CHAPTER - II

FIXED POINT THEOREMS FOR CERTAIN CLASS OF SINGLE-VALUED MAPPINGS

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2.1 **INTRODUCTION**

In recent years, several authors have attempted to study the fixed point theorems for mappings with contractive condition in the form of rational inequalities. A comprehensive account of various contractive conditions in the form of rational inequalities can be found in a recent survey made by Rhoades [99]. To study the common fixed point for more than one mappings some sort of weak conditions of commutativity are needed to prove the results.

In an attempt to generalize the notion of commutativity for single-valued mappings, Sessa [103] introduced the idea of weak commutativity for the self mappings $S$ and $I$ of a metric space $(X, d)$ i.e. $d(SIx, ISx) \leq d(Sx,Ix)$ for all $x$ in $X$. Under this concept he extended theorem 2.1 of Jungck [55]. Subsequently Jungck [56] made an extension of weak commutativity by introducing a new concept of compatible mappings and generalized some results of Singh and Singh [113] and Fisher [35].

Later on, in [57], Jungck, Murthy and Cho introduced another class of mappings called $(A)$-type-compatible mappings which under certain conditions is equivalent to compatible mappings, but otherwise
bear no relation with each other.

Recently, Pathak et. al. [90] in an attempt to extend the notion of $(A)$type-compatibility have introduced a class of $(P)$type-compatible mappings and show that under certain conditions $(A)$type-compatible and $(P)$type-compatible are equivalent.

In Section 2.2, we have obtained some results for compatible and $(A)$type-compatible mappings satisfying the contractive condition in the form of a rational inequality introduced by Fisher [31], we have remarked that with slight modification in the proof the results can also to be extended for $(P)$type-compatible mappings. The results obtained here generalize the earlier results of Fisher ([30], [31]) and many others.

Section 2.3 we have studied some results for a mapping and sequence of mappings, which in turn, generalize Banach contraction principle and a multitude of earlier known results as Fisher [31], Maia [78] and Edelstein [29], etc.

Finally, in Section 2.4 we prove some coincidence point and fixed point theorems for expansion type mappings which in turn, generalize several previously results due to Taniguchi [118], Wang et. al. [120] and Gillespie et. al. [41].

2.2 FIXED POINT THEOREMS OF MAPPINGS WITH WEAK COMMUTATIVITY CONDITIONS

In this Section, using the concepts of compatible and $(A)$type
compatible mappings, we present common fixed point theorems for four mappings satisfying a rational inequality. Our results extend the result of Fisher ([30], [31]).

**Theorem 2.2.1** Let \(\{S, I\}\) and \(\{T, J\}\) be compatible pairs of mappings of a complete metric space \((X, d)\) into itself such that

(a) \(T(X) \subseteq I(X), S(X) \subseteq J(X),\)

(b) for all \(x, y \in X,\) with \(\alpha, \beta \geq 0,\) \(2\alpha + \beta < 1,\) either

\[
d(Sx, Ty) \leq \alpha \frac{[d(Sx, Jy)]^2 + [d(Ix, Ty)]^2}{d(Sx, Jy) + d(Ix, Ty)} + \beta \ d(Ix, Jy)
\]

whenever \(d(Sx, Jy) + d(Ix, Ty) \neq 0\) or

\[d(Sx, Ty) = 0\]

whenever \(d(Sx, Jy) + d(Ix, Ty) = 0\)

If one of \(S, T, I\) and \(J\) is continuous then \(S, T, I,\) and \(J\) have a unique common fixed point \(z\) in \(X.\) Further \(z\) is the unique common fixed point of \(S\) and \(I\) and of \(T\) and \(J.\)

**Proof.** We construct the sequence as follows. Let \(x_0\) be an arbitrary point in \(X.\) Since \(S(X) \subseteq J(X)\) we can choose a point \(x_1\) in \(X\) such that \(Sx_0 = Jx_1.\) Again, since \(T(X) \subseteq I(X),\) we can choose a point \(x_2\) in \(X\) such that \(Tx_1 = Ix_2.\) In general for the point \(x_{2n}\) we can choose a point \(x_{2n+1}\) such that \(Sx_{2n} = Jx_{2n+1}\) and then a point \(x_{2n+2}\) such that \(Tx_{2n+1} = Ix_{2n+2}\) for \(n = 0, 1, 2, \ldots.\)

We denote \(U_{2n} = d(Sx_{2n}, Tx_{2n-1})\) and \(U_{2n-1} = d(Tx_{2n-1}, Sx_{2n-2})\)

We distinguish two cases :
Case-I. Suppose \( d(Sx^n, Jx^n) + d(Ix^n, Tx^n) \neq 0 \) for \( n = 0, 1, 2, \ldots \), then on using inequality (I), we get

\[
d(Tx^n, Sx^n) \leq \frac{[d(Sx^n, Jx^n)]^2 + [d(Ix^n, Tx^n)]^2}{d(Sx^n, Jx^n) + d(Ix^n, Tx^n)} + \beta d(Ix^n, Jx^n)
\]

or

\[
U_{2n+1} \leq \alpha [U_{2n+1} + U_{2n}] + \beta U_{2n}
\]

So that

\[
U_{2n+1} \leq K U_{2n} \leq \ldots \leq K^{n-1} U_0, \text{ for } n = 0, 1, 2, \ldots
\]

where \( K = \frac{\alpha + \beta}{1 - \alpha} \). Since \( 2\alpha + \beta < 1 \), it follows that \( K < 1 \) and hence the sequence.

\[\{Sx^n, Tx^n, Sx^n, \ldots, Tx_{2n-1}, Sx_{2n}, Tx_{2n-1}, \ldots\}\] is Cauchy and so gets a limit point \( z \) in \( X \). Consequently the subsequences \( \{Ix^n\} = \{Tx_{2n-1}\} \) and \( \{Jx_{2n-1}\} = \{Sx_{2n}\} \) also converge to the same point \( z \) in \( X \).

Let us now suppose that \( I \) is continuous so that sequences \( \{I^n x^n\} \) and \( \{I^n Sx^n\} \) converge to the point \( Iz \). Since \( S \) and \( I \) are compatible, we have

\[
\lim_{n \to \infty} d(I^n Sx^n, I^n Sx^n) = 0
\]

and so it follows that the sequence \( \{Sx^n\} \) converges to \( Iz \).

Now to show that \( z = Iz \) we consider

\[
d(I^n Sx^n, Tx_{2n-1}) \leq \frac{[d(I^n Sx^n, Jx_{2n-1})]^2 + [d(I^n Sx^n, Tx_{2n-1})]^2}{d(I^n Sx^n, Jx_{2n-1}) + d(I^n Sx^n, Tx_{2n-1})} + \beta d(I^n Sx^n, Jx_{2n-1})
\]
which on letting \( n \to \infty \), We get

\[
d(Iz, z) \leq (\alpha + \beta) d(Iz, z)
\]

a contradiction, hence \( Iz = z \). Again we consider

\[
d(Sz, Tx_{2n}) \leq \alpha \frac{[d(Sz, Jx_{2n})]^2 + [d(Iz, Tx_{2n})]^2}{d(Sz, Jx_{2n}) + d(Iz, Tx_{2n})} + \beta d(Iz, Jx_{2n})
\]

which on letting \( n \to \infty \), yields \( Sz = z \).

This means that \( z \) is in the range of \( S \) and since \( S(X) \subseteq J(X) \) there exists a point \( z' \) in \( X \) such that \( Jz' = z \). Thus

\[
d(z, Tz') = d(Sz, Tz') \leq \alpha \frac{[d(Sz, Jz')]^2 + [d(Iz, Tz')]^2}{d(Sz, Jz') + d(Iz, Tz')} + \beta d(Iz, Jz')
\]

\[= \alpha d(z, Tz') \]

a contradiction. Thus we have shown that \( z = Jz' = Tz' \). by using the compatibility of \( T \) and \( J \) we have \( d(TJz', JTz') = 0 \) giving thereby \( Tz = Jz \). Now consider

\[
d(z, Tz) = d(Sz, Tz) \leq \alpha \frac{[d(Sz, Jz)]^2 + [d(Iz, Tz)]^2}{d(Sz, Jz) + d(Iz, Tz)} + \beta d(Iz, Jz)
\]

\[= (\alpha + \beta) d(z, Tz) \]

a contradiction. Hence \( Tz = Jz = z \).

Thus we have proved \( z = Iz = Sz = Tz = Jz \) and so \( z \) is a common fixed point of \( S, I, T \) and \( J \).
Now suppose that $S$ is continuous so that \{$S^nx_n$\} and \{$I^nx_n$\} converge to $Sz$ and using the compatibility of $S$ and $I$ and arguing as above it follows that the sequence \{I$^nx_n$\} also converges to $Sz$. Now we consider

$$d(S^2x_{2n}, Tx_{2n+1}) \leq \alpha \frac{\left[d(S^2x_{2n}, Jx_{2n-1})\right]^2 + \left[d(I^nx_n, Tx_{2n-1})\right]^2}{d(S^2x_{2n}, Jx_{2n-1}) + d(I^nx_n, Tx_{2n-1})} + \beta d(I^nx_n, Jx_{2n-1})$$

and letting $n \to \infty$, we get

$$d(Sz, z) \leq (\alpha + \beta) d(Sz, z)$$

a contradiction, it follows that $Sz = z$.

And from above arguments, again there exists a point $z'$ in $X$ such that $Jz' = z$ and in the same way we can show that $z = Tz'$ and $Jz = Tz$.

Further we consider

$$d(Sx_{2n}, Tz) \leq \alpha \frac{\left[d(Sx_{2n}, Jz)\right]^2 + \left[d(Ix_{2n}, Tz)\right]^2}{d(Sx_{2n}, Jz) + d(Ix_{2n}, Tz)} + \beta d(Ix_{2n}, Jz)$$

and on letting $n \to \infty$, we get

$$d(z, Tz) \leq (\alpha + \beta) d(z, Tz)$$

a contradiction. And so it follows that $z = Tz = Jz (= Sz)$.

The point $z$ therefore is in the range of $T$ and since $T(X) \subseteq I(X)$ there exists a point $z''$ in $X$ such that $Iz'' = z$. Thus
\[ d(Sz'', z) = d(Sz'', tz) \]
\[ \leq \alpha \frac{[d(Sz'', Jz)]^2 + [d(tz'', tz)]^2}{d(S'', Jz) + d(tz'', tz)} + \beta \ d(tz'', Jz) \]
\[ = \alpha \ d(Sz'', z) \]

a contradiction, and so \( Sz'' = z \). Thus we have shown that \( z = Iz'' = Sz'' \) and from the compatibility of \( S \) and \( I \) it follows that \( d(Slz'', Isz'') = 0 \), giving thereby \( Sz = Iz \). Thus once again we have proved that \( z = Sz' = Iz = Tz = Jz \) and so \( z \) is a common fixed point of \( S, T, I \) and \( J \).

If the mapping \( T \) or \( J \) is continuous instead of \( S \) or \( I \) then the proof can be produced in the same way.

Case - II : If
\[ d(Sx_{2n-2}, Jx_{2n-1}) + d(Ix_{2n-2}, Tx_{2n-1}) = 0 \]
for some \( n \), then on using inequality (I) we can show that
\[ U_{2n-1} = 0. \] This implies that
\[ Sx_{2n} = Tx_{2n-1} = Sx_{2n-2} = \ldots = z \]

We assert that there exists a point \( w \) such that \( Sw = Iw = Tw = Jw = z \), otherwise if \( Sw = Iw \neq z \), then
\[ d(Iw, z) = d(Sw, Tx_{2n-1}) \]
\[ \leq \alpha \frac{[d(Sw, Jx_{2n-1})]^2 + [d(Iw, Tx_{2n-1})]^2}{d(Sw, Jx_{2n-1}) + d(Iw, Tx_{2n-1})} + \beta \ d(Iw, Jx_{2n-1}) \]
on letting \( n \to \infty \), we get
\[ = (\alpha + \beta) \ d (Iw, z) < d(Iw, z) \]

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a contradiction which yields that $lw = z = Sw$. Similarly, we can show that $Tw = Jw = z$

Now, suppose that $S$ or $I$ is continuous. Proceeding as above, it can be shown that $lw = z$ is a common fixed point of $S$, $T$, $I$ and $J$.

Furthermore, if $T$ or $J$ is continuous, then the proof that $z$ is common fixed point of $S,T,I$ and $J$ is similar.

In order to prove that $z$ is a unique common fixed point of $S$ and $I$, let $w$ be another fixed point of $S$ and $I$ (i.e. $w = Sw = lw$) then

$$d(w, z) = d(Sw, Tz)$$

$$\leq \alpha \frac{[d(Sw, Jz)]^2 + [d(lw, Tz)]^2}{d(Sw, Jz) + d(lw, Tz)} + \beta \ d(lw, Jz)$$

$$= (\alpha + \beta) \ d(w, z)$$

a contradiction, which yields that $w = z$. Similarly we can prove that $z$ is a unique common fixed point of $T$ and $J$. This completes the proof.

It is shown in Jungck et. al. [57] that if $S$ and $I$ (resp. $T$ and $J$) are both continuous then the pair $\{S,I\}$ (resp. $\{T,J\}$) are compatible in $X$ if and only if it is $(A)$type-compatible on $X$. Since only one of the mappings $S,T,I$ and $J$ is assumed continuous in Theorem 2.2.1, it is interesting to investigate the situation when $\{S,I\}$ and $\{T,J\}$ are $(A)$ type-compatible pairs.

Thus, we prove the following
Theorem 2.2.2  Let \( \{S,I\} \) and \( \{T,J\} \) be \((A)\) type-compatible pairs of mappings of a complete metric space \((X,d)\) into itself such that conditions \((a)\) and \((b)\) of Theorem 2.2.1 are satisfied.

If one of \(S,T,I,J\) is continuous then \(S,T,I,J\) have a unique common fixed point \(z\) in \(X\). Further \(z\) is the unique common fixed point of \(S\) and \(I\) and of \(T\) and \(J\).

Proof. Proceeding as in Theorem 2.2.1, we can show that the sequences \(\{Ix_{2n}\} = \{Tx_{2n-1}\}\) and \(\{Jx_{2n-1}\} = \{Sx_{2n}\}\) converge to some point \(z\) in \(X\).

Let us now suppose that \(J\) is continuous so that the sequences \(\{Jx_{2n}\}\) and \(\{JSx_{2n}\}\) converge to the point \(Iz\). Since \(S\) and \(I\) are \((A)\) type-compatible, we have

\[
\lim_{n \to \infty} d(SIx_{2n}, IIx_{2n}) = 0
\]

Hence it follows that the sequence \(\{SIx_{2n}\}\) also converges to \(Iz\). Now using the procedure of Theorem 2.2.1, we can show that \(z = Iz = Sz\).

This means that \(z\) is in the range of \(S\) and as \(S(X) \subseteq J(X)\) there exists a point \(z'\) in \(X\) such that \(Jz' = z\). Again using the argument of Theorem 2.2.1 we get \(Tz' = z\). Thus \(z = Tz' = Jz'\) and since \(T\) and \(J\) are \((A)\) type-compatible, we have

\[
d(TJz', JJz') = 0
\]

giving thereby \(Tz = Jz\) and arguing as in Theorem 2.2.1, we can show that \(z = Tz = Jz\).

Thus we have proved that \(z = Tz = Jz = Iz = Sz\) and so \(z\) is a common fixed point of \(S,T,I,J\).
Now, suppose that $S$ is continuous so that the sequences $\{S^n x_{2n}\}$ and $\{S^{-n} x_n\}$ converge to the point $S_z$. Since $S$ and $I$ are (A)type-compatible, we have

$$\lim_{n \to \infty} d(IS^n x_{2n}, SS^n x_{2n}) = 0$$

which implies that the sequence $\{IS^n x_n\}$ also converges to $S_z$. Again, arguing as in Theorem 2.2.1, we can show that $z = S_z$ and as $S(X) \subseteq J(X)$ there exists a point $z'$ in $X$ such that $Jz' = z$. Again, in the same way, we can show that $Tz' = z$ and the (A)type-compatibility of $T$ and $J$ yields that $z = Tz = Jz (= S_z)$.

The point $z$ therefore is in the range of $T$ as $T(X) \subseteq I(X)$ there exists a point $z''$ in $X$ such that $Iz'' = z$. Again arguing as in Theorem 2.2.1, we can show that $z = Iz'' = S z''$, and from the (A)type-compatibility of $S$ and $I$ it follows that

$$d(Sz'', Iz'') = 0$$

giving thereby $S_z = Iz$.

Thus, once again we have shown that $z$ is a common fixed point of $S, T, I$ and $J$.

If the mapping $T$ or $J$ is continuous instead of $S$ or $I$ then the proof can be given in the same way.

The remaining part of the proof is similar to that of Theorem 2.2.1 hence omitted.

Now, we give the following example for the illustration of Theorem 2.2.1.
Example 2.2.3 Consider $X = [0,1]$ with the usual metric. Define the self mappings $S$, $T$, $I$, and $J$ on $X$ as.

$T_x = x/6$, $S_x = x/4$, $I_x = x/3$ and $J_x = x/2$.

Clearly $T(X) = [0, 1/6] \subseteq I(X) = [0, 1/3]$ and $S(X) = [0, 1/4] \subseteq J(X) = [0, 1/2]$.

For any sequence $\{x_n\}$ in $X$, we have $d(Sx_n, ISx_n) = d(TJx_n, JTx_n) = 0$.

Hence the pairs $\{S, I\}$ and $\{T, J\}$ are compatible. A routine calculation shows that the inequality (1) of Theorem 2.2.1 holds for all $x, y \in X$, with $\alpha = 1/9$ and $\beta = 2/9$.

Clearly $x = 0$ is a unique common fixed point of $S, T, I,$ and $J$.

Remark - 1 If we choose $I = J =$ identify mapping and $T = S$ and $\beta = 0$, we get a result of Fisher [31].

Remark - 2 Our Theorem can also be extended by assuming $\{S, I\}$ and $\{T, J\}$ as ($P$)type-compatible pairs. The proof can be produced by using the Proposition 1.5.16.

Remark - 3 Since weakly commuting mappings are always compatible, theorem 2.2.1 also holds if $\{S, I\}$ and $\{T, J\}$ are assumed weakly commuting pairs.

Remark - 4 By suitably choosing $\alpha, \beta, S, T, I$ and $J$ we can derive a multitude of fixed point theorems. We omit the details.

Remark - 5 Theorems 2.2.1 and 2.2.2 ensure that $S, T, I,$ and $J$ have unique common fixed point. However, either $S$ or $I$ or $T$ or $J$ may have
other fixed point. Example supporting this fact can be found in Imdad et al. [49].

**Remark - 6** It follows from the proof of theorem 2.2.1 and 2.2.2 that if condition (l') is omitted from the hypothesis then we can say that \( z \) is a coincidence point of \( S, T, I \) and \( J \).

**Remark - 7** If we choose \( \alpha = 0 \) in Theorem 2.2.1 and Theorem 2.2.2, then we get an improved form of the theorem of Fisher [30] for two pairs of compatible and \((A)\) type-compatible mappings respectively.

### 2.3 FIXED POINT THEOREMS FOR A MAPPING AND SEQUENCE OF MAPPINGS

In this Section, we have studied the fixed point and common fixed point of mapping and sequence of mappings satisfying a contractive condition in the form of rational inequality, which in turn also generalize Banach contraction principle and yield some new results.

We prove the following:

**Theorem 2.3.1** Let \((X, d)\) be a metric space and \( T : X \rightarrow X \) be a self mapping of \( X \) such that

\[
d(Tx, Ty) \leq \alpha \frac{[d(Tx, y)]^2 + [d(x, Ty)]^2}{d(Tx, y) + d(x, Ty)} + \beta \ d(x, y)
\]

holds for all \( x, y \in X, \alpha, \beta \geq 0, 2\alpha + \beta < 1 \), whenever \( d(Tx, y) + d(x, Ty) \neq 0 \) and \( d(Tx, Ty) = 0 \) whenever \( d(Tx, y) + d(x, Ty) = 0 \) and

(i) there exists some point \( x_0 \in X \) such that the sequence \( \{ T^n x_0 \} \) has
a subsequence \( \{T^n x_o\} \) converging to some point \( z \) in \( X \).

(ii) \( T \) is \( x_o \)-continuous.

Then \( T \) has a unique fixed point.

**Proof.** Let \( T x_o = x_1, \ x_2 = T x_1 = T^2 x_o, \ldots, x_n = T^n x_o \). Then we have

\[
d(x_n', x_{n+1}) = d(T x_{n-1}, T x_n)
\]

\[
\leq \alpha \frac{[d(T x_{n-1}, x_n)]^2 + [d(x_{n-1}, T x_n)]^2}{d(T x_{n-1}, x_n) + d(x_{n-1}, T x_n)} + \beta d(x_{n-1}, x_n)
\]

\[
= \alpha \frac{[d(T x_n', x_n)]^2 - [d(x_{n-1}, x_{n-1})]^2}{d(x_n', x_n) + d(x_{n-1}, x_{n-1})} + \beta d(x_{n-1}, x_n)
\]

\[
\leq \alpha [d(x_{n-1}, x_n) + d(x_n', x_{n-1})] + \beta d(x_{n-1}, x_n)
\]

so that

\[
d(x_n', x_{n+1}) \leq \left( \frac{\alpha + \beta}{1 - \alpha} \right) d(x_{n-1}, x_n) \leq \ldots \leq \left( \frac{\alpha + \beta}{1 - \alpha} \right)^n d(x_o', x_1)
\]

if \( x_n' = x_n \), then the condition of the Theorem implies that \( x_n' = x_n = x_{n-1} = \ldots \), thus \( x_{n-1} \) would be the fixed point of \( T \).

Since \( (2\alpha + \beta) < 1 \), implies that \( \frac{\alpha + \beta}{1 - \alpha} < 1 \), it follows that \( \{x_n\} = \{T^n x_o\} \) is a Cauchy sequence in \( X \). In view of (i), without loss of generality we can assume that \( \{T^n x_o\} \) converges to some point \( z \) in \( X \).

Now the \( x_o \)-continuity of \( T \) implies that

\[
Tz = T (\lim_{n \to \infty} T^n x_o) = \lim_{n \to \infty} T^{n-1} x_o = z
\]

thus \( z \) is a fixed point of \( T \).
To show that $z$ is unique, let $w$ be another fixed point of such that $w = Tw$ then

$$d(w, z) = d(Tw, Tz)$$

$$\leq \alpha \frac{[d(Tw, z)]^2 + [d(w, Tz)]^2}{d(Tw, z) + d(w, Tz)} + \beta d(w, z)$$

$$= (\alpha + \beta) d(w, z)$$

a contradiction, which yields that $w = z$. This complete the proof.

**Remark - 1**

(i) For $\alpha = 0$, we get an analogue of Maia [78] result

(ii) For $\beta = 0$, we get an analogue of Fisher [31] result

We observe that if $X$ is a complete metric space then the condition of continuity of $T$ can be relaxed. Thus we have the following.

**Theorem 2.3.2** Let $(X, d)$ be a complete metric space and $T : X \to X$ such that

$$d(Tx, Ty) \leq \alpha \frac{[d(Tx, y)]^2 + [d(x, Ty)]^2}{d(Tx, y) + d(x, Ty)} + \beta d(x, y)$$

holds for all $x, y \in X$, $\alpha \geq 0.2 \alpha + \beta < 1$ whenever $d(Tx, y) + d(x, Ty) \neq 0$ and $d(Tx, Ty) = 0$ whenever $d(Tx, Ty) + d(x, Ty) = 0$. Then $T$ has a unique fixed point

**Proof.** Let $x_o$ be an arbitrary point in $X$. Define $x_n = T^n x_o$. Then proceeding as in Theorem 2.3.1. we can show that \{ $x_n$ \} = \{ $T^n x_o$ \} is a Cauchy sequence and so converges to some point $z$ in $X$, since $X$ is complete. Now
\[ d(Tz, x_{n+1}) = d(Tz, Tx_n) \]

\[ \leq \alpha \frac{[d(Tz, x_n)]^2 + [d(z, Tx_n)]^2}{d(Tz, x_n) + d(z, Tx_n)} + \beta d(z, x_n) \]

\[ = \alpha \frac{[d(Tz, x_n)]^2 + [d(z, x_{n-1})]^2}{d(Tz, x_n) + d(z, x_{n-1})} + \beta d(z, x_n) \]

and letting \( n \to \infty \), we get

\[ d(Tz, z) \leq \alpha d(Tz, z) \]

a contradiction, implying therefore \( z = Tz \). The uniqueness follows from Theorem 2.3.1.

Before proving the next result, following Iseki [52] we recall:

**Definition 2.3.3** A mapping \( T \) of a metric space \( (X, d) \) into itself is said to be orbitally continuous if \( \lim_{i \to \infty} T^n x = z \) implies that \( \lim_{i \to \infty} T(T^n x) = Tz \) for every \( x \in X \).

It is well known that every continuous mapping is orbitally continuous, but converse is not true (cf. Ciric [23]).

We prove the following

**Theorem 2.3.4** Let \( (X, d) \) be a metric space and \( T : X \to X \) be an orbitally continuous mapping such that

\[ d(Tx, Ty) \leq \alpha \frac{[d(Tx, y)]^2 + [d(x, Ty)]^2}{d(Tx, y) + d(x, Ty)} + \beta d(x, y) \]

holds for all \( x, y \in X \), \( \alpha, \beta \geq 0 \), \( 2\alpha + \beta < 1 \). whenever \( (Tx, y) + d(x, Ty) \neq 0 \) and \( d(Tx, Ty) = 0 \) whenever \( d(Tx, y) + d(x, Ty) = 0 \). and for some \( x_o \in X \).
the sequence \( \{T^n x_0\} \) has a cluster point \( z \) in \( X \). Then \( z \) is a unique fixed point.

**Proof.** If \( T^{n-1} x_0 = T^n x_0 \) for some \( m \in N \), then \( T^n x_0 = T^n x_0 = z \) for all \( n \geq m \) and the result follows.

We assume that \( T^{n+1} x_0 \neq T^n x_0 \) for all \( m \in N \), and let \( \lim_{i \to \infty} T^n x_0 = z \).

Define \( x_n = T^n x_0 \). Then proceeding as in Theorem 2.3.1 we can show that

\[
d(x_n, x_{n+1}) = d(T^n x_0, T^{n+1} x_0) \text{ is a decreasing sequence of positive real numbers.}
\]

Further since \( T \) is orbitally continuous

\[
\lim_{i \to \infty} d(T^n x_0, T^{n+1} x_0) = d(z, Tz)
\]

and the sequence

\[
\{d(T^n x_0, T^{n+1} x_0)\} \subseteq \{d(T^n x_0, T^{n+1} x_0)\}
\]

implies that

\[
\lim_{n \to \infty} d(T^n x_0, T^{n+1} x_0) = d(z, Tz).
\]

Also orbital continous of \( T \) yields \( \lim_{i \to \infty} T^{n+1} x_0 = Tz \),

\[
\lim_{i \to \infty} T^{n+1} x_0 = Tz \quad \text{and the sequence}
\]

\[
\{d(T^{n+1} x_0, T^{n+2} x_0)\} \subseteq \{d(T^n x_0, T^{n+1} x_0)\}
\]

and so the above relation implies that

\[
d(Tz, T^2 z) = d(z, Tz)
\]

If \( d(z, Tz) > 0 \) then

\[
d(z, Tz) = d(Tz, T^2 z)
\]

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\[
\frac{[d(Tz, Tz)]^2 + [d(z, T^2z)]^2}{d(Tz, Tz) + d(z, T^2z)} + \beta d(z, Tz)
\]
\[
\leq \alpha \frac{[d(z, Tz) + d(Tz, T^2z)] + \beta d(z, Tz)}{d(Tz, Tz) + d(z, T^2z)}
\]
\[
= (2\alpha + \beta) d(z, Tz)
\]
which is a contradiction. Hence \( z = Tz \). Thus \( z \) is a fixed point of \( T \) and the uniqueness follows from Theorem 2.3.1.

**Remark - 2**

(i) For \( \alpha = 0 \), the foregoing Theorem 2.3.4 extends a theorem of Edelstein [29].

(ii) For \( \beta = 0 \), we get a generalized version of Fisher [31] results.

In the following, we shall prove some results for the common fixed point of sequence of mappings.

**Theorem 2.3.5** Let \((X, d)\) be a complete metric space and \( \{T_n\} \), a sequence of mappings of \( X \) into itself. If for arbitrary chosen \( i, j \in N \) the inequality

\[
d(T_i x, T_j y) \leq \alpha \frac{[d(T_i x, y)]^2 + [d(x, T_j y)]^2}{d(T_i x, y) + d(x, T_j y)} + \beta d(x, y)
\]

holds for all \( x, y \in X, \alpha, \beta \geq 0, 2\alpha + \beta < 1 \) whenever \( d(T_i x, y) - d(x, T_j y) \neq 0 \) and \( d(T_i x, T_j y) = 0 \) whenever \( d(T_i x, y) + d(x, T_j y) = 0 \). Then the sequence \( \{T_n\} \) has a unique common fixed point.

**Proof.** Let \( x_o \in X \) be arbitrary. Construct the sequence \( x_i = T_i x_o, x_1 = T_j x_i, \ldots, x_{2n-1} = T_i x_{2n-2}, x_{2n-2} = T_j x_{2n-4} \). Then

\[
d(x_n, x_2) = d(T_i x_o, T_j x_i)
\]
\[
\begin{align*}
\leq \frac{\alpha}{d(T_i x, x_i)} \frac{[d(T_i x, x_i)]^2 + [d(x_i, T_i x)]^2}{d(T_i x, x_i) + d(x_i, T_i x)} + \beta d(x_i, x_i) \\
\leq \left( \frac{\alpha + \beta}{1 - \alpha} \right) d(x_i, x_i)
\end{align*}
\]

Inductively, it follows that

\[
d(x_n, x_{n-1}) \leq \left( \frac{\alpha + \beta}{1 - \alpha} \right)^n d(x_o, x_i)
\]

since \(2\alpha + \beta < 1\), this implies that \(\{x_n\}\) is a Cauchy sequence and converges to some point \(z\) in \(X\). Consequently the subsequences \(\{x_n\}\) and \(\{x_{2n}\}\) converge to \(z\). Now consider

\[
d(T_i x, z) \leq \frac{\alpha}{d(T_i x, z)} \frac{[d(T_i x, z)]^2 + [d(x_i, T_i z)]^2}{d(T_i x, z) + d(x_i, T_i z)} + \beta d(x_i, z)
\]

letting \(n \to \infty\), it follows that

\[
d(z, T_i z) \leq \alpha d(z, T_i z)
\]

a contradiction, giving thereby \(z = T_i z\).

In the same way we can show that \(z = T_i z\). For uniqueness, let \(w\) be another fixed point of \(T_i\) and \(T_i\), that is, \(w = T_i w = T_i w\), then

\[
d(w, z) = d(T_i w, T_i z)
\]

\[
\begin{align*}
\leq \alpha \frac{[d(T_i w, z)]^2 + [d(w, T_i z)]^2}{d(T_i w, z) + d(w, T_i z)} + \beta d(w, z) \\
= (\alpha + \beta) d(w, z)
\end{align*}
\]

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a contradiction, implying therefore \( w = z \).

To show that \( z \) is a fixed point of every number of the sequence \( \{T_n\} \), let \( T_k \) be an arbitrary element of \( \{T_n\} \) other than \( T_i \) and \( T_j \). Then

\[
d(z, T_k z) = d(T, z, T_k z)
\]

\[
\leq \alpha \frac{[d(T, z, z)]^2 + [d(z, T_k z)]^2}{d(T, z, z) + d(z, T_k z)} + \beta d(z, z)
\]

\[
= \alpha d(z, T_k z)
\]

a contradiction, thus \( z = T_k z \). Hence we have proved that \( z \) is a unique common fixed point of \( \{T_n\} \). This completes the proof.

**Remark - 3** For \( \{T_n\} = \{T_i, T_j\} \) the foregoing Theorem gives the result for a pair of mappings.

In view of the observation of Chu and Diaz [20], that for a mapping \( T \) to have a fixed point it is sufficient for some iterate \( T^n \) to be a contraction. The following result immediately follows.

**Corollary 2.3.6** Let \((X, d)\) be a complete metric space and \( \{T_n\} \) be a sequence of mappings of \( X \) into itself. Suppose for some positive integers \( p, q \) the condition

\[
d(T_i^p x, T_j^q y) \leq \alpha \frac{[d(T_i^p x, y)]^2 + [d(x, T_j^q y)]^2}{d(T_i^p x, y) + d(x, T_j^q y)} + \beta d(x, y)
\]

holds for all \( x, y \in X, \alpha, \beta \geq 0, 2\alpha + \beta < 1 \), whenever \( d(T_i^p x, y) + d(x, T_j^q y) \neq 0 \) and \( d(T_i^p x, T_j^q y) = 0 \) whenever \( d(T_i^p x, y) + d(x, T_j^q y) = 0 \).

Then the sequence \( \{T_n\} \) has a unique common fixed point.
Proof. Set $x_1 = T^p x_0$, $x_2 = T^q x_1$, ... , etc. Then proceeding as in Theorem 2.3.5, we can prove that $T^p$ and $T^q$ have a unique common fixed point $z$ (say) i.e. $z = T^p z = T^q z$. Thus $T (T^p z) = Tz$ so that $T^{p-1} z = T^p (Tz) = Tz$, giving thereby $Tz$ is the fixed point of $T^p$ and in view of the uniqueness of $z$, it follows that $Tz = z$. The remaining part of the proof is similar to that of Theorem 2.3.5 hence omitted.

Theorem 2.3.7 Let $(X, d)$ be a complete metric space and $\{T_n\}, n = 0, 1, 2, ...$, be a sequence of mappings of $X$ into itself such that

$$d(T_o x, T_n y) \leq \alpha \frac{[d(T_o x, y)]^2 + [d(x, T_n y)]^2}{d(T_o x, y) + d(x, T_n y)} + \beta d(x, y)$$
holds for all $x, y \in X, \alpha, \beta \geq 0, 2\alpha + \beta < 1$, whenever $d(T_o x, y) + d(x, T_n y) \neq 0$ and $d(T_o x, T_n y) = 0$ whenever $d(T_o x, y) + d(x, T_n y) = 0$ for $n = 1, 2, ...$.

Then $\{T_n\}, n = 0, 1, 2, ...$ has a unique common fixed point.

Proof. Let $x_0 \in X$ be arbitrary. Construct the sequence $\{x_n\}$ as follows

$x_1 = T_o x_0$, $x_2 = T_1 x_1$, $x_3 = T_o x_2$, $x_4 = T_2 x_3$, \ldots , $x_n = T_n x_{n-1}$, $x_{n+1} = T_{n+1} x_n$.

Then

$$d(x_n, x_{n+1}) = d(T_o x_n, T_1 x_1)$$

$$\leq \alpha \frac{[d(T_o x_n, x_1)]^2 + [d(x_n, T_1 x_1)]^2}{d(T_o x_n, x_1) + d(x_n, T_1 x_1)} + \beta d(x_n, x_1)$$

$$\leq (\frac{\alpha + \beta}{1 - \alpha}) d(x_o, x_1)$$

Thus by induction, it follows that
Since $2\alpha + \beta < 1$, it follows that \( \{x_n\} \) is a Cauchy sequence and hence converges to some point \( z \) in \( X \). Consequently the subsequences \( \{x_{2n}\} \) and \( \{x_{2n+1}\} \) also converge to the point \( z \). Now consider

\[
d(z, T^*_o z) \leq d(z, x_{2n}) + d(x_{2n}, T^*_o z) = d(z, x_{2n}) + d(T^*_n x_{2n-1}, T^*_o z) \leq d(z, x_{2n}) + \frac{[d(T^*_n x_{2n-1}, z)]^2 + [d(x_{2n-1}, T^*_o z)]^2}{d(T^*_n x_{2n-1}, z) + d(x_{2n-1}, T^*_o z)} + \beta d(x_{2n}, z)
\]

\[
= d(z, x_{2n}) + \alpha \frac{[d(x_{2n}, z)]^2 + [d(x_{2n-1}, T^*_o z)]^2}{d(x_{2n}, z) + d(x_{2n-1}, T^*_o z)} + \beta d(x_{2n}, z)
\]

Letting \( n \to \infty \), it yields

\[
d(z, T^*_o z) \leq \alpha \ d(z, T^*_o z)
\]

which is a contradiction, giving thereby \( z = T^*_o z \). Again for \( n = 1, 2, \ldots \)

\[
d(z, T^n z) = d(T^*_o z, T^n z) \leq \alpha \frac{[d(T^*_o z, z)]^2 + [d(z, T^n z)]^2}{d(T^*_o z, z) + d(z, T^n z)} + \beta d(z, z)
\]

\[
= \alpha \ d(z, T^n z)
\]

a contradiction, which implies that \( z = T^n z, n = 1, 2, \ldots \). Thus \( z = T^n z = T^*_o z, n = 1, 2, \ldots \). The uniqueness can be shown as in Theorem 2.3.5.

This completes the proof.
2.4 FIXED POINT THEOREMS FOR EXPANSION TYPE MAPPINGS

The well known Banach fixed point theorem has been generalized by many authors for different type of mappings and employing several more general contractive conditions. However, recently fixed point theorem for expansion type mappings have proved by Wang et. al. [120] Gillespie et. al. [41] and Taniguchi [118] etc.

In this Section, we have obtained some results concerning the coincidence and fixed point for expansion type mappings satisfying some weak condition of commutativity. Our results are more general and yields several earlier results as special cases.

Now, we prove the following results.

**Theorem 2.4.1** Let \((X, d)\) be a complete metric space. If \(F, G, T : X \rightarrow X\) satisfy the following

1. \(F(X) = T(X), \ G(X) = T(X)\)
2. \(d(Fx, Gy) \geq a \ d(Fx, Tx) + b \ d(Gy, Ty) + c \ d(Tx, Ty)\)

for each \(x, y \in X\) with \(x \neq y\), where \(a, b, c \geq 0; \ a + b + c > 1, \ a, b < 1\).

3. \(\{F, T\}\) and \(\{G, T\}\) are weakly commuting pairs. and
4. \(T\) is continuous at \(X\).

Then \(F, G\) and \(T\) have a common coincidence point. Further if \(c > 1\) then \(F, G\) and \(T\) have a unique common fixed point.
Proof Let $x_0$ be an arbitrary point in $X$. Since $F(X) = T(X)$ we can find a point $x_1$ in $X$ such that $Tx_1 = Fx_1$. Also since $G(X) = T(X)$ we can find $x_2$ in $X$ such that $Tx_2 = Gx_2$. In general for a point $x_{2n}$ in $X$ we can find $x_{2n-1}$ in $X$ such that $Tx_{2n} = Fx_{2n-1}$, and then a point $x_{2n-2}$ in $X$ such that $Tx_{2n-1} = Gx_{2n-2}$.

Now, for $n \geq 0$, suppose that $Tx_{2n} = Tx_{2n-1}$. If $Tx_{2n-1} \neq Tx_{2n-2}$, we obtain from condition (2)

$$d(Tx_{2n}, Tx_{2n-1}) = d(Fx_{2n-1}, Gx_{2n-2})$$

$$\geq a d(Fx_{2n-1}, Tx_{2n-1}) + b d(Gx_{2n-2}, Tx_{2n-2}) + c d(Tx_{2n-1}, Tx_{2n-2})$$

or

$$d(Tx_{2n}, Tx_{2n-1}) \geq \frac{b + c}{1 - a} d(Tx_{2n-1}, Tx_{2n-2})$$

which is contradiction and so $Tx_{2n-1} = Tx_{2n-2}$. Thus it can be shown that $Tx_{2n} = Tx_{2n-1} = Tx_{2n-2} = Tx_{2n-3} = \ldots$. For no loss of generality let $n = 0$ then we have $Tx_0 = Tx_1 = Tx_2 = Fx = Gx$. Since $\{F, T\}$ is weakly commuting we have

$$d(FTx, TFx) \leq d(Tx, Fx) = 0,$$

giving thereby $FTx = TFx = TTx$, which means that $Tx_1$ is a coincidence point of $F$ and $T$. Similarly the weak commutativity of $\{G, T\}$ yields $GTx_2 = Tgx_2 = TTx_2$, which means that $Tx_2$ is a coincidence point of $G$ and $T$. Since $Tx_1 = Tx_2$ we get thereby $Tx_1$ is a common coincidence point of $F$, $G$ and $T$.

Suppose that for $n \geq 0$, $Tx_{2n} \neq Tx_{2n-1}$ and $Tx_{2n-1} \neq Tx_{2n-2}$ then from condition (2) we get
\[ d(Tx_{2n}, Tx_{2n-1}) = d(Fx_{2n-1}, Gx_{2n-2}) \]

\[ \geq a \, d(Fx_{2n-1}, Tx_{2n-1}) + b \, d(Gx_{2n-2}, Tx_{2n-2}) + c \, d(Tx_{2n-1}, Tx_{n-2}) \]

or

\[ d(Tx_{2n}, Tx_{2n-1}) \geq \left( \frac{b + c}{1 - a} \right) d(Tx_{2n-1}, Tx_{2n-2}) \]

similarly

\[ d(Tx_{2n-1}, Tx_{2n-2}) \geq \left( \frac{a + c}{1 - b} \right) d(Tx_{2n-2}, Tx_{2n-3}) \]

since \( a + b + c > 1 \) and \( a, b < 1 \). We get \( 1 > \left( \frac{1 - a}{b + c} \right), \left( \frac{1 - b}{a + c} \right) > 0 \).

it follows that \( \{Tx_n\} \) is a Cauchy sequence and so converges to a point \( z \) in \( X \). Consequently the subsequences \( \{Fx_{2n-1}\} \) and \( \{Gx_{2n-2}\} \) converge also to a point \( z \) in \( X \). As \( T \) is continuous, the sequences \( \{TTx_n\}, \{TF_{2n-1}\} \) and \( \{TGx_{2n-2}\} \) converge to the point \( Tz \).

Since \( F \) and \( T \) are weakly commuting, we have

\[ d(FTx_{2n-1}, TFx_{2n-1}) \leq d(Tx_{2n-1}, Fx_{2n-1}) = d(Tx_{2n-1}, Tx_{2n}) \]

on letting \( n \to \infty \), we get \( d(FTx_{2n-1}, Tz) \to 0 \). and so the sequence \( \{FTx_{2n-1}\} \) also converges to a point \( Tz \). Similarly by weak commutativity of \( G \) and \( T \) the sequence \( \{GTx_{2n-2}\} \) converges to \( Tz \).

Since \( F(X) = T(X) \) there exists a point \( z' \) in \( X \) such that \( Tz = Fz' \).

Again using condition (2)

\[ d(Fz', GTx_{2n-2}) \geq a \, d(Fz', Tz') + b \, d(GTx_{2n-2}, TTx_{2n-2}) \]

\[ + \, cd(Tz', TTx_{2n-2}) \]

On letting \( n \to \infty \), we get
\[ 0 \geq (a + c) d(Tz, Tz') \]

which yields that \( Tz = Tz' \). By weak commutativity of \( \{F, T\} \) we find

\[ d(TTz, FTz) = d(TFz', FTz') \leq d(Fz', Tz') = 0 \]

giving thereby \( TTz = FTz \). Similarly we can show that \( GTz = TTz \). Thus we have shown that \( TTz = FTz = GTz \)

Further, for \( c > 1 \) we consider

\[
d(FTx_{2n-1}, Tx_{2n-1}) = d(FTx_{2n-1}, Gx_{2n-2}) \\
\geq a d(FTx_{2n-1}, TTx_{2n-1}) + b d(Gx_{2n-2}, Tx_{2n-2}) + c d(TTx_{2n-1},Tx_{2n-2})
\]

On letting \( n \to \infty \), we get

\[ d(Tz, z) \geq c d(Tz, z), \]

a contradiction, which gives that \( Tz = z \). Thus we have shown that \( z = Tz = Fz = Gz \).

To show that \( z \) is unique, let \( w \) be another common fixed point of \( F, G \) and \( T \), such that \( w = Tw = Fw = Gw \). Consider

\[
d(w, z) = d(Fw, Gz) \geq a d(Fw, Tw) + b d(Tz, Gz) + c d(Tw, Tz) = c d(w, z)
\]

which is a contradiction, implying therefore \( w = z \). This completes the proof.

**Remark - 1** (i) Our foregoing Theorem is more general as it guarantees the existence of a unique common fixed point when \( c > 1 \). The theorem - 1 of Taniguchi [118] is a corollary of our Theorem 2.4.1 with \( T = I \) (The identity mapping) and also improves in the sense that the fixed point is unique.
(ii) Our result also generalizes the results obtained by Wang et. al. [120] and Gillespie et. al. [41].

The foregoing Theorem can be further generalized for a pair of compatible mappings. Thus we prove the following.

**Theorem 2.4.2** Theorem 2.4.1 holds good if $F$, $G$ and $T$ are assumed continuous and condition (3) is replaced by

\[(3)' \quad \{F, T\} \text{ and } \{G, T\} \text{ are compatible pairs}\]

**Proof.** Proceeding as in Theorem 2.4.1, we have that \(\{Tx_n\}\) is a Cauchy sequence and so converges to a point \(z\) in \(X\). Where as the subsequences \(\{Fx_{2n-1}\}\) and \(\{Gx_{2n-1}\}\) converge also to a point \(z\) in \(X\).

Now
\[
\lim_{n \to \infty} Fx_{2n-1} = \lim_{n \to \infty} Tx_{2n-1} = z.
\]

It follows from the compatibility of \(\{F, T\}\) that
\[
\lim_{n \to \infty} d(FTx_{2n-1}, TFx_{2n-1}) = 0
\]
and from the continuity of $F$ and $T$ it follows that $Fz = Tz$ as $n \to \infty$.

Similarly we can show that $Gz = Tz$. Thus we have shown that $z$ is the common coincidence point of $F$, $G$ and $T$, and further if $c > 1$, we can argue as in Theorem 2.4.1, that $z$ is a unique common fixed point of $F$, $G$ and $T$. This completes the proof.

**Theorem 2.4.3** Let \((X, d)\) be a complete metric space. If $F, G, T : X \to X$ satisfy the following conditions.

(i) \(F(X) = T(X), \quad G(X) = T(X)\).
(ii) \( d(Fx, Gy) \geq a \min \{d(Tx, Fx), \ d(Ty, Fy), \ d(Tx, Ty)\} \)

for each \( x, y \in X \), where \( a > 1 \).

(iii) \( \{F, T\} \) and \( \{G, T\} \) are weakly commuting pairs.

(iv) \( F, G \) and \( T \) are continuous at \( X \).

Then \( F, G, \) and \( T \) have a coincidence point in \( X \).

**Proof.** Proceeding as in Theorem 2.4.1, we construct the sequence 
\[ \{Tx\} \] such that \( Tx_{2n} = Fx_{2n+1} \) and \( Tx_{2n-1} = Gx_{2n-2} \).

By condition (ii), we get
\[
d(Tx_{2n}, Tx_{2n-1}) = d(Fx_{2n+1}, Gx_{2n-2})
\geq a \min \{d(Tx_{2n+1}, Fx_{2n+1}), d(Tx_{2n+2}, Gx_{2n+2})
\mid d(Tx_{2n-1}, Tx_{2n-2})\}
\geq a \min \{d(Tx_{2n+1}, Tx_{2n+2}), d(Tx_{2n-1}, Tx_{2n-2})\},
\]

or
\[
d(Tx_{2n-2}, Tx_{2n-1}) \leq \frac{1}{a} \ d(Tx_{2n}, Tx_{2n-1})
\]

Similarly,
\[
d(Tx_{2n}, Tx_{2n+1}) \leq \frac{1}{a} \ d(Tx_{2n-1}, Tx_{2n})
\]

it follows that \( \{Tx\} \) is a Cauchy sequence, hence converges to a point \( z \) in \( X \). Consequently the subsequences \( \{Fx_{2n}\} \) and \( \{Gx_{2n-2}\} \) also converge to a point \( z \).

Since \( \{F, T\} \) is a weakly commuting pair, we have
\[
d(FTx_{2n}, TFx_{2n+1}) \leq d(Tx_{2n+1}, Fx_{2n+1})
\]
On letting $n \to \infty$ and by continuity of $F$ and $T$ we get $d(Tz, Fz) = 0$, giving thereby $Tz = Fz$. Similarly, by weakly commuting of $\{G, T\}$ and $F$ and $G$ are continuous we can get $Gz = Tz$. Hence we have shown that $Tz = Fz = Gz$. This completes the proof.

**Remark - 2** Our foregoing Theorem is more general as it involves three mappings. Further by setting $T = I$ (the identity mapping) we get Theorem - 2 of Taniguchi [118] as a corollary.

**Remark - 3** It will be interesting to investigate whether or not the continuity of mappings $T$, $F$ and $G$ is really needed in the proof.

**Theorem 2.4.4** Theorem 2.4.3 holds good if we replace condition (iii) by (iii)' $\{F, T\}$ and $\{G, T\}$ are compatible pairs.

**Proof.** The proof is similar to that of Theorem 2.4.3 except for some minor changes, hence it is omitted.

**Theorem 2.4.5** Let $(X, d)$ be a compact metric space. If $F, G, T : X \to X$ satisfy the following

(I)" either $\{F, T\}$ or $\{G, T\}$ are continuous

(II)" $F(X) = T(X)$, $G(X) = T(X)$, and

(III)" $d(Fx, Gy) > \min \{d(Tx, Fx), d(Ty, Gy), d(Tx, Ty)\}$

for all $x \neq y$ in $X$. Then $F$ and $T$ or $G$ and $T$ have a coincidence point.

**Proof.** Consider the case that $F$ and $T$ are continuous. Since $X$ is compact, there exists a point $z$ in $X$ such that

$$\max \{d(Tx, Fx) : x \in X\} = d(Tz, Fz).$$
If $Tz = Fz$, the proof is complete. Hence we suppose that $Tz \neq Fz$ then we have a sequence $\{y_n\}$ in $X$ such that

$$d(Ty_n, Gy_n) \to \sup\{d(Tx, Gx) : x \in X\} \text{ whenever } n \to \infty.$$  Set $Tx_n = Gy_n$ for each $n > 0$. If $x_n \neq y_n$ for all $n > 0$. From (III)"

$$d(Fz, Tz) \geq d(Fx_n, Tx_n) = d(Fx_n, Gy_n)$$

$$> \min \{d(Tx_n, Fx_n), d(Ty_n, Gy_n), d(Tx_n, Ty_n)\}$$

$$= \min \{d(Tx_n, Fx_n), d(Ty_n, Gy_n), d(Gy_n, Ty_n)\}$$

$$= d(Ty_n, Gy_n)$$

On letting $n \to \infty$, we getting thereby

$$d(Fz, Tz) > \sup\{d(Tz, Gz) : x \in X\}.$$  

Now, since $F(X) = T(X)$, then there exists $z'$ in $X$ such that $Fz = Tz'$. Consider

$$\sup\{d(Tx, Gx) : x \in X\} \geq d(Tz', Gz') = d(Fz, Gz')$$

$$> \min \{d(Tz, Fz), d(Tz', Gz') . d(Tz, Tz')\}$$

$$= \min \{d(Tz, Fz), d(Tz', Gz') . d(Tz, Fz)\}$$

$$= d(Tz, Fz).$$

This is a contradiction. Therefore $F$ and $T$ have a coincidence point. Similary if we choose that $G$ and $T$ are continuous, we get a coincidence point of $G$ and $T$. This completes the proof.