Chapter 3

Riemannian Manifolds - I

The subject of this thesis is to extend the combinatorial curve reconstruction approach to curves embedded in Riemannian manifolds. A Riemannian manifold is an abstraction of a curved space in which it is possible to measure geometric quantities such as length of a curve segment, the area of an enclosed region, angle between two curves at a point, etc. Our interest is to equip the manifold with the metric structure, which involves the idea of geodesics and the shortest distance between two point on the given manifold. Riemannian manifolds arise in variety of engineering applications. For example, meteorological studies involve the surface of earth which is a surface like a sphere, it is indeed a Riemannian manifold. The euclidean motion group used in graphics applications is also an example of a Riemannian manifold. In this chapter, we will define a few basic terms and state some of the results from Riemannian geometry which will serve as the building blocks for the development of the ideas ahead. At the end we will present examples of Riemannian manifolds relevant to the problem we will deal with. The focus will be on the sphere - $S^2$ and surface patches, euclidean motion group - $SE(3)$, and $SE(2)$ with scaling.
3.1 Differentiable Manifolds

A manifold $\mathcal{M}$ can be considered as a topological space which locally resembles a Euclidean space. A differentiable manifold allows partial differentiation and consequently all the features of differential calculus on $\mathcal{M}$. We will briefly present the essentials of manifold theory to the extent that is required for our work. The vector space $\mathbb{R}^n$ is a topological space, and the vector operations are continuous with respect to the topology. In addition we have the notion of differentiability for real valued functions on $\mathbb{R}^n$, i.e. $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable if the partial derivatives

$$\frac{\partial^{i_1 + \ldots + i_r} f}{\partial u^{i_1}_1 \ldots \partial u^{i_r}_r}$$

of all order exist and are continuous. Such functions are called $C^\infty$ functions.

The *natural coordinate functions* of $\mathbb{R}^n$ are mappings $u_i : \mathbb{R}^n \to \mathbb{R}$ defined by

$$u_i(x_1, \ldots, x_n) = x_i$$

for $i = 1, 2, \ldots, n$. A function $\phi : \mathbb{R}^m \to \mathbb{R}^n$ is differentiable, continuous or linear if and only if each $u_i \cdot \phi$ is differentiable, continuous or linear, respectively.

**Definition 3.1.** A patch (or chart) on a topological space $\mathcal{M}$ is a pair $(x, \mathcal{U})$, where $\mathcal{U}$ is an open subset of $\mathbb{R}^n$ and

$$x : \mathcal{U} \to x(\mathcal{U}) \subset \mathcal{M}$$

is a homeomorphism of $\mathcal{U}$ onto an open set $x(\mathcal{U})$ of $\mathcal{M}$. Let

$$x_i = u_i \circ x^{-1} : x(\mathcal{U}) \to \mathbb{R}$$

for $i = 1, \ldots, n$. Then $(x_1, \ldots, x_n)$ is called a system of local coordinates for $\mathcal{M}$. 


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3.1. Differentiable Manifolds

**Figure 3.1:** A coordinate patch \( x : \mathbb{R}^n \to \mathcal{M} \), with \( x(q) = p \).

**Definition 3.2.** An atlas \( \mathcal{A} \) on a topological space \( \mathcal{M} \) is a collection of patches \((x_\alpha, U_\alpha)\) such that \( x_\alpha : U_\alpha \subset \mathbb{R}^n \to \mathcal{M} \) and \( \bigcup_{\alpha} x_\alpha(U_\alpha) = \mathcal{M} \). A topological space \( \mathcal{M} \) equipped with an atlas is called a topological manifold.

**Figure 3.2:** Change of coordinates \( y^{-1} \circ x \). If it is differentiable for every pair of intersecting patches on \( \mathcal{M} \) then the manifold \( \mathcal{M} \) is called a differentiable manifold.

Let \( \mathcal{A} \) be an atlas on a topological space \( \mathcal{M} \). If \((x, U)\) and \((y, V)\) are any two patches in \( \mathcal{A} \) such that \( x(U) \cap y(V) = \mathcal{W} \) is a nonempty subset of \( \mathcal{M} \), then the map

\[
y^{-1} \circ x : x^{-1}(\mathcal{W}) \to y^{-1}(\mathcal{W})
\]  \hspace{1cm} (3.1)
is a homeomorphism between open subsets of $\mathbb{R}^n$. We call $y^{-1} \circ x$ a change of coordinates.

**Definition 3.3.** A differentiable manifold is a topological space $\mathcal{M}$ equipped with an atlas $\mathcal{A}$ such that the change of coordinates (3.1) is differentiable (that is, of class $C^\infty$) in the ordinary Euclidean sense*. The dimension of the manifold $\mathcal{M}$ (denoted by $\text{dim } \mathcal{M}$) is $n$.

**Example 3.1.** The Euclidean space $\mathbb{R}^n$ is a differentiable manifold. The identity map $1 : \mathbb{R}^n \to \mathbb{R}^n$

$$(u_1, \ldots, u_n) : \mathbb{R}^n \to \mathbb{R}^n$$

constitutes an atlas $\mathcal{A} := \{(1, \mathbb{R}^n)\}$ for $\mathbb{R}^n$ by itself.

**Example 3.2.** A regular surface $\mathcal{M}$ in $\mathbb{R}^n$ is a differentiable manifold.

1. A Monge patch is a patch $x : \mathcal{U} \to \mathbb{R}^3$ of the form

$$x(u, v) = (u, v, h(u, v)),$$

where $\mathcal{U}$ is an open set in $\mathbb{R}^2$ and $h$ is a differentiable function. Paraboloid defined as $h(u, v) := (u, v, au^2 + bv^2)$, where $a, b \neq 0$, and monkey saddle defined as $h(u, v) := (u, v, u^3 - 3uv^2)$ are examples of regular surfaces each parametrized by a single patch.

2. A sphere is an example of a regular surface which needs an atlas with at least two patches to cover it. Refer to section 4.3 for more details.

### 3.2 Tangent Vectors and Tangent Space

The tangent space to a differentiable manifold $\mathcal{M}$ at a point $p \in \mathcal{M}$ can be thought of as the best linear approximation to $\mathcal{M}$ at $p$. For surfaces in $\mathbb{R}^3$, a tangent vector at a point $p$ of the surface is

*In order to ensure uniqueness of convergence and avoid pathological situations, we will always take $\mathcal{M}$ to be a connected, Hausdorff topological space.
defined as the velocity in $\mathbb{R}^3$ of a curve in the surface passing through $p$. But one of the main aims of modern differential geometry is to present ideas in a way that is intrinsic to the manifold itself, in particular, is not dependent on an embedding in some higher dimensional vector space. For example, the definition of a differentiable manifold itself made no reference to such an embedding, and neither should the definition of a tangent space. In the following, we define the tangent vector as an equivalence class of curves on the manifold.

**Definition 3.4.** Let $M$ be a differentiable manifold. A differentiable function $\alpha : (-\varepsilon, \varepsilon) \rightarrow M$ is called a differentiable curve in $M$. Suppose that $\alpha(0) = p \in M$, and let $\mathcal{D}$ be the set of functions on $M$ that are differentiable at $p$. The tangent vector to the curve $\alpha$ at $t = 0$ is a function $\alpha'(0) : \mathcal{D} \rightarrow \mathbb{R}$ given by

$$
\alpha'(0)f = \frac{d(f \circ \alpha)}{dt}
$$

$$
\bigg|_{t=0}, \ f \in \mathcal{D}.
$$

A tangent vector at $p$ is the tangent vector at $t = 0$ of some curve $\alpha$ with $\alpha(0) = p$. The set of all tangent vectors to $M$ at $p$ will be indicated by $T_pM$.

If we choose a parametrization $x : U \rightarrow M$ at $p = x(0)$, we can express the function $f$ and the curve $\alpha$ in this parametrization by

$$
f \circ x(q) = f(x_1, \ldots, x_n), \ q = (x_1, \ldots, x_n) \in U\]

and

$$
x^{-1} \circ \alpha(t) = (x_1(t), \ldots, x_n(t)),
$$

respectively. Therefore, restricting $f$ to $\alpha$, we obtain

$$
\alpha'(0)f = \left. \frac{d}{dt}(f \circ \alpha) \right|_{t=0} = \left. \frac{d}{dt}f(x_1(t), \ldots, x_n(t)) \right|_{t=0} = \sum_{i=1}^{n} x_i'(0) \left( \frac{\partial f}{\partial x_i} \right) = \left( \sum_{i=1}^{n} x_i'(0) \left( \frac{\partial}{\partial x_i} \right)_0 \right) f.
$$
So the vector $\alpha'(0)$ can be expressed in the parametrization $x$ by

$$\alpha'(0) = \sum_i x_i'(0) \left( \frac{\partial}{\partial x_i} \right)_0.$$

where $\left( \frac{\partial}{\partial x_i} \right)_0$ is the tangent vector at $p$ of the coordinate curve $x_i \to x(0, \ldots, 0, x_i, 0, \ldots, 0)$.

The expression 3.2 shows that the tangent vector to the curve $\alpha$ at $p$ depends only on the derivative of $\alpha$ in a coordinate system. It follows from 3.2 that the set $T_p M$, with the usual operations of functions, forms a vector space of dimension $n$, and that the choice of parametrization $x : U \to M$ determines an associated basis $\left\{ \left( \frac{\partial}{\partial x_i} \right)_0, \ldots, \left( \frac{\partial}{\partial x_n} \right)_0 \right\}$ in $T_p M$. The linear structure in $T_p M$ defined above does not depend on the parametrization. The vector space $T_p M$ is called the tangent space of $M$ at $p$.

**Example 3.3.** Tangent space at a point on a 2-sphere is the tangent plane at a point. Depending upon the parametrization of a 2-sphere we can find orthogonal vectors spanning the tangent space at a given point. See section 4.3 for a particular parametrization of 2-sphere and the corresponding tangent space at a point on a 2-sphere.

We state the definition of differential of a differentiable mapping without proof. It will be used when we will discuss issues related to isometric embedding.

**Definition 3.5.** Let $M$ and $N$ be differentiable manifolds and let $\phi : M \to N$ be a differentiable mapping. For every $p \in M$ and for each $v \in T_p M$, choose a differentiable curve $\alpha : (-\varepsilon, \varepsilon) \to M$ with $\alpha(0) = p$, $\alpha'(0) = v$. Take $\beta = \phi \circ \alpha$. The mapping $d\phi_p : T_p M \to T_{\phi(p)} N$ given by $d\phi_p(v) = \beta'(0)$ is a linear mapping that does not depend on the choice of $\alpha$. The linear mapping $d\phi_p$ is called the differential of $\phi$ at $p$. Some times $\phi_p^*$ is also used in place of $d\phi_p$ to denote the differential.

In addition it can be shown that if $m$ and $n$ are the dimensions of $M$ and $N$ respectively,

$$\beta'(0) = d\phi_p(v) = \left( \frac{\partial y_i}{\partial x_j} \right) (x_j'(0)),$$
where \( i = 1, \ldots, m \) and \( j = 1, \ldots, n \) and \( \left( \frac{\partial y_i}{\partial x_j} \right) \) denotes an \( m \times n \) matrix and \( x'_j(0) \) denotes a column matrix with \( n \) elements obtained from the parametrizations \( x \) and \( y \).

**Figure 3.3**: Differential of map \( \phi : M \to N \) and representation of \( \phi \) in local coordinates.

**Definition 3.6.** Let \( M \) and \( N \) be differentiable manifolds. A mapping \( \phi : M \to N \) is called a diffeomorphism if it is differentiable, bijective and its inverse is also differentiable. The concept of a diffeomorphism is the natural idea of equivalence between differentiable manifolds.

**Definition 3.7.** Let \( M \) be an \( n \) dimensional differentiable manifold and let \( TM = \{ (p, v); p \in M, v \in T_pM \} \). \( TM \) is called the tangent bundle of \( M \).

**Definition 3.8.** A vector field \( X \) on a differentiable manifold \( M \) is a correspondence that associates to each point \( p \in M \) a vector \( X(p) \in T_pM \). In terms of mappings, \( X \) is a mapping of \( M \) into the tangent bundle \( TM \). The field is differentiable if the mapping \( X : M \to TM \) is differentiable.
Considering a parametrization $x : U \subset \mathbb{R}^n \rightarrow M$ we can write

$$X(p) = \sum_{i=1}^{n} a_i(p) \frac{\partial}{\partial x_i}, \quad (3.3)$$

where each $a_i : U \rightarrow \mathbb{R}$ is a function on $U$ and $\left\{ \frac{\partial}{\partial x_i} \right\}_{i=1,...,n}$ is the basis associated to $x$. It is clear that $x$ is differentiable if and only if the functions $a_i$ are differentiable for some parametrization.

A vector field can also be thought of as a mapping $X : \mathcal{D}(M) \rightarrow \mathcal{D}(M)$. $\mathcal{D}(M) := C^\infty(M)$ denotes the ring of smooth functions on $M$.

### 3.3 Riemannian Manifolds

**Definition 3.9.** A Riemannian metric (or Riemannian structure) on a differentiable manifold $M$ is a correspondence which associates to each point $p \in M$ an inner product $\langle \cdot, \cdot \rangle_p$, a symmetric, bilinear, positive-definite form on the tangent space $T_pM$, which varies differentiably in the sense that if $x : U \subset \mathbb{R}^n \rightarrow M$ is a system of coordinates around $p$, with $x(x_1, \ldots, x_n) = q \in x(U)$ and $\frac{\partial}{\partial x_i}(q) = dx_q(0, \ldots, 1, \ldots, 0)$, then $\langle \frac{\partial}{\partial x_i}(q), \frac{\partial}{\partial x_j}(q) \rangle_q = g_{ij}(x_1, \ldots, x_n)$ is a differentiable function on $U$.

Whenever there is no possibility of confusion the index $p$ in the function $\langle \cdot, \cdot \rangle_p$ is discarded. The function $g_{ij} = g_{ji}$ is called the local representation of the Riemannian metric in the coordinate system $x : U \subset \mathbb{R}^n \rightarrow M$. A differentiable manifold with a given Riemannian metric will be called a Riemannian manifold.

Let $M$ and $N$ be Riemannian manifolds. A diffeomorphism $f : M \rightarrow N$ is called an isometry if:

$$\langle u, v \rangle_p = \langle df_p(u), df_p(v) \rangle_{f(p)}, \quad (3.4)$$

for all $p \in M$ and $u, v \in T_pM$.

**Example 3.4.** $\mathbb{R}^2$ with the usual inner product defined as the vector dot product is a Riemannian
manifold. Suppose $\mathbb{R}^2$ is equipped with a different inner product, say $g_{ij} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. It is certainly a valid Riemannian metric since it is symmetric and positive definite.

Now since $g_{ij}$ is constant at all points on $\mathbb{R}^2$, we can find a linear isometry $f : (\mathbb{R}^2, \langle \cdot, \cdot \rangle_{g_{ij}}) \rightarrow (\mathbb{R}^2, \langle \cdot, \cdot \rangle)$, given by, $f(x, y) = (x, x + y)$.

The differential, i.e. the Jacobian of $f$, of the isometry is $f_* = df = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

Moreover it can be shown that every differentiable manifold $\mathcal{M}$ (Hausdorff with countable basis) has a Riemannian metric. Now by using the Riemannian inner product we will proceed to calculate the lengths of curves in Riemannian manifolds.

**Definition 3.10.** A differentiable mapping $c : I \rightarrow \mathcal{M}$ of $I := [0, 1] \subset \mathbb{R}$ into a differentiable manifold $\mathcal{M}$ is called a curve.

A parametrized curve can admit self-intersection as well as corners. To avoid difficulties at singularities we assume that the curve under study is always a smooth (infinitely differentiable) curve.

**Definition 3.11.** A vector field $V$ along a curve $c : I \rightarrow \mathcal{M}$ is a differentiable mapping that associates to every $t \in I$ a tangent vector $V(t) \in T_{c(t)}\mathcal{M}$. To say that $V$ is differentiable means that for any differentiable function $f$ on $\mathcal{M}$, the function $t \rightarrow V(t)f$ is a differentiable function on $I$.

The vector field $dc(\frac{d}{dt})$, denoted by $\frac{dc}{dt}$, is called the velocity field of $c$. The restriction of a curve $c$ to a closed interval $[a, b] \subset I$ is called a segment. If $\mathcal{M}$ is a Riemannian manifold, we define the length of the segment by

$$l^b_a(c) = \int_a^b \left\langle \frac{dc}{dt}, \frac{dc}{dt} \right\rangle^{1/2} dt.$$  \hfill (3.5)
3.4 Covariant Derivative, Parallel Transport and Geodesics

Let $S \subset \mathbb{R}^3$ be a surface and let $c : I \to S$ be a curve in $S$, with $X : I \to \mathbb{R}^3$ a vector field along $c$ tangent to $S$. The vector $\frac{dX}{dt}(t), \ t \in I$ does not in general belong to $T_{c(t)}S$. To make sure that the differentiation is an intrinsic geometric notion on $S$, instead of usual derivative $\frac{dX}{dt}(t)$, the orthogonal projection of $\frac{dX}{dt}(t)$ on $T_{c(t)}S$ is considered. This is called the covariant derivative and is denoted by $\nabla_t X(t)$. Still it is interesting to note that this process of taking the derivative is valid only by making use of the ambient space in which the manifold is embedded. To overcome the orthogonal projection, we need to define a so called Levi-Civita connection, denoted as $\nabla$, on a Riemannian manifold. In fact it can be shown that $\nabla$ is uniquely determined from a given metric $\langle \cdot, \cdot \rangle$.

It is occasionally helpful to visualize covariant derivative as the intrinsic directional derivative, where we take the definition of directional derivative and replace the vector difference by vector difference of parallel translated vector. For more on the above notions, the reader may refer to [17].

**Definition 3.12.** On a differential manifold $\mathcal{M}$, an affine connection $\nabla$ is defined as a mapping

$$
\nabla : \mathcal{V}(\mathcal{M}) \times \mathcal{V}(\mathcal{M}) \to \mathcal{V}(\mathcal{M})
$$

(3.6)

where $\mathcal{V}(\mathcal{M})$ is the set of all vector fields of class $C^\infty$ on $\mathcal{M}$. It is denoted by $(X, Y) \to \nabla_X Y$ and satisfies the following properties:

1. $\nabla_{fX+gY}Z = f\nabla_X Z + g\nabla_Y Z$.
2. $\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z$.
3. $\nabla_X (fY) = f\nabla_X Y + X(f)Y$,

where $X, Y, Z \in \mathcal{V}(\mathcal{M})$ and $f, g \in \mathcal{D}(\mathcal{M})$. 
Let $\mathcal{M}$ be a differentiable manifold with an affine connection $\nabla$. There exists a unique correspondence which associates to a vector field $X$ along the differentiable curve $c: I \to \mathcal{M}$ an another vector field $\frac{DX}{dt}$ along $c$, called the covariant derivative of $X$ along $c$, such that

1. $\frac{D}{dt}(X + Y) = \frac{DX}{dt} + \frac{DY}{dt}$.

2. $\frac{D}{dt}(fX) = \frac{df}{dt}X + f\frac{DX}{dt}$, where $Y$ is a vector field along $c$ and $f \in \mathcal{D}(\mathcal{M})$.

3. If $X$ is induced by a vector field $V \in \mathfrak{X}(\mathcal{M})$, i.e., $X(t) = V(c(t))$, then $\frac{DX}{dt} = \nabla_{dc/dt} V$.

Let $x: U \subset \mathbb{R}^n \to \mathcal{M}$ be a system of coordinates with $c(I) \cap x(U) \neq \emptyset$ and let $(x_1(t), \ldots, x_n(t))$ be the local expression of $c(t)$, $t \in I$. Let $X_i = \frac{\partial}{\partial x_i}$. then we can express the field $X$ locally as $X = \sum_j x^j X_j$, $j = 1, \ldots, n$, where $x^j = x^j(t)$ and $X_j = X_j(c(t))$.

\[
\frac{DX}{dt} = \sum_j \frac{dx^j}{dt} X_j + \sum_{i,j} \frac{dx_i}{dt} x^j \nabla_{X_i} X_j. \tag{3.7}
\]

The correspondence expressed in (3.7) satisfies the above three conditions. The above expression is the expression of covariant derivative in terms of a connection.

**Definition 3.13.** Let $\mathcal{M}$ be a differentiable manifold with an affine connection $\nabla$. A vector field $X$ along curve $c: I \to \mathcal{M}$ is called parallel if $\frac{DX}{dt} = 0$.

Suppose there exists such a parallel field $X$ in $\mathfrak{X}(U)$ along $c$ with $X(t_0) = X_0$. Then $X = \sum_j x^j X_j$ satisfies

\[0 = \frac{DX}{dt} = \sum_j \frac{dx^j}{dt} X_j + \sum_{i,j} \frac{dx_i}{dt} x^j \nabla_{X_i} X_j.
\]

Putting $\nabla_{X_i} X_j = \sum_k \Gamma^k_{ij}$, called the Christoffel symbols, and replacing $j$ with $k$ in the first sum, we obtain

\[
\frac{DX}{dt} = \sum_k \left\{ \frac{dx^k}{dt} + \sum_{i,j} x^j \frac{dx_i}{dt} \Gamma^k_{ij} \right\} X_k = 0.
\]
The system of \( n \) differential equations in \( x^k(t) \),

\[
\frac{dx^k}{dt} + \sum_{i,j} x^j \frac{dx_i}{dt} \Gamma^k_{ij} = 0, \quad k = 1, \ldots, n, \tag{3.8}
\]

posses a unique solution satisfying the initial condition. Moreover, since the system is linear, any solution is defined for all \( t \in I \). As an example, we present parallel transport of a vector on a sphere in Appendix C.

**Theorem 3.14. (Levi-Civita).** Given a Riemannian manifold \( M \), there exists a unique affine connection \( \nabla \) on \( M \) satisfying the conditions;

1. \( \nabla \) is symmetric, i.e. \( \nabla_X Y - \nabla_Y X = [X, Y] \) for all \( X, Y \in \mathfrak{X}(M) \). \([X, Y] = XY - YX\) is called the bracket.

2. \( \nabla \) is compatible with the Riemannian metric.

We will encourage reader to follow the definition of compatibility from [17]. A connection is defined in terms of the Christoffel symbols. Observe that for Euclidean space \( \mathbb{R}^n \) the \( \Gamma^k_{ij} = 0 \). Observer also that the covariant derivative (3.7) differs from the usual derivative in \( \mathbb{R}^n \) by terms which involve the Christoffel symbols.

**Definition 3.15.** A parametrized curve \( \gamma : I \to M \) is a geodesic at \( t_0 \in I \) if \( \frac{D}{dt} \left( \frac{d\gamma}{dt} \right) = 0 \) at the point \( t_0 \); if \( \gamma \) is geodesic \( \forall t \in I \), we say \( \gamma \) is a geodesic.

This definition describes the geodesic as a curve with zero acceleration. In other words the magnitude of the velocity vector is constant. We will see that a geodesic minimizes the arc length for points which are close enough. Geodesics are local length minimizers.
3.5 Riemannian Manifold as a Metric Space

Suppose in a system of coordinates \((x, \mathcal{U})\) about \(\gamma(t_0), \gamma\) is a geodesic. From the definition above the curve \(\gamma(t) = (x_1(t), \ldots, x_n(t))\) is a geodesic iff,

\[
\frac{D}{dt} \left( \frac{d\gamma}{dt} \right) = \sum_k \left( \frac{d^2 x^k}{dt^2} + \sum_{i,j} \frac{dx_i}{dt} \frac{dx_j}{dt} \Gamma^k_{ij} \right) \frac{\partial}{\partial x^k} = 0.
\]

So we get a second order system of nonlinear ordinary differential equations,

\[
\frac{d^2 x^k}{dt^2} + \sum_{i,j} \frac{dx_i}{dt} \frac{dx_j}{dt} \Gamma^k_{ij} = 0, \quad k = 1, \ldots, n.
\] (3.9)

Existence of solution for a given initial value problem is guaranteed locally. One may refer to the results on existence in [17]. Geodesic starting from a point \(q\) in the direction \(v \in T_q \mathcal{M}\) within small interval \((-\delta, \delta)\) will be denoted by \(\gamma(t, q, v)\). In regard to this the exponential map is defined next.

Let \(p \in \mathcal{M}\) and let \(U \subset T \mathcal{M}\) be a suitable open set. Then the map \(\exp : U \to \mathcal{M}\) given by

\[
\exp(q, v) = \gamma(1, q, v), \quad (q, v) \in U
\]

is called the exponential map on \(U\). It is a differentiable map. If we restrict it to tangent space \(T_q \mathcal{M}\), we get

\[
\exp_q : B_\varepsilon(0) \subset T_q \mathcal{M} \to \mathcal{M}
\]

denoted by \(\exp_q(v) = \exp(q, v)\), where \(B_\varepsilon(0)\) is an open ball with center at the origin 0 of \(T_q \mathcal{M}\).

On manifold \(\mathcal{M}\), \(\exp_q(v)\) is a point obtained by traveling the length equal to \(|v|\), starting from \(q\), along a geodesic which passes through \(q\) with velocity \(\frac{v}{|v|}\). We state the following result which will play a crucial role in development of tubular neighborhood for a curve on manifold without proof.

**Proposition 3.16.** Given \(p \in \mathcal{M}\), there exists an \(\varepsilon > 0\) such that \(\exp_p : B_\varepsilon \subset T_p \mathcal{M} \to \mathcal{M}\) is a
diffeomorphism of $B_\varepsilon(0)$ onto an open subset of $\mathcal{M}$.

For Lie groups the exponential map plays an important role. The elements of lie algebra (tangent space at the group identity) are mapped to group elements via $\exp$.

**Definition 3.17.** A Riemannian manifold $\mathcal{M}$ is geodesically complete if for all $p \in \mathcal{M}$, the $\exp_p$ is defined for all $v \in T_p\mathcal{M}$, i.e. geodesic $\gamma(t)$ starting from $p$ is defined for all values of $t \in \mathbb{R}$.

Now we define distance function on a Riemannian manifold $\mathcal{M}$. Given two points $p, q \in \mathcal{M}$, consider all the piecewise differentiable curves joining $p$ and $q$. Such curves exist since $\mathcal{M}$ is connected.

**Definition 3.18.** The distance $d(p, q)$ is defined by $d(p, q) = \infimum$ of the lengths of all curves $c_{p,q}$, where $c_{p,q}$ is a piecewise differentiable curve joining $p$ and $q$. With the distance $d$, $\mathcal{M}$ is a metric space, i.e.,

1. $d(p, r) \leq d(p, q) + d(q, r)$,
2. $d(p, q) = d(q, p)$,
3. $d(p, q) \geq 0$, and $d(p, q) = 0$ iff $p = q$.

If there exists a minimizing geodesic $\gamma$ joining $p$ to $q$ then $d(p, q) = \text{length of } \gamma$. To be in a position to work with manifolds as metric spaces we need the Hopf-Rinow-de Rham Theorem which is stated here without proof:

**Theorem 3.19.** Let $\mathcal{M}$ be a Riemannian manifold and let $p \in \mathcal{M}$. The following are equivalent:

1. $\exp_p$ is defined on all of $T_p\mathcal{M}$.
2. $\mathcal{M}$ is complete as a metric space.
3. $\mathcal{M}$ is geodesically complete.
4. For any $q \in \mathcal{M}$ there exists a geodesic $\gamma$ joining $p$ to $q$ with $\ell(\gamma) = d(p, q)$.

The Riemannian manifolds we are concerned with, which arise in engineering applications, are complete in the above sense. Having equipped Riemannian manifolds with a metric structure, we will turn our attention to suitable examples in the next chapter.