}{u^q \sin^\beta q u} du \right\}^{1/q} \\
\leq \bigcirc \left( \frac{1}{n} \right) \bigcirc \left( \psi \left( \frac{1}{n} \right) \right) \left\{ \int_0^{\infty} \left( \frac{1}{u} \right)^q \frac{1}{u^q} du \right\}^{1/q} \\
= \bigcirc \left( \frac{1}{n} \right) \bigcirc \left( \psi \left( \frac{1}{n} \right) \right) \bigcirc \left\{ \int_0^{\infty} \left( \frac{1}{u} \right)^{q-\beta} du \right\}^{1/q} \\
= \bigcirc \left( \frac{1}{n} \right) \bigcirc \left( \psi \left( \frac{1}{n} \right) \right) \bigcirc \left( \frac{1}{n} \right)^{-q-\beta} + \frac{1}{q} \\
= \bigcirc \left( \psi \left( \frac{1}{n} \right) \right) \bigcirc \left( \frac{1}{n} \right)^{-q-\beta} + \frac{1}{p} - 1 \\
= \bigcirc \left( \psi \left( \frac{1}{n} \right) \right)^{\beta + \frac{1}{p} - 1}, \quad (4.4.1)
\]

and
\[
\frac{1}{\lambda/n} \int_{\lambda/n}^{\lambda} \frac{1}{u} \phi_x(u) - \phi_x(u+h) \, du \leq \frac{1}{\lambda/n} \int_{\lambda/n}^{\lambda} \frac{1}{u \sin \beta u} \, du,
\]

\[
\leq \left\{ \int_{\lambda/n}^{\lambda} \frac{du}{u^q \sin \beta u} \right\}^{1/q} \times \left\{ \int_{\lambda/n}^{\lambda} \frac{1/p}{u^q \sin \beta u} \, du \right\}^{1/q}
\]

\[
\leq \int_{\lambda/n}^{\lambda} |f(u) - f(u+h)|^p \sin \beta u \, du \times \left( \frac{n^\beta}{q-1} \right) \left( \frac{1}{n} \right)^{q-1} = o \left( \psi \left( \frac{1}{n} \right) \right) \circ (n^{\beta + \frac{1}{p}}),
\]

\[\text{(4.4.2)}\]

holds uniformly almost everywhere. Now (4.4.1) and (4.4.2) complete the proof of our theorem (4.4.1).

It may be noted that the following result is an immediate corollary of the above theorem.

\[\text{Corollary }^* (4.4.2): \text{ If } f(x) \in \text{Lip } (\kappa, p) \text{ where}\]

\[\text{* Shin-Ichi-Izumi [?] }\]
$0 < \alpha \leq 1, \quad p > 1, \quad \alpha > 1$

then

$$f(x) - s_n(x) = \mathcal{O} \left( \frac{1}{n^{\frac{1}{p}} - \frac{1}{p}} \right)$$

uniformly almost everywhere.