Chapter 6

Zero-divisor Graph of Non-commutative Ring

The concept of zero-divisor graph $\Gamma(R)$ of a commutative ring $R$ was introduced by Beck [19]. However, he let all the elements of a commutative ring be vertices of the graph and was mainly interested in colorings. In Anderson and Livingston [12], the vertex set of $\Gamma(R)$ was chosen to be $\mathbb{Z}(R)$, set of zero-divisors of $R$, and the authors studied the interplay between the ring-theoretic properties of a commutative ring $R$ and the graph-theoretic properties of $\Gamma(R)$. In this chapter, we have studied the zero-divisor directed graph of non-commutative ring. Six relations have been defined in directed graph and their connections with right as well as left annihilator of non-commutative rings have been discussed. We have also defined various orthogonal conditions and have investigated their relations with annihilators of non-commutative rings. Some interesting properties of these orthogonal conditions with regular elements of non-commutative ring have been obtained. Again using orthogonal conditions and annihilators, necessary conditions for getting directed circuit of length 4, have been obtained in zero-divisor directed graph of non-commutative ring. By using orthogonal condition, we have obtained some results on reduced rings. Finally, we have defined weak complemented graph and have obtained two important results of zero-divisor directed graph of non-commutative ring. The chapter is the outcome of our paper "Zero-divisor graphs of non-commutative rings" which is
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6.1 Definitions and notations

Throughout this discussion $R$ denotes a non-commutative ring and $G$ is a directed graph unless otherwise stated.

**Definition 6.1.1.** [71] The zero-divisor graph of $R$, denoted by $\Gamma(R)$, is the directed graph with vertices $Z(R)^* = Z(R) - \{0\}$, the set of non-zero zero-divisors of $R$, and for distinct $x, y \in Z(R)^*$, $x$ is adjacent to $y$ or $y$ is adjacent from $x$, $(x \text{ adj } y)$ if and only if $xy = 0$. If $x$ is not adjacent to $y$ then we write $x \text{ nadj } y$. We say that $x$ and $y$ are not adjacent if $x$ is not adjacent to $y$ and $y$ is also not adjacent to $x$.

**Definition 6.1.2.** For $a, b \in V(G)$, $a$ is related to $b$ under the relation $R_1$ if and only if $a$ and $b$ are not adjacent but $a$ and $b$ are adjacent to exactly the same vertices i.e. $a$ is adjacent to $x$ if and only if $b$ is adjacent to $x$ and we write $aR_1b$. Similarly we define five other relations $R_2, R_3, R_4, R_5,$ and $R_6$, on $V(G)$, as follows:

(i) $aR_2b$ if and only if $a$ and $b$ are not adjacent and $x$ is adjacent to $a$ if and only if $x$ is adjacent to $b$.

(ii) $aR_3b$ if and only if $a$ is not adjacent to $b$ and $a$ is adjacent to $x$ if and only if $x$ is adjacent to $b$.

(iii) $aR_4b$ if and only if $a$ is not adjacent to $b$ and $a$ is adjacent to $x$ if and only if $b$ is adjacent to $x$.

(iv) $aR_5b$ if and only if $a$ is not adjacent to $b$ and $x$ is adjacent to $a$ if and only if $x$ is adjacent to $b$. 

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(v) \(aR_b b\) if and only if \(a\) and \(b\) are not adjacent and \(a\) is adjacent to \(x\) if and only if \(x\) is adjacent to \(b\).

Then we observe that \(R_1\) and \(R_2\) are two equivalence relations on \(V(G)\).

**Definition 6.1.3.** For \(a, b \in V(G)\), \(a\) is strong orthogonal to \(b\), written \(a \perp_s b\), if \(a\) is adjacent to \(b\) and there is no vertex \(c\) of \(G\) such that \(b\) is adjacent to \(c\) and \(c\) is adjacent to \(a\). Similarly, we define other orthogonal conditions as follows:

(i) \(a\) and \(b\) are orthogonal, written \(a \perp b\), if \(a\) and \(b\) are adjacent and there is no vertex \(c\) of \(G\) such that \(b\) is adjacent to \(c\) and \(c\) is adjacent to \(a\).

(ii) \(a\) is 1\(^{st}\) weak orthogonal to \(b\), written \(aw_1 b\), if \(a\) is adjacent to \(b\) and there is no vertex \(c\) of \(G\) such that \(c\) is adjacent to \(a\) and \(c\) is adjacent to \(b\).

(iii) \(a\) is 2\(^{nd}\) weak orthogonal to \(b\), written \(aw_2 b\), if \(a\) is adjacent to \(b\) and there is no vertex \(c\) of \(G\) such that \(a\) is adjacent to \(c\) and \(b\) is adjacent to \(c\).

(iv) \(a\) and \(b\) are said to be 1\(^{st}\) weak orthogonal, written \(a \perp_1 b\), if \(a\) and \(b\) are adjacent and there is no vertex \(c\) of \(G\) such that \(c\) is adjacent to \(a\) and \(c\) is adjacent to \(b\).

(v) \(a\) and \(b\) are said to be 2\(^{nd}\) weak orthogonal, written \(a \perp_2 b\), if \(a\) and \(b\) are adjacent and there is no vertex \(c\) of \(G\) such that \(a\) is adjacent to \(c\) and \(b\) is adjacent to \(c\).

**Definition 6.1.4.** \(G\) is said to be 1\(^{st}\) weak complemented, if for every vertex \(a\) in \(G\), there is a vertex \(b\) with \(aw_1 b\) and that \(G\) is uniquely complemented, whenever \(aw_1 b\) and \(aw_1 c\), then \(bR_1 c\).

**Remark 6.1.1.** For any subset \(A\) of \(R\), \(A^*\) contains all non-zero elements of \(R\).
6.2 Annihilator on $\Gamma(R)$

In this section, we discuss some interesting results of the relations $R_1, R_2, R_3, R_4, R_5,$ and $R_6$ with the annihilators (left as well as right) of a non-commutative ring $R$. Imposing conditions like reduced and von-Neumann regularity in non-commutative rings, equivalence of relations $R_1, R_2, R_3, R_4, R_5,$ and $R_6$ with the annihilators are established. Finally, we relate various orthogonal conditions with the annihilators of $R$.

**Theorem 6.2.1.** The relation $R_1$ is an equivalence relation on $V(G)$, where $G$ is any directed graph.

*Proof.* Since $G$ has no any directed self-loop, so $aR_1a$, for every $a \in V(G)$ Again, for $a, b \in V(G)$, $aR_1b$, then $bR_1a$. Also if $aR_1b$ and $bR_1c$, for $c \in V(G)$, then $aR_1c$. Since $a$ and $c$ are adjacent implies $a \text{ adj } c$ i.e. $b \text{ adj } c$, which is a contradiction. Also for $x \in V(G)$, $a \text{ adj } x \Leftrightarrow b \text{ adj } x \Leftrightarrow c \text{ adj } x$. Hence $R_1$ is an equivalence relation on $V(G)$. $\blacksquare$

**Theorem 6.2.2.** The relation $R_2$ is an equivalence relation on $V(G)$, where $G$ is any directed graph.

*Proof.* Since $G$ has no any directed self-loop, so $aR_2a$, for every $a \in V(G)$ Again, for $a, b \in V(G)$, $aR_2b$, then $bR_2a$. Also if $aR_2b$ and $bR_2c$, for $c \in V(G)$, then $aR_2c$. Since $a$ and $c$ are adjacent implies $a \text{ adj } c$ i.e. $a \text{ adj } b$, which is a contradiction. Also for $x \in V(G)$, $x \text{ adj } a \Leftrightarrow x \text{ adj } b \Leftrightarrow x \text{ adj } c$. Hence $R_2$ is an equivalence relation on $V(G)$. $\blacksquare$

**Theorem 6.2.3.** For distinct $a, b \in Z(R)^*$, $aR_1b$ in $\Gamma(R)$ if and only if $\text{Rann}(a) - \{a\} = \text{Rann}(b) - \{b\}$.

*Proof.* First we assume that $aR_1b$ in $\Gamma(R)$. Let $x \in \text{Rann}(a) - \{a\}$. This implies $ax = 0, x \neq a$ i.e. $a \text{ adj } x$. Since $aR_1b$, so $b \text{ adj } x$. From this, $bx = 0$ and so $x \in \text{Rann}(b)$. Also...
$x \neq b$. This gives

$$Rann(a) - \{a\} \subseteq Rann(b) - \{b\}.$$

Similarly,

$$Rann(b) - \{b\} \subseteq Rann(a) - \{a\}.$$

Conversely, we suppose that $Rann(a) - \{a\} = Rann(b) - \{b\}$. If $a \text{ adj } b$, then we get $b \in Rann(b) - \{b\}$, a contradiction. So $a \text{ nadj } b$. Similarly $b \text{ nadj } a$. Again for $x \in Z(R)^*$,

$$a \text{ adj } x \iff ax = 0 \iff x \in Rann(b) - \{b\} \iff b \text{ adj } x.$$

Hence $aR_1b$. 

**Theorem 6.2.4.** For distinct $a, b \in Z(R)^*$, $aR_2b$ in $\Gamma(R)$ if and only if $Lann(a) - \{a\} = Lann(b) - \{b\}$.

**Proof.** First we assume that $aR_2b$ in $\Gamma(R)$. Let $x \in Lann(a) - \{a\}$. This implies $xa = 0, x \neq a$ i.e. $x \text{ adj } a$. Since $aR_2b$, so $x \text{ adj } b$. From this, $xb = 0$ and so $x \in Lann(b)$. Also $x \neq b$. This gives

$$Lann(a) - \{a\} \subseteq Lann(b) - \{b\}.$$

Similarly,

$$Lann(b) - \{b\} \subseteq Lann(a) - \{a\}.$$

Conversely, we suppose that $Lann(a) - \{a\} = Lann(b) - \{b\}$. If $a \text{ adj } b$, then we get
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\[ a \in \text{Lann}(a) - \{a\}, \text{ a contradiction. So } a \nadj b. \text{ Similarly } b \nadj a. \text{ Again for } x \in Z(R)^*, \]

\[ x \adj a \iff xa = 0 \]

\[ \iff x \in \text{Lann}(b) - \{b\} \]

\[ \iff x \adj b. \]

Hence \( aR_2b \).

\textbf{Theorem 6.2.5.} For \( a, b \in Z(R)^* \), \( aR_3b \) in \( \Gamma(R) \) if and only if \( \text{Rann}(a) - \{a\} = \text{Lann}(b) - \{b\} \).

\textit{Proof.} For \( a = b \), the result is obvious. First we assume that \( aR_3b \) in \( \Gamma(R) \). Let \( x \in \text{Rann}(a) - \{a\} \). This implies \( ax = 0, x \neq a \) i.e. \( a \adj x \). Since \( aR_3b \), so \( x \adj b \). From this, \( xb = 0 \) and so \( x \in \text{Lann}(b) \). Also \( x \neq b \). This gives

\[ \text{Rann}(a) - \{a\} \subseteq \text{Lann}(b) - \{b\}. \]

Similarily,

\[ \text{Lann}(b) - \{b\} \subseteq \text{Rann}(a) - \{a\}. \]

Conversely, we suppose that \( \text{Rann}(a) - \{a\} = \text{Lann}(b) - \{b\} \). If \( a \adj b \), then we get \( b \in \text{Lann}(b) - \{b\}, \) a contradiction. So \( a \nadj b. \text{ Again for } x \in Z(R)^*, \]

\[ a \adj x \iff ax = 0 \]

\[ \iff x \in \text{Lann}(b) - \{b\} \]

\[ \iff x \adj b. \]

Hence \( aR_3b. \)

\textbf{Theorem 6.2.6.} For distinct \( a, b \in Z(R)^* \), \( aR_4b \) in \( \Gamma(R) \) if and only if \( \text{Rann}(a) - \{a\} = \text{Rann}(b) - \{b\} \).
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**Proof.** First we assume that \( aR_4b \) in \( \Gamma(R) \). Let \( x \in Rann(a) - \{a\} \). This implies \( ax = 0 \), \( x \neq a \) i.e. \( a \ adj x \). Since \( aR_4b \), so \( b \ adj x \). From this, \( bx = 0 \) and so \( x \in Rann(b) \). Also \( x \neq b \). This gives

\[
Rann(a) - \{a\} \subseteq Rann(b) - \{b\}.
\]

Similarly,

\[
Rann(b) - \{b\} \subseteq Rann(a) - \{a\}.
\]

Conversely, we suppose that \( Rann(a) - \{a\} = Rann(b) - \{b\} \). If \( a \ adj b \), then we get \( b \in Rann(b) - \{b\} \), a contradiction. So \( a \ nadj b \). Again for \( x \in Z(R)^* \),

\[
a \ adj x \iff ax = 0
\]

\[
\iff x \in Rann(b) - \{b\}
\]

\[
\iff b \ adj x.
\]

Hence \( aR_4b \). \( \blacksquare \)

**Theorem 6.2.7.** For distinct \( a, b \in Z(R)^* \), \( aR_5b \) in \( \Gamma(R) \) if and only if \( Lann(a) - \{a\} = Lann(b) - \{b\} \).

**Proof.** First we assume that \( aR_5b \) in \( \Gamma(R) \). Let \( x \in Lann(a) - \{a\} \). This implies \( xa = 0 \), \( x \neq a \) i.e. \( x \ adj a \). Since \( aR_5b \), so \( x \ adj b \). From this, \( xb = 0 \) and so \( x \in Lann(b) \). Also \( x \neq b \). This gives

\[
Lann(a) - \{a\} \subseteq Lann(b) - \{b\}.
\]

Similarly,

\[
Lann(b) - \{b\} \subseteq Lann(a) - \{a\}.
\]
Conversely, we suppose that $\text{Lann}(a) - \{a\} = \text{Lann}(b) - \{b\}$. If $a \text{ adj } b$, then we get $a \in \text{Lann}(a) - \{a\}$, a contradiction. So $a \text{ nadj } b$. Again for $x \in Z(R)^*$,

\[
x \text{ adj } a \iff xa = 0 \\
\iff x \in \text{Lann}(b) - \{b\} \\
\iff x \text{ adj } b.
\]

Hence $aR_b b$.

**Theorem 6.2.8.** For $a, b \in Z(R)^*$, $aR_b b$ in $\Gamma(R)$ if and only if $\text{Rann}(a) - \{a\} = \text{Lann}(b) - \{b\}$.

**Proof.** For $a = b$, the result is obvious. First we assume that $aR_b b$ in $\Gamma(R)$. Let $x \in \text{Rann}(a) - \{a\}$. This implies $ax = 0$, $x \neq a$ i.e. $a \text{ adj } x$. Since $aR_b b$, so $x \text{ adj } b$. From this, $xb = 0$ and so $x \in \text{Lann}(b)$. Also $x \neq b$. This gives

\[
\text{Rann}(a) - \{a\} \subseteq \text{Lann}(b) - \{b\}.
\]

Similarly,

\[
\text{Lann}(b) - \{b\} \subseteq \text{Rann}(a) - \{a\}.
\]

Conversely, we suppose that $\text{Rann}(a) - \{a\} = \text{Lann}(b) - \{b\}$. If $a \text{ adj } b$, then we get $b \in \text{Lann}(b) - \{b\}$, a contradiction. So $a \text{ nadj } b$. Similarly, $b \text{ nadj } a$. Again for $x \in Z(R)^*$,

\[
a \text{ adj } x \iff ax = 0 \\
\iff x \in \text{Lann}(b) - \{b\} \\
\iff x \text{ adj } b.
\]

Hence $aR_b b$.

**Theorem 6.2.9.** For the idempotent elements $a, b \in R$ with $aR_b b$, $a = b$. 

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Proof. Since \(a(1-a) = 0\), so \(a \text{ adj } (1-a)\). This gives \(b = ab\). Similarly \((1-b)b = 0\) implies \(a = ab\). Hence \(a = b\).

Similarly, for the idempotent elements \(a, b \in R\) with \(aRb, a = b\).

**Theorem 6.2.10.** Consider the following statements for a non-commutative ring \(R\) and \(a, b \in Z(R)^*\).

1. (i) \(aR_1b\), (ii) \(aR_2b\), (iii) \(aR_3b\), (iv) \(aR_4b\), (v) \(aR_5b\), (vi) \(aR_6b\).
2. (i) \(\text{Rann}(a) = \text{Rann}(b)\), (ii) \(\text{Lann}(a) = \text{Lann}(b)\), (iii) \(\text{Rann}(a) = \text{Lann}(b)\)

(a) If \(R\) is reduced, then (1)(i) and (2)(i) are equivalent, (1)(ii) and (2)(ii) are equivalent, (1)(iii) and (2)(iii) are equivalent, (1)(iv) and (2)(i) are equivalent, (1)(v) and (2)(ii) are equivalent and (1)(vi) and (2)(iii) are equivalent.

(b) If \(R\) is von-Neumann regular, then (1)(i) and (2)(i) are equivalent, (1)(ii) and (2)(ii) are equivalent, (1)(iii) and (2)(iii) are equivalent, (1)(iv) and (2)(i) are equivalent, (1)(v) and (2)(ii) are equivalent, (1)(vi) and (2)(iii) are equivalent.

**Theorem 6.2.11.** For \(a, (\neq) b \in Z(R)^*\), \(a \perp_s b\) in \(\Gamma(R)\) if and only if \(ab = 0\) and \(\text{Lann}(a) \cap \text{Rann}(b) \subseteq \{0, a, b\}\).

Proof. First we assume \(a \perp_s b\). So \(ab = 0\). Let \(x \in \text{Lann}(a) \cap \text{Rann}(b)\). This gives \(xa = 0\), \(bx = 0\). If \(x \notin \{0, a, b\}\) then we get a contradiction to the fact that \(a \perp_s b\). This implies \(x \in \{0, a, b\}\). Conversely we assume \(ab = 0\) and \(\text{Lann}(a) \cap \text{Rann}(b) \subseteq \{0, a, b\}\). If possible, suppose there exists a non-zero element \(x\) distinct from \(a\) and \(b\) with \(x \text{ adj } a\), \(b \text{ adj } x\). Then we get \(x \in \text{Lann}(a) \cap \text{Rann}(b)\), a contradiction. This contradiction implies that \(a \perp_s b\). Hence the result.
Theorem 6.2.12. For \(a, (\neq)b \in Z(R)^*\), \(a \perp b\) in \(\Gamma(R)\) if and only if \(ab = 0, ba = 0\) and \(\text{Lann}(a) \cap \text{Rann}(b) \subseteq \{0, a, b\}\).

Proof. First we assume \(a \perp b\). So \(ab = 0, ba = 0\). Let \(x \in \text{Lann}(a) \cap \text{Rann}(b)\). This gives \(xa = 0, bx = 0\). If \(x \not\in \{0, a, b\}\) then we get a contradiction to the fact that \(a \perp b\). This implies \(x \in \{0, a, b\}\). Conversely we assume \(ab = 0, ba = 0\) and \(\text{Lann}(a) \cap \text{Rann}(b) \subseteq \{0, a, b\}\). If possible, suppose there exists a non-zero element \(x\) distinct from \(a\) and \(b\) with \(x \adj a, b \adj x\). Then we get \(x \in \text{Lann}(a) \cap \text{Rann}(b)\), a contradiction. This contradiction implies that \(a \perp b\). Hence the result.

Theorem 6.2.13. For \(a, (\neq)b \in Z(R)^*\), \(aw_1b\) in \(\Gamma(R)\) if and only if \(ab = 0\) and \(\text{Lann}(a) \cap \text{Lann}(b) \subseteq \{0, a, b\}\).

Proof. First we assume \(aw_1b\). So \(ab = 0\). Let \(x \in \text{Lann}(a) \cap \text{Lann}(b)\). This gives \(xa = 0, xb = 0\). If \(x \not\in \{0, a, b\}\) then we get a contradiction to the fact that \(aw_1b\). This implies \(x \in \{0, a, b\}\). Conversely we assume \(ab = 0\) and \(\text{Lann}(a) \cap \text{Lann}(b) \subseteq \{0, a, b\}\). If possible, suppose there exists a non-zero element \(x\) distinct from \(a\) and \(b\) with \(x \adj a, x \adj b\). Then we get \(x \in \text{Lann}(a) \cap \text{Lann}(b)\), a contradiction. This contradiction implies that \(aw_1b\). Hence the result.

Theorem 6.2.14. For \(a, (\neq)b \in Z(R)^*\), \(aw_2b\) in \(\Gamma(R)\) if and only if \(ab = 0\) and \(\text{Rann}(a) \cap \text{Rann}(b) \subseteq \{0, a, b\}\).

Proof. First we assume \(aw_2b\). So \(ab = 0\). Let \(x \in \text{Rann}(a) \cap \text{Rann}(b)\). This gives \(ax = 0, bx = 0\). If \(x \not\in \{0, a, b\}\) then we get a contradiction to the fact that \(aw_2b\). This implies \(x \in \{0, a, b\}\). Conversely we assume \(ab = 0\) and \(\text{Rann}(a) \cap \text{Rann}(b) \subseteq \{0, a, b\}\). If possible, suppose there exists a non-zero element \(x\) distinct from \(a\) and \(b\) with \(a \adj x, b \adj x\). Then we get \(x \in \text{Rann}(a) \cap \text{Rann}(b)\), a contradiction. This contradiction implies that \(aw_2b\). Hence the result.
Theorem 6.2.15. For \(a, (\neq) b \in Z(R)^*\), \(a \perp_1 b\) in \(\Gamma(R)\) if and only if \(ab = 0, ba = 0\) and \(\text{Lann}(a) \cap \text{Lann}(b) \subseteq \{0, a, b\}\).

Proof. First we assume \(a \perp_1 b\). So \(ab = 0, ba = 0\). Let \(x \in \text{Lann}(a) \cap \text{Lann}(b)\). This gives \(xa = 0, xb = 0\). If \(x \nsubseteq \{0, a, b\}\) then we get a contradiction to the fact that \(a \perp_1 b\). This implies \(x \in \{0, a, b\}\). Conversely we assume \(ab = 0, ba = 0\) and \(\text{Lann}(a) \cap \text{Lann}(b) \subseteq \{0, a, b\}\). If possible, suppose there exists a non-zero element \(x\) distinct from \(a\) and \(b\) with \(x \operatorname{adj} a, x \operatorname{adj} b\). Then we get \(x \in \text{Lann}(a) \cap \text{Lann}(b)\), a contradiction. This contradiction implies that \(a \perp_1 b\). Hence the result.

Theorem 6.2.16. For \(a, (\neq) b \in Z(R)^*\), \(a \perp_2 b\) in \(\Gamma(R)\) if and only if \(ab = 0, ba = 0\) and \(\text{Rann}(a) \cap \text{Rann}(b) \subseteq \{0, a, b\}\).

Proof. First we assume \(a \perp_2 b\). So \(ab = 0, ba = 0\). Let \(x \in \text{Rann}(a) \cap \text{Rann}(b)\). This gives \(ax = 0, bx = 0\). If \(x \nsubseteq \{0, a, b\}\) then we get a contradiction to the fact that \(a \perp_2 b\). This implies \(x \in \{0, a, b\}\). Conversely we assume \(ab = 0, ba = 0\) and \(\text{Rann}(a) \cap \text{Rann}(b) \subseteq \{0, a, b\}\). If possible, suppose there exists a non-zero element \(x\) distinct from \(a\) and \(b\) with \(a \operatorname{adj} x, b \operatorname{adj} x\). Then we get \(x \in \text{Rann}(a) \cap \text{Rann}(b)\), a contradiction. This contradiction implies that \(a \perp_2 b\). Hence the result.

6.3 Orthogonality in \(\Gamma(R)\)

The main goal of this section is to establish the results 6.3.21 and 6.3.22. We continue the investigation of the zero-divisor graph \(\Gamma(R)\) and obtain some results which are related with (right) regular and (left) invertible elements of \(R\). The first result of this section asserts the significance of the strong orthogonal condition in directed graphs. Further, investigations are carried out for establishing different orthogonal conditions established with the annihilators of \(R\). Using the orthogonal conditions, three interesting results are
being established. We also determine the distance \( d(x, y) \) (\( d(y, x) \)) of an element \( x \) of \( \text{nil}(R)^* \) and an element \( y \) of \( Z_1(R)^* (Z_1(R)^*)^* \).

**Theorem 6.3.1.** Let \( G \) be a directed graph containing the vertices \( a_1, a_2, a_3, ..., a_p \) and every vertex is strong orthogonal to all of its succeeding vertices. Then \( G \) has no directed circuits.

**Proof.** From the given condition, we have \( a_1\text{adj}a_2, a_1\text{adj}a_3, a_1\text{adj}a_4, ..., a_1\text{adj}a_p; a_2\text{adj}a_3, a_2\text{adj}a_4, ..., a_{p-2}\text{adj}a_{p-1}, a_{p-1}\text{adj}a_p. \)

**Case - 1**: Suppose there is no arc other than the given arcs. Let \( a_{j_1} \rightarrow a_{j_2} \rightarrow ...a_{j_{q-1}} \rightarrow a_{j_q} \rightarrow a_{j_1} \) be a directed circuit of length \( q \leq p \) where \( a_{j_k} \in \{a_1, a_2, ..., a_p\}, k \in \{1, 2, 3, ..., q\}. \)

Then \( a_{j_1}\text{adj}a_{j_3}, a_{j_2}\text{adj}a_{j_4}, ..., a_{j_q}\text{adj}a_{j_1}. \) Also \( a_{j_1}\text{adj}a_{j_3}, a_{j_2}\text{adj}a_{j_5}, ..., a_{j_q}\text{adj}a_{j_1}, a_{j_2}\text{adj}a_{j_1}. \)

Since \( a_{j_1}\text{adj}a_{j_3}, a_{j_4}\text{adj}a_{j_5}, a_{j_q}\text{adj}a_{j_1}; \) which is a contradiction. This contradiction implies that there is no directed circuit in \( G \).

**Case - 2**: Suppose there are some vertices which are adjacent to other vertices in the opposite direction. Then each of these vertices is adjacent to immediate preceding vertex of the ordered sequence \( a_1, a_2, a_3, ..., a_p \), otherwise we get a contradiction i.e. \( a_2\text{adj}a_1, a_3\text{adj}a_2, a_4\text{adj}a_3, ..., a_{p-1}\text{adj}a_{p-2}, a_p\text{adj}a_{p-1}. \) Also if \( a_2\text{adj}a_1 \), then \( a_3 \) can’t be adjacent to \( a_2. \) But \( a_4 \) can be adjacent to \( a_3 \) and if so, \( a_5 \) can’t be adjacent to \( a_4 \) and so on. We consider the extreme case to construct a directed circuit in \( G \) i.e. we consider maximum number of directed arcs in \( G \). If \( p \) is even, then we have to consider \( a_2\text{adj}a_1, a_4\text{adj}a_3, ..., a_{p}\text{adj}a_{p-1} \) and \( p \) is odd, then \( a_2\text{adj}a_1, a_4\text{adj}a_3, ..., a_{p-1}\text{adj}a_{p-2}. \) If possible suppose there is no arc other than the given arcs. Let \( a_{j_1} \rightarrow a_{j_2} \rightarrow ...a_{j_{q-1}} \rightarrow a_{j_q} \rightarrow a_{j_1} \) be a directed circuit of length \( q \leq p \) where \( a_{j_k} \in \{a_1, a_2, ..., a_p\}, k \in \{1, 2, 3, ..., q\}. \) If \( a_{j_1} = a_1 \), then \( a_{j_{q}}\text{adj}a_{j_1}, \) which is not possible. So \( a_{j_1} \neq a_1. \) Now \( a_{j_1}\text{adj}a_{j_3}, a_{j_2}\text{adj}a_{j_4}, ..., a_{j_{q}}\text{adj}a_{j_1}. \) Also \( a_{j_2}\text{adj}a_{j_4}, a_{j_3}\text{adj}a_{j_5}, ..., a_{j_{q}}\text{adj}a_{j_1}. \)

But \( a_{j_1}\text{adj}a_{j_3}, a_{j_2}\text{adj}a_{j_4}, a_{j_q}\text{adj}a_{j_1}; \) which is a contradiction. Hence in this case \( a_{j_1} \rightarrow a_{j_2} \rightarrow ...a_{j_{q-1}} \rightarrow a_{j_q} \rightarrow a_{j_1} \) can’t be a directed...
circuit. Similarly if we consider \(a_3 \text{adj} a_2, a_5 \text{adj} a_4, a_7 \text{adj} a_6, \ldots, a_{p-1} \text{adj} a_{p-2}\), (for \(p\) is even as well as odd) then also we can’t construct a directed circuit in \(G\). Hence there is no directed circuit in \(G\).

Theorem 6.3.2. Let \(R\) be a non-commutative ring with unity 1 such that every non zero element is left invertible. If for \(a, b \in Z(R)^*\), \(a \perp_s b, a^2 \neq 0, b^2 \neq 0\) then \(ab = 0\) and \(a + b\) is a right regular element of \(R\).

Proof. Let \(a \perp_s b, a^2 \neq 0, b^2 \neq 0\), for \(a, b \in Z(R)^*\). Let \(c(a + b) = 0\) with \(c \neq 0\). If \(d\) is the left multiplicative inverse of \(c\), then \(dc = 1\). This gives \(dc(a + b) = 0\), so \(a = -b\) or \(b = -a\). From this, we get \(a^2 = 0, b^2 = 0\), a contradiction. So \(c = 0\) i.e. \(a + b\) is a right regular element in \(R\).

Theorem 6.3.3. Let \(R\) be a non-commutative ring with unity 1 such that every non zero element is left invertible. If for \(a, b \in Z(R)^*\), \(a \perp b, a^2 \neq 0, b^2 \neq 0\) then \(ab = 0\) and \(a + b\) is a right regular element of \(R\).

Proof. Let \(a \perp b, a^2 \neq 0, b^2 \neq 0\), for \(a, b \in Z(R)^*\). Let \(c(a + b) = 0\) with \(c \neq 0\). If \(d\) is the left multiplicative inverse of \(c\), then \(dc = 1\). This gives \(dc(a + b) = 0\), so \(a = -b\) or \(b = -a\). From this, we get \(a^2 = 0, b^2 = 0\), a contradiction. So \(c = 0\) i.e. \(a + b\) is a right regular element in \(R\).

Theorem 6.3.4. Let \(R\) be a non-commutative ring with unity 1 such that every non zero element is left invertible. If for \(a, b \in Z(R)^*\), \(aw_1 b, a^2 \neq 0, b^2 \neq 0\) then \(ab = 0\) and \(a + b\) is a right regular element of \(R\).

Proof. Let \(aw_1 b, a^2 \neq 0, b^2 \neq 0\), for \(a, b \in Z(R)^*\). Let \(c(a + b) = 0\) with \(c \neq 0\). If \(d\) is the left multiplicative inverse of \(c\), then \(dc = 1\). This gives \(dc(a + b) = 0\), so \(a = -b\) or \(b = -a\). From this, we get \(a^2 = 0, b^2 = 0\), a contradiction. So \(c = 0\) i.e. \(a + b\) is a right regular element in \(R\).
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Theorem 6.3.5. Let $R$ be a non-commutative ring with unity 1 such that every non zero element is left invertible. If for $a, b \in Z(R)^*$, $aw_2b$, $a^2 \neq 0$, $b^2 \neq 0$ then $ab = 0$ and $a + b$ is a right regular element of $R$.

Proof. Let $aw_2b$, $a^2 \neq 0$, $b^2 \neq 0$, for $a, b \in Z(R)^*$. Let $c(a + b) = 0$ with $c \neq 0$. If $d$ is the left multiplicative inverse of $c$, then $dc = 1$. This gives $dc(a + b) = 0$, so $a = -b$ or $b = -a$. From this, we get $a^2 = 0$, $b^2 = 0$, a contradiction. So $c = 0$ i.e. $a + b$ is a right regular element in $R$.

Theorem 6.3.6. Let $R$ be a non-commutative ring with unity 1 such that every non zero element is left invertible. If for $a, b \in Z(R)^*$, $ab = 0$ and $a + b$ is a regular element, then $a \perp b$, $a^2 \neq 0$ and $b^2 \neq 0$.

Proof. Since $ab = 0$ and $a + b$ is a regular element, therefore $a \neq b$. If $a^2 = 0$, then $a(a + b) = 0$. So $a^2 \neq 0$. Similarly $b^2 \neq 0$. Suppose that $bc = 0 = ca$ for some $c \in R$. If $c \neq 0$ then $b = a = 0$ i.e. $c = 0$. Also $ab = 0$. Hence $a \perp b$.

Theorem 6.3.7. Let $R$ be a non-commutative ring with unity 1 such that every non zero element is left invertible. If for $a, b \in Z(R)^*$, $ab = 0$ and $a + b$ is a regular element, then $aw_1b$, $a^2 \neq 0$ and $b^2 \neq 0$.

Proof. Since $ab = 0$ and $a + b$ is a regular element, therefore $a \neq b$. If $a^2 = 0$, then $a(a + b) = 0$. So $a^2 \neq 0$. Similarly $b^2 \neq 0$. Suppose that $cb = 0 = ca$ for some $c \in Z(R)$. If $c \neq 0$ then $b = a = 0$ i.e. $c = 0$. Also $ab = 0$. Hence $aw_1b$.

Theorem 6.3.8. Let $R$ be a non-commutative ring with unity 1 such that every non zero element is left invertible. If for $a, b \in Z(R)^*$, $ab = 0$ and $a + b$ is a regular element, then $aw_2b$, $a^2 \neq 0$ and $b^2 \neq 0$. 

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Proof. Since \( ab = 0 \) and \( a + b \) is a regular element, therefore \( a \neq b \). If \( a^2 = 0 \), then \( a(a + b) = 0 \). So \( a^2 \neq 0 \). Similarly \( b^2 \neq 0 \). Suppose that \( cb = 0 = ca \) for some \( c \in Z(R) \).

If \( c \neq 0 \) then \( b = a = 0 \) i.e. \( c = 0 \). Also \( ab = 0 \). Hence \( aw_2b \).

Theorem 6.3.9. Let \( R \) be a non-commutative ring and \( a, b, c \in Z(R)^* \). If \( a \perp_s b \), \( c \perp_s a \) and \( x \) is a common element of \( Rann(b) \) and \( Lann(c) \). Then \( \Gamma(R) \) has a directed circuit of length 4.

Proof. As \( a \perp_s b \), \( c \perp_s a \), so \( ab = 0 \) and \( ca = 0 \). Also \( x \) is a common element of \( Rann(b) \) and \( Lann(c) \), therefore \( bx = 0 \) and \( xc = 0 \). This gives \( a \rightarrow b \rightarrow x \rightarrow c \rightarrow a \) is a directed circuit of length 4. Hence the result.

Theorem 6.3.10. Let \( R \) be a non-commutative ring and \( a, b, c \in Z(R)^* \). If \( aw_1b \), \( cw_1a \) and \( x \) is a common element of \( Rann(b) \) and \( Lann(c) \). Then \( \Gamma(R) \) has a directed circuit of length 4.

Proof. As \( aw_1b \), \( cw_1a \), so \( ab = 0 \) and \( ca = 0 \). Also \( x \) is a common element of \( Rann(b) \) and \( Lann(c) \), therefore \( bx = 0 \) and \( xc = 0 \). This gives \( a \rightarrow b \rightarrow x \rightarrow c \rightarrow a \) is a directed circuit of length 4. Hence the result.

Theorem 6.3.11. Let \( R \) be a non-commutative ring and \( a, b, c \in Z(R)^* \). If \( aw_2b \), \( cw_2a \) and \( x \) is a common element of \( Rann(b) \) and \( Lann(c) \). Then \( \Gamma(R) \) has a directed circuit of length 4.

Proof. As \( aw_2b \), \( cw_2a \), so \( ab = 0 \) and \( ca = 0 \). Also \( x \) is a common element of \( Rann(b) \) and \( Lann(c) \), therefore \( bx = 0 \) and \( xc = 0 \). This gives \( a \rightarrow b \rightarrow x \rightarrow c \rightarrow a \) is a directed circuit of length 4. Hence the result.

Theorem 6.3.12. Let \( R \) be a non-commutative reduced ring and \( a, b, c \in Z(R)^* \). If \( aw_1b \), \( cw_1a \), then \( c \) is not adjacent to \( b \) in \( \Gamma(R) \).
Proof. Since \( aw_1b, cw_1a \). So \( ab = 0 \) and \( ca = 0 \). If possible, let \( cb = 0 \). Then \( c = b \). If \( c = b \) then \( cb = 0 \) gives \( b^2 = 0 \) and so \( b = 0 \), as \( R \) is reduced. Therefore \( c \neq b \). This implies \( cb \neq 0 \). Hence \( b \) can’t be adjacent to \( c \).

**Theorem 6.3.13.** Let \( R \) be a non-commutative reduced ring and \( a, b, c \in Z(R)^* \). If \( aw_2b, aw_2c \), then \( b \) is not adjacent to \( c \) in \( \Gamma(R) \).

Proof. Since \( aw_2b, aw_2c \). So \( ab = 0 \) and \( ac = 0 \). If possible, let \( bc = 0 \). Then \( b = c \). If \( b = c \) then \( bc = 0 \) gives \( b^2 = 0 \) and so \( b = 0 \), as \( R \) is reduced. Therefore \( b \neq c \). This implies \( bc \neq 0 \). Hence \( b \) can’t be adjacent to \( c \).

**Theorem 6.3.14.** Let \( R \) be a non-commutative reduced ring and \( a, b, c \in Z(R)^* \). If \( a \bot_s b, c \bot_s a \), then \( bR_3c \).

Proof. Since \( a \bot_s b, c \bot_s a \). So \( ab = 0 \) and \( ca = 0 \). If possible, let \( bc = 0 \). Then \( c = b \). If \( c = b \) then \( bc = 0 \) gives \( b^2 = 0 \) and so \( b = 0 \), as \( R \) is reduced. Therefore \( c \neq b \). This implies \( bc \neq 0 \). So \( b \) can’t be adjacent to \( c \). Suppose that \( bd = 0 \) for \( d \in Z(R)^* \). Now \( b(dc) = (bd)c = 0, (dc)a = d(ca) = 0 \). As \( R \) is reduced, so \( dc \neq a \) and \( dc \neq b \). This gives \( a \rightarrow b \rightarrow dc \rightarrow a \) is a directed circuit, which is a contradiction. This contradiction implies that \( d \ adj c \). Also suppose that \( x \ adj c \). If \( b \) is not adjacent to \( x \), then \( a(bx) = (ab)x = 0 \) and \( (bx)c = b(xc) = 0 \). This gives \( a \rightarrow bx \rightarrow c \rightarrow a \) is a directed circuit, a contradiction. Therefore \( b \) is adjacent to \( x \). Hence \( bR_3c \).
Proof. (a) implies (b). Suppose that (1) holds. Then \( ab = 0, ba = 0 \) since \( a \perp_1 b \). Suppose that \( c(a + b) = 0 \) for some \( c \in R \). Let \( y = ca = -cb \). Then \( ya = yb = 0 \). Thus \( y \in \{0, a, b\} \), since \( a \perp_1 b \). If \( y = a \), then \( a^2 = ya = 0 \), a contradiction. Similarly \( y = b \) yields \( b^2 = 0 \). Hence \( y = 0 \). Thus \( ca = cb = 0 \) and thus \( c \in \{0, a, b\} \), since \( a \perp_1 b \). If \( c = a \), then \( a^2 = ca = 0 \), a contradiction. Similarly, \( b^2 = 0 \) if \( c = b \). Thus \( c = 0 \), and hence \( a + b \) is a right regular element of \( R \).

(b) implies (a). Suppose that (2) holds. First we observe that \( a \neq b \), since \( a + b \) is a right regular element of \( R \). If \( a = 0 \). Then \( a(a + b) = a^2 + ab = 0 \), a contradiction. Thus \( a^2 \neq 0 \), and similarly \( b^2 \neq 0 \). Suppose that \( ca = cb = 0 \), for some \( c \in R \). Then \( c(a + b) = 0 \), for some \( c \in R \). Then \( c(a + b) = 0 \) and hence \( c = 0 \), since \( a + b \) is a right regular element of \( R \). Since \( ab = 0 \) and \( ba = 0 \), we have \( a \perp_1 b \).

Theorem 6.3.16. Let \( R \) be a non-commutative ring and \( a, b \in \text{Z}(R)^* \). Then the following statements are equivalent.

(a) \( a \perp_2 b, a^2 \neq 0 \) and \( b^2 \neq 0 \).

(b) \( ab = 0, ba = 0 \) and \( a + b \) is a left regular element of \( R \).

Proof. (a) implies (b). Suppose that (1) holds. Then \( ab = 0, ba = 0 \) since \( a \perp_2 b \). Suppose that \( (a+b)c = 0 \) for some \( c \in R \). Let \( y = ac = -bc \). Then \( ay = by = 0 \). Thus \( y \in \{0, a, b\} \), since \( a \perp_2 b \). If \( y = a \), then \( a^2 = ay = 0 \), a contradiction. Similarly \( y = b \) yields \( b^2 = 0 \). Hence \( y = 0 \). Thus \( ac = bc = 0 \) and thus \( c \in \{0, a, b\} \), since \( a \perp_2 b \). If \( c = a \), then \( a^2 = ac = 0 \), a contradiction. Similarly, \( b^2 = 0 \) if \( c = b \). Thus \( c = 0 \), and hence \( a + b \) is a left regular element of \( R \).

(b) implies (a). Suppose that (2) holds. First we observe that \( a \neq b \), since \( a + b \) is a left regular element of \( R \). If \( a = 0 \) then \( (a+b)a = a^2 + ba = 0 \), a contradiction. Thus \( a^2 \neq 0 \), and similarly \( b^2 \neq 0 \). Suppose that \( ac = bc = 0 \), for some \( c \in R \). Then \( (a+b)c = 0 \) and
hence $c = 0$, since $a + b$ is a left regular element of $R$. Since $ab = 0$ and $ba = 0$, we have $a \perp_{2} b$.

**Theorem 6.3.17.** For the vertices $a, x$ and $y$ of a directed graph $G$ with $aw_1x$ and $xR_2y$, $aw_1y$.

**Proof.** Since $aw_1x$, so $a$ is adjacent to $x$. As $xR_2y$, therefore $a$ is adjacent to $y$. Also if $c \adj y$, $c \adj a$, for a vertex $c$ in $V(G)$, then $c \adj x$, which is a contradiction to $aw_1x$. This contradiction implies that there is no $c$ with $c \adj y$, $c \adj a$. This gives $aw_1y$.

**Theorem 6.3.18.** For the vertices $a, x$ and $y$ of a directed graph $G$ with $xw_2a$ and $xR_1y$, $yw_2a$.

**Proof.** Since $xw_2a$, so $x$ is adjacent to $a$. As $xR_1y$, therefore $y$ is adjacent to $a$. Also if $a \adj c$, $y \adj c$, for a vertex $c$ in $V(G)$, then $x \adj c$, which is a contradiction to $aw_2x$. This contradiction implies that there is no $c$ with $a \adj c$, $y \adj c$. This gives $aw_2y$.

**Theorem 6.3.19.** Let $R$ be a non-commutative ring with $x \in \text{nil}(R)^*$ and $y \in Z_r(R)^*$. Then $d(x, y) \leq 2$.

**Proof.** We assume that $x \neq y$ and $xy \neq 0$. As $x = y$ or $xy = 0$, $d(x, y) = 0$ or $1$ respectively. Now $y \in Z_r(R)^*$ and $xy \neq 0$, so there is an element $z \in Z_l(R)^* \setminus \{x\}$ with $zy = 0$. Let $n$ be the least positive integer such that $x^n z = 0$. Then $x \rightarrow x^{n-1} z \rightarrow y$ is a path of length 2 from $x$ to $y$. Thus $d(x, y) \leq 2$.

**Theorem 6.3.20.** Let $R$ be a non-commutative ring with $x \in \text{nil}(R)^*$ and $y \in Z_l(R)^*$. Then $d(y, x) \leq 2$.

**Proof.** We assume that $x \neq y$ and $yx \neq 0$. As $x = y$ and $yx = 0$, $d(y, x) = 0$ or $1$ respectively. Now $y \in Z_l(R)^*$ and $yx \neq 0$, so there is an element $z \in Z_r(R)^* \setminus \{x\}$ with
\[ yz = 0. \] Let \( n \) be the least positive integer such that \( zx^n = 0 \). Then \( y \to zx^{n-1} \to x \) is a path of length 2 from \( y \) to \( x \). Thus \( d(y, x) \leq 2 \). 

**Theorem 6.3.21.** Let \( R \) be a non-commutative ring with \( \text{nil}(R) \) non-zero. If \( \Gamma(R) \) is 1\textsuperscript{st} weak complemented, then either \( |R| = 8 \), \( |R| = 9 \), or \( |R| > 8 \), and \( \text{nil}(R) = \{0, x\} \) for some \( 0 \neq x \in R \).

**Proof.** Case 1 : Suppose \( \Gamma(R) \) is 1\textsuperscript{st} weak complemented and let \( a \in \text{nil}(R) \) have index of nilpotence \( n \geq 3 \). Let \( y \in Z(R)^* \) be a weak complement of \( a \). Then \( ay = 0 \). Now \( a^{n-1}y = 0 = a^{n-1}a \). Therefore \( y = a^{n-1} \), since \( aw_1y \). Thus \( Lann(a) = \{0, a^{n-1}\} \), since \( za = 0 \), then \( za^{n-1} = 0 \) and \( a^i w_1 a^{n-1} \), for \( 1 \leq i \leq n - 2 \). Suppose that \( n > 3 \). Then \( a^{n-2} + a^{n-1} \) is a left zero-divisor of both \( a^{n-1} \) and \( a^{n-2} \), a contradiction. Since \( a^{n-2} w_1 a^{n-1} \) and \( a^{n-2} + a^{n-1} \notin \{0, a^{n-1}, a^{n-2}\} \). Thus if \( R \) has a nilpotent element with index \( n \geq 3 \), then \( n = 3 \). In this case, \( Ra^2 = \{0, a^2\} \), since each \( z \in Ra^2 \) is a left zero-divisor of \( a \) and \( a^2 \) and \( aw_1 a^2 \). Also \( Lann(a^2) = \{0, a, a^2, a + a^2\} (= Rann(a^2)) \). Thus \( R \) is local with \( |R| = 8 \), \( \text{nil}(R) = Z(R) = Lann(a^2) \) its maximal ideal.

Case 2 : Suppose that each non-zero nilpotent element of \( R \) has index of nilpotence 2. Let \( 0 \neq y \in \text{nil}(R) \) have weak complement \( z \in Z(R)^* \). Now \( (ry)y = 0 = (ry)z \), for every \( r \in R \). Therefore \( Ry \subseteq \{0, y, z\} \). First suppose that \( 2y \neq 0 \). Then \( z = 2y \). Since \( 2y \in Ry \subseteq \{0, y, z\} \). Also \( Lann(y) = \{0, y, 2y\} \), since \( yw_1 2y \). Thus \( Ry = \{0, y, 2y\} \), so we have \( |R| = 9 \). In this case, \( R \) is local with maximal ideal \( Z(R) = \text{nil}(R) = Lann(y) \) and \( \Gamma(R) \) is a star graph with one arc. Next suppose that each non-zero element of \( R \) has index of nilpotence 2 and \( |R| \neq 9 \). Therefore \( 2y = 0 \). We show that \( \text{nil}(R) = \{0, y\} \). Suppose that \( z \) is another non-zero nilpotent element of \( R \); so \( z^2 = 0 \). Then \( y + z \) is a nilpotent element of index 2. First \( Ry = \{0, y, y'\} \) and \( Rz = \{0, z, z'\} \), where \( y' \) and \( z' \) are 1\textsuperscript{st} weak complement of \( y \) and \( z \) respectively. Next \( zy = 0 \). For if \( zy \neq 0 \), then \( zy = y' \). Therefore \( Ry = \{0, y, zy\} = Lann(y) \) and hence \( |R| = 9 \), a contradiction. Similarly
$yz = 0$, if $yz \neq 0$ then $yz = z'$. So $Rz = \{0, z, yz\} = \text{Lann}(z)$ and hence $|R| = 9$, which is also a contradiction. Let $w$ be a 1$^\text{st}$ weak complement of $y + z$. Clearly $w$ is neither $y$ nor $z$. If $w = y$, then $(y + z)y = 0$ implies $z \in \text{Lann}(y)$, a contradiction. Then either $yw = y$ or $yw = y'$. If not then $yw = 0$. Also $y(y + z) = 0$, a contradiction. However if $y' = yw$, then $|R| = 9$, again a contradiction. Thus $yw = y$. Similarly $zw = z$. But then $(y + z)w = yw + zw = y + z$, which contradicts $(y + z)w_1w$. Hence $R$ has a unique non-zero nilpotent element.

**Theorem 6.3.22.** Let $R$ be a non-commutative ring with $\text{nil}(R)$ non-zero. If $\Gamma(R)$ is 1$^\text{st}$ weak complemented and $|R| > 9$ then any 1$^\text{st}$ complement of the non-zero nilpotent element of $R$ is an end.

**Proof.** Suppose $\Gamma(R)$ is uniquely complemented and $|R| > 9$. Let $x$ be the unique non-zero nilpotent. Let $y$ be a complement of $x$. We first show $xw_1(x + y)$. Clearly $x(x + y) = 0$, since $x^2 = 0$ and $xw_1y$. Also $x = -x$ and $x + y \in Z(R)^*$. Suppose $wx = 0 = w(x + y)$ for some $w \in Z(R)^*$. Then $wy = 0$ implies $w = y$ or $w = x$. If $w = y$ then $y^2 = 0$, a contradiction. Thus $w = x$. So $xw_1(x + y)$. By uniqueness of complements, we have $yR_1(x + y)$. Suppose that $zy = 0$ for some $z \in Z(R)^* - \{x\}$. Then $z \neq y$ since, $yR_1(x + y)$. Hence $zx = 0$, a contradiction to $xw_1y$. Thus no such $z$ can exist; so $y$ is an end.

### 6.4 Examples on $\Gamma(R)$

In this section, we produce five examples of zero-divisor directed graph of non-commutative ring.

**Example. 6.4.1.** Let $R = \{a + bi + cj + dk : a, b, c, d \in \mathbb{Z}_2\}$ with $i^2 = j^2 = k^2 = -1$ and $ij = k$, $ji = -k$. Then $R$ forms a ring.

$\therefore R = \{0, k, j + k, i, i + k, i + j, i + j + k, 1, 1 + k, 1 + j, 1 + j + k, 1 + i, 1 + i + k, 1 + i + j, 1 + i + j + k\}$.
$j, 1 + i + j + k \}.$

Let $M_2(R) = \{ \begin{bmatrix} 0 & 0 \\ 0 & p \end{bmatrix} : p \in R \} = \{ A_1, A_2, ..., A_8, B_1, B_2, ..., B_8 \}$, where

\[
A_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 \\ 0 & k \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 \\ 0 & j \end{bmatrix}, \quad A_4 = \begin{bmatrix} 0 & 0 \\ 0 & j + k \end{bmatrix}, \quad A_5 = \begin{bmatrix} 0 & 0 \\ 0 & i \end{bmatrix},
\]

\[
A_6 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_7 = \begin{bmatrix} 0 & 0 \\ 0 & i + j \end{bmatrix}, \quad A_8 = \begin{bmatrix} 0 & 0 \\ 0 & i + j + k \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},
\]

\[
B_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 + j \end{bmatrix}, \quad B_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 + j + k \end{bmatrix}, \quad B_5 = \begin{bmatrix} 0 & 0 \\ 0 & 1 + i \end{bmatrix},
\]

\[
B_6 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_7 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_8 = \begin{bmatrix} 0 & 0 \\ 0 & 1 + i + k \end{bmatrix}.
\]

Then, we have $A_1A_2 = 0 = A_1A_3 = A_1A_4 = A_1A_5 = A_1A_6 = A_1A_7 = A_1A_8 = A_1B_1 = A_1B_2 = A_1B_3 = A_1B_4 = A_1B_5 = A_1B_6 = A_1B_7 = A_1B_8$. So $Z(M_2(R))^\ast = \{ A_2, A_3, A_4, A_5, A_6, A_7, A_8, B_1, B_2, B_3, B_4, B_5, B_6, B_7, B_8 \}$.

**Example. 6.4.2.** Consider the ring $R = \{0, a, b, c\}$ with addition and multiplication operations defined as follows:

\[
\begin{array}{cccc}
+ & 0 & a & b & c \\
0 & 0 & a & b & c \\
a & a & 0 & c & b \\
b & b & c & 0 & a \\
c & c & b & a & 0 \\
\end{array}
\]

and

\[
\begin{array}{cccc}
. & 0 & a & b & c \\
0 & 0 & 0 & 0 & 0 \\
a & 0 & a & 0 & a \\
b & 0 & b & 0 & b \\
c & 0 & c & 0 & c \\
\end{array}
\]

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Here \( Z(R)^* = \{a, b, c\} \) and arcs of \( \Gamma(R) \) are \( ab \) and \( cb \).

**Example. 6.4.3.** Consider the ring \( R = \{0, a, b, c\} \) with addition and multiplication operations defined as follows:

\[
\begin{array}{cccc}
+ & 0 & a & b \\
0 & 0 & a & b \\
a & a & 0 & c \\
b & b & c & 0 \\
c & c & b & a \\
\end{array}
\]

and

\[
\begin{array}{cccc}
. & 0 & a & b \\
0 & 0 & 0 & 0 \\
a & 0 & 0 & 0 \\
b & 0 & a & b \\
c & 0 & a & b \\
\end{array}
\]

Here \( Z(R)^* = \{a, b, c\} \) and arcs of \( \Gamma(R) \) are \( ab \) and \( ac \). \( \Gamma(R) \) is shown below:

\[
\begin{array}{cccc}
c & a \\
0 & b \\
\end{array}
\]

**Example. 6.4.4.** Consider the non-commutative ring \( R = \{(a_{ij})_{2\times 2} : a_{11}, a_{12} \in \mathbb{Z}_2, a_{21} = 0 = a_{22}\} = \{A_1, A_2, A_3, A_4\} \), where

\[
A_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, A_4 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.
\]

Here \( Z(R)^* = \{A_2, A_3, A_4\} \) and arcs of \( \Gamma(R) \) are \( A_2A_3 \) and \( A_2A_4 \). \( \Gamma(R) \) is shown below:

\[
\begin{array}{c}
A_3 \quad A_2 \\
A_1 \quad A_4 \\
\end{array}
\]
The following example illustrates some of our results.

Example. 6.4.5. Consider the non-commutative ring

\[ R = \{(a_{ij})_{2 \times 2} : a_{ij} \in \mathbb{Z}_2\} = \{A_1, A_2, \ldots, A_8, B_1, B_2, \ldots, B_8\}, \]

where

\[
\begin{align*}
A_1 &= \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}, & A_2 &= \begin{bmatrix}
0 & 0 \\
0 & 1
\end{bmatrix}, & A_3 &= \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}, & A_4 &= \begin{bmatrix}
0 & 0 \\
1 & 1
\end{bmatrix}, & A_5 &= \begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}, \\
A_6 &= \begin{bmatrix}
0 & 0 \\
0 & 1
\end{bmatrix}, & A_7 &= \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}, & A_8 &= \begin{bmatrix}
0 & 0 \\
1 & 1
\end{bmatrix}, & B_1 &= \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}, & B_2 &= \begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix}, \\
B_3 &= \begin{bmatrix}
1 & 0 \\
1 & 0
\end{bmatrix}, & B_4 &= \begin{bmatrix}
1 & 0 \\
1 & 1
\end{bmatrix}, & B_5 &= \begin{bmatrix}
1 & 1 \\
0 & 0
\end{bmatrix}, & B_6 &= \begin{bmatrix}
1 & 1 \\
0 & 1
\end{bmatrix}, & B_7 &= \begin{bmatrix}
1 & 1 \\
1 & 0
\end{bmatrix}, \\
B_8 &= \begin{bmatrix}
1 & 1 \\
1 & 1
\end{bmatrix}.
\end{align*}
\]

Here \( Z(R)^* = \{A_2, A_3, A_4, A_5, A_6, B_1, B_3, B_5, B_8\} \) and the arcs of \( \Gamma(R) \) are

\[ A_2A_5, A_2B_1, A_2B_5, A_3A_2, A_3A_4, A_4A_6, A_4B_3, A_4B_8, A_5B_1, A_5B_5, A_6A_5, A_6B_1, A_6B_5, B_1A_2, B_1A_3, B_1A_4, B_3A_2, B_3A_3, B_3A_4, B_5A_6, B_5B_3, B_5B_8, B_8A_6, B_8B_3. \]

We observe that \( A_4 \) and \( B_5 \) establish the existence of the Theorem 6.2.3. Since \( A_4 \) and \( B_5 \) are not adjacent and both \( A_4 \) and \( B_5 \) are adjacent to \( A_6, B_3, \) and \( B_8 \). Also \( Rann(A_4) = \{A_1, A_6, B_3, B_8\} = Rann(B_5) \) i.e. \( A_4R_1B_5 \) and \( Rann(A_4) = Rann(B_5) \).

Theorem 6.2.11 is established by the elements \( A_2 \) and \( A_5 \). As \( A_2 \) is adjacent to \( A_5 \) and there is no element \( A \) in \( Z(R)^* \) with \( A_5 \) adj \( A \) and \( A \) adj \( A_2 \). This gives \( A_2 \perp_A A_5 \). Also \( Lann(A_2) = \{A_1, A_3\} \) and \( Rann(A_5) = \{A_1, A_5, B_1, B_5\} \). From this \( Lann(A_2) \cap Rann(A_5) = \{A_1\} \subseteq \{A_1, A_2, A_5\} \).

Similarly \( A_2, A_3 \) and \( A_5 \) establish the existence of the Theorem 6.3.9. Since \( A_3 \) is adjacent to \( A_2 \) and there is no element \( A \) in \( Z(R)^* \) with \( A_2 \) adj \( A \) and \( A \) adj \( A_3 \). From this \( A_3 \perp_A A_2 \). Also \( A_2 \perp_A A_5 \) and \( Lann(A_3) = \{A_3, B_1, B_3\} \). i.e. \( B_1 \in Rann(A_5) \cap Lann(A_3) \). This gives \( A_2 \rightarrow A_5 \rightarrow B_1 \rightarrow A_3 \rightarrow A_2 \).

\( A_4 \) and \( B_3 \) establish the existence of the Theorem 6.3.15. As \( A_4 \) adj \( B_3 \) and \( B_3 \) adj \( A_4 \).
Also there is no element $A$ in $Z(R)^*$ with $A \adj B_3$ and $A \adj A_4$. From this $A_4 \perp_1 A_3$. Again

$$A_4 + B_3 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus $A_4 + B_3$ is a (right) regular element of $R$.

Other results of this chapter can also be established using this above example.

**Conclusion**: In this chapter, we have investigated different characteristics of zero divisor graph of non-commutative rings. The uniqueness character of the complements will develop many ring-theoretic concepts. Further, the idea of zero-divisor graph isomorphism of any two non-commutative rings, will give some interesting results.