CHAPTER - III

FIXED POINT THEOREMS IN CONVEX METRIC SPACES

3.1 Introduction
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3.1 **INTRODUCTION**

Markin [79] and Nadler [84] initiated the study of Fixed point theorems for a multivalued mappings. Since then there have been several extensions known fixed point theorems for multivalued mappings which generally associates each point of a metric space \((X, d)\) to a closed subset of \(X\). However, in some physical situation the mapping involved is not self-mapping of \(X\).

Assad and Kirk [4] gave sufficient conditions for such mappings to have a fixed point by proving a theorem for multivalued contraction mappings in a complete metrically convex metric space. For similar results one can also refer to Assad [5], Itoh [54], Khan [66], Hadzic and Gajic [43], Rhoades ([95], [100]), Ahmad and Khan [2], Ahmad et. al. [1], Assad et.al.[7], Chang [17], Pathak [89] etc.

In Section 3.2 basic definitions are included which find immediate use in the following sections of this Chapter.

In Section 3.3 we have studied fixed point theorems for a pair of non-self mappings with weak commutativity conditions employing Boyd and Wong [10] type contractive condition in convex metric space. Our results generalize and extend the earlier results due to Assad [6].
and Kannan [62]. An example has been provided to support the hypothesis of our results.

In Section 3.4 we are able to extend a result of Rhoades [100] for a pair of single-valued and multivalued mappings in a complete metrically convex metric space. Examples have been provided to support the validity of the hypothesis.

Finally, in Section 3.5 we prove two theorems concerning the approximation of common fixed point for a pair of weakly commuting mappings in Banach space. Our results extend the results of Khan [67], Shimi [110] and many others.

3.2 BASIC DEFINITIONS

Let $(X, d)$ be a metric space. Following Nadler [84] we define

(i) $CB(X) = \{ A : A$ is a non-empty closed and bounded subset of $X \}$

$C(X) = \{ A : A$ is a non-empty compact subset of $X \}$

(ii) For non-empty subsets of $A$ and $B$ of $X$, and $x \in X$

$D(A, B) = \inf \{ d(a, b) : a \in A, b \in B \}$

$D(x, A) = \inf \{ d(x, a) : a \in A \}$

$H(A, B) = \max \\{ \{ \sup D(a, B) : a \in A \}, \{ \sup D(A, b) : b \in B \} \}$

It is well known (Kuratowski [75]) that $CB(X)$ is a metric space with the distance function $H$ called Hausdorff metric and $(CB(X), H)$ is a complete metric space in the event that $(X, d)$ is complete.

Following Assad and Kirk [4], we recall
**Definition 3.2.1** A metric space $(X,d)$ is said to be metrically convex if for any $x, y \in X$ (with $x \neq y$) there exists $z \in X (x \neq y \neq z)$ such that
\[ d(x, z) + d(z, y) = d(x, y). \]

**Definition 3.2.2** Let $K$ be a non-empty closed subset of a metric space $(X,d)$. A mapping $F : K \to CB(X)$ is said to be continuous at $x_o \in K$ if for any $\varepsilon > 0$, there exists $\delta > 0$ such that $H(Fx, Fx_o) < \varepsilon$, whenever $d(x, x_o) < \delta$. If $F$ is continuous at every point of $K$, we say that $F$ is continuous at $K$.

In an attempt to extend the concepts of weak commutativity of Sessa [103] and compatibility of Jungck [56], Hadzic and Gajic [43] and Hadzic [44] introduced the following.

**Definition 3.2.3** Let $K$ be a non-empty subset of a metric space $(X,d)$. $F : K \to CB(X)$ and $T : K \to X$. Then the pair $\{F, T\}$ is said to be weakly commuting if for every $x, y$ in $K$ such that $x \in Fy$ and $Ty \in K$
\[ D(Tx, FTy) \leq D(Ty, Fy) \]

**Definition 3.2.4** Let $K$ be a non-empty subset of a metric space $(X,d)$. $F : K \to CB(X)$ and $T : K \to X$. Then the pair $\{F, T\}$ is said to be compatible if for every sequence $\{x_n\}$ from $K$ and from the relation
\[ \lim_{n \to \infty} D(Fx_n, Tx_n) = 0 \text{ and } Tx_n \in K, \] it follows that
\[ \lim_{n \to \infty} D(Ty_n, FTx_n) = 0, \]
for every sequence $\{y_n\}$ from $K$ such that $y_n \in Fx_n$.

For $K = X$ and $F$ a single-valued, Definitions 3.2.3 and 3.2.4
reduce to those of Sessa [103] and Jungck [56] respectively.

If we assume $F : K \to X$ (Single-valued mapping) then the above definitions can be restated as.

**Definition 3.2.5** Let $K$ be a non-empty subset of a metric space $(X,d)$. $F, T : K \to X$. The pair $\{F, T\}$ is said to be weakly commuting if for each $x, y \in K$, such that $x = Fy$ and $Ty \in K$, we have,

$$d(Tx, FTy) \leq d(Ty, Fy).$$

**Definition 3.2.6** Let $K$ be a non-empty subset of a metric space $(X,d)$ and $F, T : K \to X$. The pair $\{F, T\}$ is said to be compatible if for every sequence $\{x_n\}$ from $K$ and from the relation,

$$\lim_{n \to \infty} d(Tx_n, Fx_n) = 0 \quad \text{and} \quad Tx_n \in K, \; n \in \mathbb{N},$$

it follows that

$$\lim_{n \to \infty} d(Ty_n, FTx_n) = 0,$$

for every sequence $\{y_n\}$ from $K$ such that $y_n = Fx_n, \; n \in \mathbb{N}$.

### 3.3 RESULTS CONCERNING THE COMMON FIXED POINT FOR A PAIR OF MAPPINGS IN BANACH SPACE

In this Section, we have studied fixed point for a pair of mapping with weak conditions of commutativity employing the Boyd and Wong [10] type contractive condition. Our results extend the earlier known results of Assad [6] and many others. An example has also been given.

Let $X$ be a Banach space for all $x, y \in X$, $d(x,y)$ well represent $\| x-y \|$. 

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The following Lemma due to Assad and Kirk [4] finds immediate applications.

**Lemma 3.3.1** Let $(X,d)$ be a metrically convex metric space, and $K$ a non-empty closed subset of $X$. If $x \in K$, and $y \notin K$, then there exists a point $z \in \partial K$ (the boundary of $K$) such that

$$d(x,y) = d(x,z) + d(z,y).$$

It is well known that every Banach space is metrically convex.

**Definition 3.3.2** Let $K$ be a non-empty subset of a metric space $(X, d)$ and $F, T : K \to X$ satisfy the condition

$$\phi[d(Fx, Fy)] \leq b\{\phi[d(Tx, Fx)] + \phi[d(Ty, Fy)]\}$$

$$+ c \min \{\phi[d(Tx, Fy)], \phi[d(Ty, Fx)]\}$$

for all $x, y \in K$, with $x \neq y$, $b, c \geq 0$, $2b + c < 1$ and $\phi : R \to R$ be an increasing, continuous function for which the following property holds:

$$\phi(t) = 0 \text{ if and only if } t = 0$$

We call a function $F$ satisfying condition (3.3.1) as generalized $T$-contractive.

Motivated from Assad [6], we prove the following:

**Theorem 3.3.3** Let $X$ be a Banach space, $K$ a non-empty closed subset of $X$. Let $F, T : K \to X$ be such that $F$ is generalized $T$-contractive satisfying the conditions.

(i) $\partial K \subseteq TK, FK \subseteq TK$. 

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(ii) \(Tx \in \partial K \Rightarrow Fx \in K\).

(iii) \(F\) and \(T\) are weakly commuting.

(iv) \(T\) is continuous at \(K\).

and \(\varphi : \mathbb{R}^+ \to \mathbb{R}^+\) an increasing continuous function satisfying (3.3.2).

Then there exists a unique common fixed point \(z\) in \(K\) such that \(z = Tz = Fz\).

**Proof.** We construct the sequences \(\{x_n\}\) and \(\{y_n\}\) in the following way.

Let \(x \in \partial K\), then there exists a point \(x_n \in K\) such that \(x = T_{x_n}\) as \(\partial K \subseteq TK\). From \(Tx_{o} \in \partial K\) and the implication \(Tx \in \partial K \Rightarrow Fx \in K\), we conclude that \(Fx_{o} \in K \cap FK \subseteq TK\). Let \(x_{j} \in K\) be such that \(y_{j} = Tx_{j} = Fx_{o} \in K\). Let \(y_{j} = Fx_{j}\). Suppose \(y_{j} \in K\), then \(y_{j} \in K \cap FK \subseteq TK\), which implies that there exists a point \(x_{j} \in K\) such that \(y_{j} = Tx_{j}\). Suppose \(y_{j} \in K\), then there exists a point \(p \in \partial K\) such that (Lemma 3.3.1).

\[
d(Tx_{j}, p) + d(p, y_{j}) = d(Tx_{j}, y_{j}).
\]

since \(p \in \partial K \subseteq TK\), there exists a point \(x_{j} \in K\) such that \(p = Tx_{j}\) and so

\[
d(Tx_{j}, Tx_{j}) + d(Tx_{j}, y_{j}) = d(Tx_{j}, y_{j}).
\]

Let \(y_{j} = Fx_{j}\). Thus repeating the foregoing arguments we obtain two sequences \(\{x_n\}\) and \(\{y_n\}\) such that

(i) \(y_{n-1} = Fx_{n}\).

(ii) \(y_{n} \in K \Rightarrow y_{n} = Tx_{n}\), or

(iii) \(y_{n} \notin K \Rightarrow Tx_{n} \in \partial K\) and

\[
d(Tx_{m}, Tx_{n}) + d(Tx_{m}, y_{n}) = d(Tx_{m}, y_{n}).
\]
We denote
\[ P = \{ T_n \in \{ T_n \} : T_n = y_n \}, \]
\[ Q = \{ T_n \in \{ T_n \} : T_n \neq y_n \}. \]
Obviously the two consecutive terms of \{ T_n \} can not lie in \( Q \). Let us denote \( t_n = d(T_n, T_{n-1}) \). We have the following three cases:

**Case 1.** If \( T_{n-1} \in P \), then
\[
\phi(t_n) = \phi[d(T_n, T_{n-1})] = \phi[d(F_n, F_{n-1})] \leq b\{\phi[d(T_{n-1}, F_{n-1})] + \phi[d(T_n, F_{n-1})]\] + c \min \{ \phi[d(T_{n-1}, F_{n-1})], \phi[d(T_n, F_{n-1})]\} = b[\phi(t_{n-1}) + \phi(t_n)].
\]
and thus, \( \phi(t_n) \leq \frac{b}{1-b} \phi(t_{n-1}) \)

**Case 2.** If \( T_n \in P \), \( T_{n-1} \in Q \).

Note that
\[
d(T_n, T_{n-1}) + (T_{n-1}, y_{n-1}) = d(T_n, y_{n-1})
or \quad d(T_n, T_{n-1}) \leq d(T_{n-1}, y_{n-1}) = d(y_{n-1}, y_{n-1}).
\]

hence
\[
\phi(t_n) = \phi[d(T_n, T_{n-1})] \leq \phi[d(y_{n-1}, y_{n-1})] = \phi[d(F_{n-1}, F_{n-1})] \leq b\{\phi[d(T_{n-1}, F_{n-1})] + \phi[d(T_n, F_{n-1})]\] + c \min \{ \phi[d(T_{n-1}, F_{n-1})], \phi[d(T_n, F_{n-1})]\} = b[\phi(t_{n-1}) + \phi[d(y_{n-1}, y_{n-1})]]
\]
Therefore, \( \phi[d(y_{n-1}, y_{n-1})] \leq \frac{b}{1-b} \phi(t_{n-1}) \).
and hence $\varphi(t_n) \leq \varphi[\varphi(d(y_n, y_{n-1})] \leq \left(\frac{b}{1-b}\right) \varphi(t_{n-1})$.

**Case 3** If $T_{n} \in Q$, $T_{n-1} \in P$, and thus $T_{n-1} \in P$. Since $T_{n}$ is a convex linear combination of $T_{n-1}$ and $y_n$ it follows that

\begin{equation}
(*) \quad d(T_{n-1}, T_{n-1}) \leq \max \{d(T_{n-1}, T_{n-1}) , d(y_n, T_{n-1})\}
\end{equation}

If $d(T_{n-1}, T_{n-1}) \leq d(y_n, T_{n-1})$, then

$d(T_{n-1}, T_{n-1}) \leq d(y_n, T_{n-1})$, and hence

\[
\varphi(t_n) = \varphi[d(T_{n-1}, T_{n-1})]
\leq \varphi[d(y_n, T_{n-1})]
= \varphi[d(y_n, y_{n-1})]
= \varphi[d(F_{x_{n-1}}, F_{x_n})]
\leq b \{ \varphi[d(T_{n-1}, F_{x_{n-1}})] + \varphi[d(T_{n-1}, F_{x_n})] \} + c \min \{ \varphi[d(T_{n-1}, F_{x_n})], \varphi[d(T_{n-1}, F_{x_{n-1}})] \}
= b \{ \varphi[d(T_{n-1}, y_n)] + \varphi(t_{n-1}) \} + c \min \{ \varphi[d(T_{n-1}, T_{n-1})], \varphi[d(T_{n}, y_n)] \}.
\]

It follows that $(1-b) \varphi(t_n) \leq b \{ \varphi[d(T_{n-1}, y_n)] + c(\varphi[d(T_{n-1}, y_n)]$)

Since $d(T_{n-1}, y_n) \geq d(T_{n-1}, y_{n-1})$, (as $T_{n} \in Q$)

hence $\varphi[d(T_{n-1}, y_n)] \geq \varphi[d(T_{n-1}, y_{n-1})]$.

And therefore $(1-b) \varphi(t_n) \leq (b+c) \varphi[d(T_{n-1}, y_n)]$

or,

$\varphi(t_n) \leq \left(\frac{b+c}{1-b}\right) \varphi[d(T_{n-1}, y_n)].$

Now, proceeding as in Case 2. (because $T_{n-1} \in P$, $T_{n} \in Q$), we obtain

$\varphi(t_n) \leq \left(\frac{b+c}{1-b}\right) \left(\frac{b}{1-b}\right) \varphi(t_{n-2}).$
From (*) if \( d(y_n, Tx_{n-1}) \leq d(Tx_{n-1}, Tx_{n-2}) \), then

\[
d(Tx_n, Tx_{n-1}) \leq d(Tx_{n-1}, Tx_{n-2}), \text{ and hence}
\]

\[
\phi(t_n) = \varphi[d(Tx_n, T_n)] \leq \varphi[d(Tx_{n-1}, T_{n-1})]
\]

\[
= \varphi[d(Fx_{n-2}, Fx_n)]
\]

\[
\leq b \{ \varphi[d(Tx_{n-2}, Fx_n)] + \varphi[d(Tx_n, Fx_n)] \}
\]

\[+ c \min \{ \varphi[d(Tx_{n-2}, Fx_n)], \varphi[d(Tx_n, Fx_n)] \}
\]

\[
\leq b \{ \varphi(t_{n-2}) + \varphi(t_n) \} + c \varphi(t_{n-1}).
\]

Therefore, noting that by Case 2, \( \phi(t_{n-1}) < \phi(t_{n-2}) \), we conclude

\[
\phi(t_n) \leq \left( \frac{b + c}{l - b} \right) \phi(t_{n-2}).
\]

Thus in all cases

\[
\phi(t_n) \leq \begin{cases} 
\left( \frac{b + c}{l - b} \right) \phi(t_{n-1}), & \text{or} \\
\left( \frac{b + c}{l - b} \right) \phi(t_{n-2}), & \text{for } n = 1, \ \phi(t_1) \leq \left( \frac{b + c}{l - b} \right) \phi(t_0), \\
\left( \frac{b + c}{l - b} \right) \phi(t_1) \leq \left( \frac{b + c}{l - b} \right)^2 \phi(t_0), & \text{for } n = 2.
\end{cases}
\]

it implies by induction that

\[
\phi(t_n) \leq \left( \frac{b + c}{l - b} \right)^n \phi(t_0).
\]

On letting \( n \to \infty \), we have \( \phi(t_n) \to 0 \), and by property \( \varphi(t) = 0 \) if and only if \( t = 0 \), we have

\[
t_n = (Tx_n, Tx_{n-1}) \to 0 \text{ as } n \to \infty.
\]

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so that \( \{T_n\} \) is a Cauchy sequence and hence converges to a point \( z \) in \( K \).

Now, there exists a subsequence \( \{T_{n_k}\} \) of \( \{T_n\} \) such that \( \{T_{n_k}\} \) is contained in \( P \). For convenience, we take \( \{T_{n_k}\} = \{T_n\} \). Since \( T \) is continuous \( \{TT_n\} \) converges to \( Tz \).

We wish to show that \( Tz = Fz \). On using the weak commutativity of \( F \) and \( T \) (Definition 3.2.5), we have \( T_n = F_{n,1} \) and \( T_{n,1} \in K \), so

\[
d(TT_n, FT_{n,1}) \leq d(F_{n,1}, T_{n,1}) = d(T_n, T_{n,1})
\]

On letting \( n \to \infty \), we get \( d(TTz, FT_{n,1}) \to 0 \), which implies that \( \{FT_{n,1}\} \to Tz \) and also \( \phi[d(Tz, FT_{n,1})] \to 0 \) as \( n \to \infty \), since \( \phi \) is continuous.

Now consider

\[
\phi[d(FFT_{n,1}, Fz)] \leq b \{ \phi[d(FF_{n,1}, FFT_{n,1})] + \phi[d(Tz, Fz)] \} + c \min \{ \phi[d(FFT_{n,1}, Fz)], \phi[d(Tz, FFT_{n,1})] \}
\]

which on letting \( n \to \infty \), we get

\[
\phi[d(Tz, Fz)] \leq b \{ \phi[d(Tz, Fz)] \}
\]

a contradiction, giving thereby \( \phi[d(Tz, Fz)] = 0 \), which implies that \( d(Tz, Fz) = 0 \), and thus \( Tz = Fz \).

To show that \( Tz = z \). Consider

\[
\phi[d(T_n, Tz)] = \phi[d(F_{n,1}, Fz)] \\
\leq b \{ \phi[d(T_{n,1}, F_{n,1})] + \phi[d(Tz, Fz)] \} + c \min \{ \phi[d(T_{n,1}, Fz)], \phi[d(Tz, F_{n,1})] \}
\]

\[ \Rightarrow \quad \boxed{56} \]
and letting $n \to \infty$, we get

$$\varphi[d(z, Tz)] \leq c \varphi[d(z, Tz)]$$

which is a contradiction, giving thereby $\varphi[d(z, Tz)] = 0$, which implies that $d(z, Tz) = 0$, and hence $z = Tz$.

Thus we have shown that $z = Tz = Fz$, so $z$ is a common fixed point of $F$ and $T$. To show that $z$ is unique, let $w$ be another fixed point of $F$ and $T$, then

$$\varphi[d(w, z)] = \varphi[d(Fw, Fz)]$$

$$\leq b \{ \varphi[d(Tw, Fw)] + \varphi[d(Tz, Fz)] \}$$

$$+ c \ min \ \{ \varphi[d(Tw, Fz)] , \varphi[d(Tz, Fw)] \}$$

$$= c \ \varphi[d(w, z)]$$

a contradiction giving therefore $\varphi[d(w, z)] = 0$, which implies that $d(w, z) = 0$, thus $w = z$. This completes the proof.

Remark - 1 Our foregoing theorem is more general and extends theorem 3.1 due to Assad [6] for a pair of weakly commuting mappings.

Remark - 2 If we set $c = 0$, $b < 1/2$ and $\varphi(t) = t$, then also the result so obtained extends a theorem of Kannan [62] for a pair of weakly commuting non-self mappings.

Remark - 3 The condition $FK \subseteq TK$ can not be relaxed. Therefore Theorem 3.1 of Assad [6] can not be obtained by putting $T = I$ (the identity mapping). For if we put $T = I$ then $FK \subseteq IK = K$ which contradicts the hypothesis that $F$ is a mapping from $K$ into $X$. 

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Now, we extend Theorem 3.3.3 further by assuming that a pair \{F, T\} is compatible (Definition 3.2.6) and \(T\) is continuous at \(K\). We prove the following.

**Theorem 3.3.4** Let \(X\) be a Banach space, \(K\) a non-empty closed subset of \(X\). Let \(F, T : K \to X\) be such that \(F\) is generalized \(T\)-contractive satisfying the conditions:

(i) \(\partial K \subseteq TK, FK \subseteq TK\),
(ii) \(Tx \in \partial K \Rightarrow Fx \in K\),
(iii) \(F\) and \(T\) are compatible,
(iv) \(T\) is continuous at \(K\)

and \(\varphi : \mathbb{R}^+ \to \mathbb{R}^+\) an increasing continuous function satisfying (3.3.2). Then there exists a unique common fixed point \(z\) in \(K\) such that \(z = Tz = Fz\).

**Proof** Proceeding as in Theorem 3.3.3 we can show that the sequence \(\{Tx_n\}\) converges to a point \(z\) in \(K\). Again we assume that there exists a subsequence \(\{Tx_{n_k}\}\) is contained in \(P\). We again denote for convenience, \(\{Tx_{n_k}\} = \{Tx_n\}\).

Since \(Tx_n = Fx_{n-1}\) and \(Tx_{n-1} \in K\) and \(d(Fx_{n-1}, Tx_{n-1}) = d(Tx_n, Tx_{n-1}) \to 0\) as \(n \to \infty\), it follows that from the compatibility of \(F\) and \(T\) that

\[ \lim_{n \to \infty} d(TTx_n, FTx_{n-1}) = 0 \]

and from the continuity of \(T\) it follows that \(\{FTx_{n-1}\} \to Tz\) as \(n \to \infty\).
Now, arguing in the same manner as in Theorem 3.3.3 we can show that \( z = Tz = Fz \). The uniqueness follows from Theorem 3.3.3. This completes the proof.

**Example 3.3.4** Let \( X \) be the set of reals equipped with the Euclidean metric and \( K = \{-2\} \cup [0,1] \), we define the mappings \( F, T : K \to X \) as follows:

\[
T_x = \begin{cases} 
-2x, & x \in [0,1] \\
1, & x = -2 
\end{cases}
\]

\[
F_x = \begin{cases} 
-x/2, & x \in [0,1) \\
0, & x \in \{-2, 1\}
\end{cases}
\]

It is easy to see that \( \partial K = \{-2, 0, 1\} \subseteq [-2, 0] \cup \{1\} = T(K) \) and \( F(K) = (-1/2, 0) \subseteq T(K) \). Furthermore

\[
T(1) = -2 \in \partial K \Rightarrow F(1) = 0 \in K
\]

\[
T(0) = 0 \in \partial K \Rightarrow F(0) = 0 \in K
\]

\[
T(-2) = 1 \in \partial K \Rightarrow F(-2) = 0 \in K
\]

By routine calculation we can show that \( \{F, T\} \) is \( \varphi \)-contraction with \( \varphi(t) = t^2 \) where \( b = \frac{59}{1520} \), \( c = \frac{125}{152} \) and that the pair \( \{F, T\} \) is weakly commuting.

Here we note that \( 0 = T(0) = F(0) \).

**3.4. FIXED POINT THEOREMS IN METRICALLY CONVEX METRIC SPACE**

In this Section, we study some fixed point theorems for a pair of
multivalued and single-valued mappings satisfying the contractive condition of Rhoades [100] thereby extending a result obtained by him. Two illustrative examples are also furnished to demonstrate the validity of the hypothesis.

Following Rhoades [100] we introduce the following

**Definition 3.4.1** Let $K$ be a non-empty closed subset of a metric space $(X,d)$. Let $F: K \rightarrow CB(X)$ and $T: K \rightarrow X$. then $F$ is said to be generalized $T$-contraction of $K$ into $CB(X)$ if for each $x, y$ in $K$, such that

$$H(Fx, Fy) \leq h \max \left\{ \frac{d(Tx, Ty)}{a}, D(Tx, Fx), D(Ty, Fy), \frac{[D(Tx, Fy) + D(Ty, Fx)]}{a + h} \right\}$$

where $0 < h < (1 + \sqrt{5})/2$, $a \geq 1 + [2h^2/(1 + h)]$.

We need the following Lemma due to Nadler [84]

**Lemma 3.4.2** Let $A, B \in CB(X), x \in A$. Then, for each positive number $\theta$, there exists $y \in B$ such that

$$d(x, y) \leq H(A, B) + \theta$$

If $A, B \in C(X)$ then one can find $b \in B$ such that $d(a, b) \leq H(A, B)$.

Now we prove the following

**Theorem 3.4.3** Let $(X,d)$ be a complete metrically convex metric space. $K$ a non-empty closed subset of $X$. If $F$ is a generalized $T$-contraction of $K$ into $CB(X)$ satisfying $x \neq y, Fx \neq Fy, h(h + 1) < 1$. and (i) $\partial K \subseteq TK, FK \subseteq TK; T \in \partial K \Rightarrow Fx \subseteq K$. 

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(ii) \( \{F, T\} \) is weakly commuting pair,

(iii) \( T \) is continuous at \( K \)

Then there exists a point \( z \) in \( K \) such that \( z = Tz \in Fz \).

**Proof.** If \( \theta = h(1+h) = 0 \), then the theorem holds trivially. Thus without loss of generality we assume \( \theta > 0 \). We construct the sequences \( \{x_n\} \) and \( \{y_n\} \) in the following way.

Let \( x \in \partial K \), then there exists a point \( x_o \in K \) such that \( x = Tx_o \) as \( \partial K \subseteq TK \). From \( Tx_o \in \partial K \) and the implication \( Tx \in \partial K \Rightarrow Fx \subseteq K \), we conclude that \( Fx_o \in K \cap FK \subseteq TK \).

Let \( x_1 \in K \) be such that \( y_1 = Tx_1 \in Fx_o \subseteq K \). Since \( y_1 \in Fx_o \), there exists a point \( y_2 \in Fx_j \) such that by Lemma 3.4.2

\[
d(y_1, y_2) \leq H(Fx_o, Fx_j) + \theta.
\]

Suppose \( y_2 \in K \), then \( y_2 \in K \cap FK \subseteq TK \) which implies that there exists a \( x_2 \in K \) such that \( y_2 = Tx_2 \). Suppose \( y_2 \not\in K \). Then there exists a point \( q \in \partial K \) such that

\[
d(Tx_2, q) + d(q, y_2) = d(Tx_1, y_2).
\]

Since \( K \in \partial K \subseteq TK \), there exists a point \( x_2 \in K \) such that \( q = Tx_2 \) and so.

\[
d(Tx_1, TTx_2) + d(Tx_2, y_2) = d(Tx_1, y_2).
\]

Let \( y_3 \in Fx_2 \) be such that

\[
d(y_3, y_2) \leq H(Fx_o, Fx_j) + \theta.
\]

Thus repeating the foregoing arguments, we obtain two sequences \( \{x_n\} \) and \( \{y_n\} \) such that
(i)' \( y_{n-1} = F_{x_n} \),

(ii)' \( y_n \in K \Rightarrow y_n = T_{x_n} \) or \( y_n \notin K \Rightarrow T_{x_n} \in \partial K \) and 
\[ d(T_{x_{n-1}}, T_{x_n}) + d(T_{x_n}, y_n) = d(T_{x_{n-1}}, y_n). \]

(iii)' \( d(y_n, y_{n+1}) \leq H(F_{x_{n-1}}, F_{x_n}) + \theta^n \).

We denote
\[ P = \{ T_{x_i} \in \{ T_{x_n} \} : T_{x_i} = y_i \}, \]
\[ Q = \{ T_{x_i} \in \{ T_{x_n} \} : T_{x_i} \neq y_i \}. \]

Obviously two consecutive terms of \( \{ T_{x_n} \} \) cannot lie in \( Q \). We consider the following three cases:

**Case - 1** If \( T_{x_n} \in P, T_{x_{n-1}} \in P \) then
\[
\begin{align*}
&d(T_{x_n}, T_{x_{n-1}}) \\
&= d(y_n, y_{n-1}) \\
&\leq H(F_{x_{n-1}}, F_{x_n}) + \theta^n \\
&\leq h \max \left\{ \frac{d(T_{x_{n-1}}, T_{x_n})}{a}, D(T_{x_{n-1}}, F_{x_{n-1}}) + D(T_{x_n}, F_{x_n}) \right\}
\frac{[D(T_{x_{n-1}}, F_{x_n}) + D(T_{x_n}, F_{x_{n-1}})]}{a + h} + \theta^n
\end{align*}
\]
\[
\begin{align*}
&\leq h \max \left\{ \frac{d(T_{x_{n-1}}, T_{x_n})}{a}, d(T_{x_{n-1}}, T_{x_n}) + d(T_{x_n}, T_{x_{n+1}}) \right\}
\frac{[d(T_{x_{n-1}}, T_{x_n}) + d(T_{x_n}, T_{x_n})]}{a + h} + \theta^n
\end{align*}
\]
\[
\begin{align*}
&\leq \max \left\{ h d(T_{x_{n-1}}, T_{x_n}) + \theta^n, \frac{\theta^n}{1-h} \right\}
\frac{h d(T_{x_{n-1}}, T_{x_n}) + (a + h)\theta^n}{a}
\end{align*}
\]

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Case - 2 If $T_{x_n} \in P$, $T_{x_{n-1}} \in Q$, then by (ii)', we obtain

$$d(T_{x_n}, T_{x_{n-1}}) \leq d(T_{x_n}, y_{n-1})$$
$$= d(y_{n-1}, y_{n-1})$$
$$\leq H(F_{x_{n-1}}, F_{x_n}) + \theta^n$$

$$\leq h \max \left\{ \frac{d(T_{x_{n-1}}, T_{x_n})}{a}, \frac{D(T_{x_{n-1}}, F_{x_{n-1}}) + D(T_{x_n}, F_{x_n})}{a + h}, \frac{D(T_{x_n}, T_{x_{n-1}}) + D(T_{x_{n-1}}, F_{x_{n-1}})}{a + h} \right\} + \theta^n$$

$$\leq h \max \left\{ \frac{d(T_{x_{n-1}}, T_{x_n})}{a}, d(T_{x_{n-1}}, T_{x_n}), d(T_{x_n}, y_{n-1}) \right\}$$

$$\leq h \max \left\{ \frac{d(T_{x_{n-1}}, T_{x_n})}{a}, d(T_{x_{n-1}}, T_{x_n}), d(T_{x_n}, y_{n-1}) \right\}$$

$$\leq \max \left\{ h d(T_{x_{n-1}}, T_{x_n}) + \theta^n, \frac{\theta^n}{1-h}, \frac{h d(T_{x_{n-1}}, T_{x_n}) + (a + h) \theta^n}{a} \right\}$$

$$\leq h d(T_{x_{n-1}}, T_{x_n}) + \max \left\{ \frac{1}{1-h}, \frac{a + h}{a} \right\} \theta^n$$

$$\leq h d(T_{x_{n-1}}, T_{x_n}) + \frac{\theta^n}{1-h}$$

Case - 3 If $T_{x_n} \in Q$, $T_{x_{n-1}} \in P$, then $T_{x_{n-1}} = y_{n-1}$. Hence by the convexity of $X$. 

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Suppose that the maximum of the right hand side of (*) is \( d(y, Tx_n) \) then

\[
d(Tx, Tx_n) \leq d(y, Tx_n) = d(y, y_{n-1})
\]

\[
\leq H(Fx_{n-1}, Fx_n) + \theta^n
\]

\[
\leq h \max \left\{ \frac{d(Tx_{n-1}, Tx_n)}{a}, D(Tx_{n-1}, Fx_{n-1}), D(Tx_n, Fx_n), \right. \\
\left. \frac{[D(Tx_{n-1}, Fx_n) + D(Tx_n, Fx_n)]}{a + h} \right\} + \theta^n
\]

\[
\leq h \max \left\{ \frac{d(Tx_{n-1}, Tx_n)}{a}, d(Tx_{n-1}, y_n), d(Tx_n, Tx_{n-1}), \right. \\
\left. \frac{[d(Tx_{n-1}, Tx_n) + d(Tx_n, y_n)]}{a + h} \right\} + \theta^n
\]

Since \( Tx_n \in Q \), then

\[
d(Tx_{n-1}, y_n) > d(Tx_{n-1}, Tx_n)
\]

we note that

\[
d(Tx_{n-1}, Tx_n) + d(Tx_n, y_n) \leq d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n-1}) + d(Tx_n, y_n)
\]

\[
= d(Tx_{n-1}, y_n) + d(Tx_n, Tx_{n-1})
\]

So that

\[
d(Tx_n, Tx_{n-1}) \leq h \max \left\{ d(Tx_{n-1}, y_n), d(Tx_n, Tx_{n-1}), \right. \\
\left. \frac{[d(Tx_{n-1}, y_n) + d(Tx_n, Tx_{n-1})]}{a + h} \right\} + \theta^n
\]
\[
\leq h \max \left\{ h d(T_{x_{n-1}}, y_n) + \theta^n, \frac{h d(T_{x_{n-1}}, y_n) + (a+h)\theta^n}{1-h} \right\}.
\]

\[
\leq h d(T_{x_{n-1}}, y_n) + \frac{\theta^n}{1-h}
\]

\[
\leq h^2 d(T_{x_{n-1}}, T_{x_{n-1}}) + h \frac{\theta^{n-1}}{1-h} + \frac{\theta^n}{1-h},
\]

by Case 2.

If the maximum of the right hand side of (*) is

\[d(T_{x_{n-1}}, T_{x_{n-1}}),\]

then

\[
(\ast\ast) \quad d(T_x, T_{x_{n-1}}) \leq d(T_{x_{n-1}}, T_{x_{n-1}})
\]

\[
\leq d(T_{x_{n-1}}, y_n) + d(y_n, T_{x_{n-1}})
\]

\[
= d(T_{x_{n-1}}, y_n) + d(y_n, y_{n-1})
\]

\[
\leq d(T_{x_{n-1}}, y_n) + H(F_{x_{n-1}}, F_{x_n}) + \theta^n
\]

\[
\leq d(T_{x_{n-1}}, y_n) + h \max \left\{ \frac{d(T_{x_{n-1}}, T_{x_n})}{a}, D(T_{x_{n-1}}, F_{x_{n-1}}) \right\} + \theta^n
\]

\[
D(T_x, F_{x_n}), \quad \frac{D(T_{x_{n-1}}, F_{x_n}) + D(T_{x_n}, F_{x_{n-1}})}{a + h}
\]

\[
\leq d(T_{x_{n-1}}, y_n) + h \max \left\{ \frac{d(T_{x_{n-1}}, T_{x_n})}{a}, d(T_{x_{n-1}}, y_n) \right\}
\]

\[
d(T_x, T_{x_{n-1}}), \quad \frac{[d(T_{x_{n-1}}, T_{x_{n-1}}) + d(T_{x_n}, y_n)]}{a + h}
\]

\[
\leq \max \left\{ (1+h) d(T_{x_{n-1}}, y_n) + \theta^n, \frac{\theta^n}{1-h} \right\}
\]

\[
\frac{[d(T_{x_{n-1}}, T_{x_{n-1}}) + d(T_{x_n}, y_n)]}{a + h} + \theta^n \right\}.
\]
Using (*) if the maximum of quantity in braces is the third term, then as $T_{x_n} \in Q$,
\[
d(T_{x_{n-1}}, T_{x_n}) \leq \frac{h d(T_{x_n}, y_n) + (a+h)\theta^n}{a}
\]
\[
\leq \frac{h d(T_{x_n}, y_n) + (a+h)\theta^n}{a}
\]

Therefore
\[
d(T_{x_{n-1}}, T_{x_n}) \leq \max \left\{(1+h) d(T_{x_{n-1}}, y_n) + \theta^n, \frac{\theta^n}{1-h},\right\}
\]
\[
\leq (1+h) d(T_{x_{n-1}}, y_n) + \frac{\theta^n}{1-h}
\]
\[
\leq h(1+h) d(T_{x_{n-2}}, T_{x_{n-1}}) + \frac{h \theta^{n-1}}{1-h} + \frac{\theta^n}{1-h}
\]
Thus in any case, we have
\[
d(T_{x_n}, T_{x_{n-1}}) \leq \begin{cases} 
  h d(T_{x_{n-1}}, T_{x_n}) + \frac{\theta^n}{1-h} & \text{or} \\
  h(1+h) d(T_{x_{n-2}}, T_{x_{n-1}}) + \frac{h \theta^{n-1}}{1-h} + \frac{\theta^n}{1-h}
\end{cases}
\]

Now, proceeding as in Rhoades [100], it is easy to show that $\{T_{x_n}\}$ is a Cauchy sequence and hence converges to a point $z$ in $K$. Now, there exists a subsequence $\{T_{x_{n_k}}\}$ of $\{T_{x_n}\}$ such that $\{T_{x_{n_k}}\}$ is contained in $P$. For convenience, we take $\{T_{x_{n_k}}\} = \{T_{x_n}\}$. Since $T$ is continuous $\{TT_{x_n}\}$ is converges to $Tz$. Thus using the weak commutativity of $F$ and $T$ (Definition 3.2.3) we have

\[\therefore\]
\( T_x \in F_{x_{n-1}} \cap K \) and \( T_x \in K \), so

\[
D(T_{T_x}, FT_{x_{n-1}}) \leq D(Fx_{n-1}, Tx_{n-1}) \leq d(Tx, Tx_{n-1})
\]

On letting \( n \to \infty \), we get

\[
D(Tz, FT_{x_{n-1}}) \to 0
\]

Now, consider

\[
D(T_{T_x}, Fz) \leq D(T_{T_x}, FT_{x_{n-1}}) + H(FT_{x_{n-1}}, Fz)
\]

\[
\leq D(T_{T_x}, FT_{x_{n-1}}) + h \max \left\{ \frac{D(T_{T_x_{n-1}}, Tz)}{a}, D(T_{T_x_{n-1}}, FT_{x_{n-1}}) \right\}
\]

\[
D(Tz, Fz), \frac{[D(T_{T_x_{n-1}}, Fz) + D(Tz, FT_{x_{n-1}})]}{a + h}
\]

and letting \( n \to \infty \), we get

\[
D(Tz, Fz) \leq h \max \left\{ 0, 0, D(Tz, Fz), \frac{D(Tz, Fz)}{a + h} \right\}
\]

which implies that

\[
D(Tz, Fz) \leq h D(Tz, Fz)
\]

a contradiction, yielding thereby \( Tz \in F_z \), as \( F_z \) is closed.

Again, consider

\[
d(Tx, Tz) \leq H(Fx_{n-1}, Fz)
\]

\[
\leq h \max \left\{ \frac{d(Tx_{n-1}, Tz)}{a}, D(Tx_{n-1}, Fx_{n-1}), D(Tz, Fz), \frac{[D(Tx_{n-1}, Fz) + D(Tz, Fx_{n-1})]}{a + h} \right\}
\]
\[ \leq h \max \left\{ \frac{d(Tx_n, Tz)}{a}, \frac{d(Tx_{n-1}, Tx_n)}{a}, d(Tz, Tz), \frac{[d(Tx_n, Tz) + d(Tz, Tx_n)]}{a + h} \right\} \]

on letting \( n \to \infty \), we get

\[
d(z, Tz) \leq \max \left\{ \frac{hd(z, Tz)}{a}, 0, 0, \frac{2h d(z, Tz)}{a + h} \right\}
\]

\[
\leq \left\{ \frac{2h}{a + h} \right\} d(z, Tz)
\]

giving thereby

\[ z = Tz \in Fz. \text{This completes the proof.} \]

**Theorem 3.4.4** Theorem 3.4.3 holds good if we replace the condition (ii) and (iii) by

(ii)" \( \{F, T\} \) is a compatible pair,

(iii)" \( F \) and \( T \) are continuous at \( K \),

Then there exists a point \( z \) in \( K \) such that \( z = Tz \in Fz \).

**Proof** Proceeding as Theorem 3.4.3 we have that the sequence \( \{Tx_n\} \) is Cauchy and therefore converges to some point \( z \) in \( K \). So, as argued there, there exists a subsequence \( \{Tx_{n_k}\} \) in \( P \) that is \( y_{n_k} = Tx_{n_k} \). Again, for convenience, we denote \( Tx_{n_k} \) as \( Tx_n \).
Now, we use the compatibility of \( \{F, T\} \) (Definition 3.2.4) to show that \( Tz \in Fz \). Since \( Tx \in Fx \cap K \) and \( Tx_{n-1} \in K \), we have

\[
D(Fx_{n-1}, Tx_{n-1}) \leq d(Tx_n, Tx_{n-1}) \to 0, \quad \text{as } n \to \infty.
\]

It follows from the compatibility of \( \{F, T\} \) that

\[
\lim_{n \to \infty} D(TTx_n, FTx_{n-1}) = 0.
\]

From the inequality

\[
D(TTx_n, Fz) \leq D(TTx_n, FTx_{n-1}) + H(FTx_{n-1}, Fz)
\]

and since \( F \) is \( H \)-continuous and \( T \) is continuous, on letting \( n \to \infty \) it follows that \( D(Tz, Fz) = 0 \) giving thereby \( Tz \in Fz \) as \( Fz \) is closed.

Proceeding as Theorem 3.4.3 we have \( z = Tz \in Fz \). This completes the proof.

**Remark - 1** The condition \( FK \subseteq TK \) can not be relaxed. Therefore the result of Rhoades [100] can not be obtained by putting \( T = I \) the identity mapping. For if we put \( T = I \) then \( FK \subseteq IK = K \) which contradicts the hypothesis that \( F \) is a mapping from \( K \) to \( X \). So our theorems are new results.

**Remark - 2** The condition in the hypothesis '\( x \neq y \), \( Fx \neq Fy \)' is necessary. Since the Theorems fail for \( F \) taken as constant mapping which is demonstrated by the following example.
Example 3.4.5  Let \((X,d) = R, K = (-\infty, -1] \cup [1. + \infty), F: K \to CB(X)\) and \(T : K \to X\) two mappings so defined:

\[ F(x) = [-2, -1] \cup [1, 2] \quad \text{and} \quad Tx = -x \quad \forall x \in K \]

It is easy to verify that the mappings satisfy all the conditions of the hypothesis except \(x \neq y, Fx \neq Fy\). We see that \(T(0) = 0 \notin F(0)\) and so \(F\) and \(T\) have no common fixed point.

Now, we furnish an example demonstrating the validity of the hypothesis of our theorems.

Example 3.4.6  Let \(X\) be the set of reals equipped with euclidean metric and \(K = \{-3\} \cup [0, 1]\). We define the mappings \(T : K \to X\) and \(F : K \to CB(X)\) as follows

\[ T(x) = \begin{cases} -3x, & x \in [0, 1] \\ 1, & x = -3 \end{cases} \]

\[ F(x) = \begin{cases} [-x/2, 0], & x \in [0, 1) \\ \{0\}, & x \in \{-3\} \cup \{1\} \end{cases} \]

It easy to see that \(\partial K = \{-3\} \cup \{1\} \subset [-3, 0] \cup \{1\} = TK\) and \(FK = (-1/2, 0] \subset TK\). Furthermore

\[ T(1) = -3 \in \partial K \Rightarrow F(1) = \{0\} \subset K. \]

\[ T(0) = 0 \in \partial K \Rightarrow F(0) = \{0\} \subset K. \]
and \( T(-3) = 1 \in \partial K \Rightarrow F(-3) = \{0\} \subset K \).

Similarly be a routine calculation we can show that \( F \) is a generalized \( T \)-contractive of \( K \) into \( CB(X) \) with \( h = 0.6 \) and \( a = 1.45 \) and also \( \{F, T\} \) is weakly commuting.

Here one may note that \( 0(=T(0)) \in F(0) = \{0\} \)

\[ \text{3.5 FIXED POINT APPROXIMATION OF WEAKLY COMMUTING MAPPINGS IN BANACH SPACE} \]

In this Sections, we prove two theorems concerning the approximation of common fixed points for a pair of weakly commuting mappings in Banach spaces. Our results one indeed extension of those obtained by Khan [67] and Shimi [110] which are in tern generalizations of results due to Kannan [62] and Krasnoselskii [72].

Before presenting our results, we recall that a pair of self-mapping \( \{F, G\} \) of a normed linear space \( X \) is said to be weakly commuting (cf. Sessa [103]) if

\[ \| FGx - GFx \| \leq \| Gx - Fx \| \text{ for all } x \in X. \]

Clearly commuting pair is weakly commuting but the converse need not to be true in general.

Now, we prove the following
Theorem 3.5.1  Let $X$ be a Banach space and $x_o \in X$ be arbitrary. Let $F, G : X \to X$ such that

(i) $F$ and $G$ are weakly commuting

(ii) $F(X) \subseteq G(X)$

(iii) $G$ is continuous and Linear

(iv) for all $x, y \in X$, we have

$$||Fx - Fy|| \leq a||Gx - Gy|| + b(||Gx - Fx|| + ||Gy - Fy||)$$

$$+ c(||Gx - Fy|| + ||Gy - Fx||)$$

where $a, c \geq 0$, $b > 0$, $a + 2b + 2c \leq 1$.

Let $x_o \in X$ be arbitrary. If $\{x_n\}$ is a sequence in $X$ satisfying

$$Gx_{n-1} = \frac{1}{2} [Gx_n + Fx_n], n = 0, 1, 2, \ldots$$

and for which $Gx_n$ converges. Then there exists a unique common fixed point of $F$ and $G$.

Proof. Define a mapping $F_y$ by setting $F_y(x) = \frac{1}{2} (Gx + Fx)$

Then $F_y(x_o) = G(x_{n-1}), n = 0, 1, 2, \ldots$ Also since $F(X) \subseteq G(X)$ it follows that $F_y$ is a self mapping on $X$, $F_y(X) \subseteq G(X)$. the sequence $\{Gx_n\}$ is a sequence of G-iteration of $x_o$ under $F_y$ (c.f Park [87]) and we claim that a pair $\{F_y, G\}$ is weakly commuting. For that as $G$ is linear we define

$$F_yG = \frac{1}{2} [G^2 + FG], GF_y = \frac{1}{2}(G^2 + GF)$$

Now for all $x, y \in X$, and using weak commutativity of $F$ and $G$
we have

\[
\| F_j Gx - GF_j x \| = 1/2 \| Gx + FGx - Gx - GFx \|
\leq 1/2 \| Gx - Fx \|
= 1/2 \| Gx - 2F_j x + Gx \|
= \| Gx - F_j x \|
\]

which means that a pair \{F_j, G\} is weakly commuting.

Now for any \( x, y \in X \), we have

\[
\| F_j x - F_j y \| \leq 1/2 \| Gx - Gy \| + 1/2 \| Fx - Fy \|
\]

Now, since

\[
\|Fx - Fy\| \leq a \| Gx - Gy \| + b[\|Gx - Fx\| + \|Gy - Fy\|]
+ c[\|Gx - Fy\| + \|Gy - Fx\|]
\]

\[
\leq (a + 2c) \| Gx - Gy \| + (b + c) [\|Gx - Fx\| + \|Gy - Fx\|]
\]

Therefore

\[
(*) \quad \| F_j x - F_j y \| \leq 1/2 (1 - a + 2c) \| Gx - Gy \|
+ (b+c) [(\| Gx - Fx \|)/2 + (\| Gx - Fy \|)/2]
\leq \| Gx - G'y \| + 1/2 [\| Gx - F_j x \| + \| G'y - F_j y \|]
\]

If \( \{Gx_n\} \) converges to \( u \), then for \( n = 1, 2, 3, \ldots \). By using (*) and weak commutativity of \( F_j \) and \( G \) we have

\[
\| G(Gx_n) - F_j u \| = \| GF_j (x_n) - F_j u \|
\]
Thus

\[ \frac{1}{2} ||G^2x_{n+1} - F_ju|| \leq \frac{3}{2} ||Gx_{n+1} - Gx_n|| + ||G^2x_n - Gu|| + ||G^2x_n - G^2x_{n-1}|| + ||G^2x_{n-1} - F_ju|| \]

and letting \( n \to \infty \), and by using continuity of \( G \) and the fact that \( Gx_n \), \( Gx_{n-1} \), \( \to u \), we get

\[ ||Gu - F_ju|| = 0 \]

getting therefore, \( Gu = F_ju \) and thus \( Gu = Fu \). Next by weak commutativity of \( F \) and \( G \) we get

\[ ||FGu - GFu|| \leq ||Gu - Fu|| = ||Gu - Gu|| = 0 \]

giving thereby, \( FGu = GFu = G(Gu) = F(Fu) \)

Now, consider
\[\|Fu - F^2 u\| \leq a \| Gu - GFu\| + b (\| Gu - Fu\| + \|GFu - F^2 u\|)\]
\[+ c (\| Gu - F^2 u\| + \| GFu - Fu\|)\]

which yields

\[F(Fu) = F(u)\]

But \(G(Fu) = F(Gu) = F(Fu) = Fu\), and so \(Fu\) is a common fixed point of \(F\) and \(G\). For uniques let \(Fw\) be another fixed point of \(F\) and \(G\) such that \(FFw = GFw = Fw\). Now

\[\| FFu - FFw \| \leq a \| GFu - GFw\| + b (\|GFu - FFu\| + \|GFu - FFw\|)\]

so that,

\[\| Fu - Fw \| \leq (a + 2c) \| Fu - Fw \|\]

a contradiction as \(b > 0\) implies \(a + 2c < 1\). Hence \(FU = Fw\) is a unique fixed point of \(F\) and \(G\).

**Remark - 1** We extend a theorem 3.1 of Khan [67] by involving weak commutativity instead of commutativity.

**Remark - 2** Putting \(G = I\) in Theorem 3.5.1 we get an extension of the result of Shimi [110].

Now we wish to investigate the solvability of certain non-linear functional equations in Banach space.
**Theorem 3.5.2** Let \( \{u_n\} \) be a sequence of elements in a Banach space \( X \), and let \( \{v_n\} \) be a sequence of solutions to the equation \( Gx - Fx = G(u_n), \ n = 1, 2, \ldots \), where \( F \) and \( G \) are as in Theorem 3.5.2 except that \( G \) need not be linear. Then if \( \{Gu_n\} \to 0 \) as \( n \to \infty \) the sequence \( \{Gv_n\} \) converges to the solution of the equation \( Fx = Gx \).

**Proof.** Firstly we observe that

\[
||Fv_n - Fv_m|| \leq a ||Gv_n - Gv_m|| + b \left( ||Gv_n - Fv_n|| + ||Gv_m - Fv_m||\right)
+ c \left( ||Gv_n - Gv_m|| + ||Gv_m - Fv_m||\right)
\leq a \left( ||Gv_n - Fv_n|| + ||Fv_n - Fv_m|| + ||Fv_m - Gv_m||\right)
+ b \left( ||Gu_n|| + ||Gu_m||\right) + c \left( ||Gv_n - Fv_n|| + ||Fv_n - Fv_m||\right)
+ ||Gv_m - Fv_m|| + ||Fv_m - Fv_n||
\]

so that, we have

\[
||Fv_n - Fv_m|| \leq \left( \frac{a + b + c}{1 - a - 2c} \right) (||Gu_n|| + ||Gu_m||)
\]

Now

\[
||Gv_n - Gv_m|| \leq ||Gv_n - Fv_n|| + ||Fv_n - Fv_m|| + ||Fv_m - Gv_m||
\leq ||Gu_n|| + \left( \frac{a + b + c}{1 - a - 2c} \right) (||Gu_n|| + ||Gu_m||) + ||Gu_m||
\]

and letting \( n \to \infty \), we get \( ||Gv_n - Gv_m|| \to 0 \) and hence \( \{Gv_n\} \) is a Cauchy sequence and so it will converge to some point, say \( u \).
Now, consider

\[ ||u-Fv_n|| \leq ||u-Gv_n|| + ||Gv_n-Fv_n|| \]

\[ = ||u-Gv_n|| + ||Gu_n|| \]

which on letting \(n \to \infty\), we get that \(\{Fv_n\}\) also converges to \(u\). Now by using weak commutativity of \(F\) and \(G\) and continuity of \(G\) we get.

\[ ||Gu-Fu|| \leq ||Gu - GFv_n|| + ||GFv_n - FGv_n|| + ||FGv_n - Fu|| \]

\[ \leq ||Gu - GFv_n|| + ||Fv_n - Gv_n|| + a ||GGv_n - Gu|| \]

\[ + b[||GGv_n - FGv_n|| + ||Gu-Fu||] + c[||GGv_n-Fu||] \]

\[ + ||Gu - FGv_n|| \]

\[ \leq ||Gu - GFv_n|| + ||Fv_n - Gv_n|| + a ||GGv_n - Gu|| \]

\[ + b[||GGv_n - GFv_n|| + ||GFv_n - FGv_n|| + ||Gu - Fu||] \]

\[ + c[||GGv_n - Fu|| + ||Gu - GFv_n|| + ||GFv_n - FGv_n||] \]

\[ \leq ||Gu - GFv_n|| + ||Fv_n - Gv_n|| + a ||GGv_n - Gu|| \]

\[ + b[||GGv_n - GFv_n|| + ||Fv_n - Gv_n|| + ||Gu - Fu||] \]

\[ + c[||GGv_n - Fu|| + ||Gu - GFv_n|| + ||Fv_n - Gv_n||] \]

and letting \(n \to \infty\), we get

\[ ||Gu-Fu|| \leq (b+c) ||Gu-Fu|| \]

a contradiction. Hence \(Gu = Fu\). This completes the proof.