CHAPTER IV

SOME STRUCTURE PROPERTIES OF Q-CYCLIC FUZZY GROUP FAMILY AND BCK-ALGEBRA

Introduction: Ray [1993] established subgroups and normal subgroups of a fuzzy group and their criteria's. The fuzzy order of an element of a group also defined and its relationship with the order of the element is examined. The construction of the smallest fuzzy group containing a given arbitrary fuzzy set is systematized. A cyclic fuzzy group that is a restriction of a fuzzy group or that is generated by a fuzzy point is brought to discussion. Finally he explained the usual interactions between a cyclic and abelian fuzzy group. He also pointed out the possibility of an abelian fuzzy group might be isomorphic to a direct product of some of its cyclic subgroups are touched upon.

Roventa [2001] showed that a crisp environment the notions of a normal subgroup and group operating on a set. He studied extensions of these classical notions to the larger universe of fuzzy sets. Finally he proved that a characterization of operations of a fuzzy group on a fuzzy set in terms of homomorphism's of crisp groups.

Kim [2006] considered the intuitionistic Q- fuzzification of the concept of sub algebra's in BCK/BCI algebra and he explained (i) Let A be an intuitionistic Q- fuzzy sub algebra of X. Then $X_A^{(\alpha,\beta)}$ is a sub algebra of X with $\alpha + \beta \leq 1$. (ii) Any sub algebra of X can be realized as both a μ- level sub algebra and a γ-level algebra of some intuitionistic Q-fuzzy sub algebra of X.

(iii) Let f : X →Y be a homomorphism from a BCK/BCI algebra X onto a BCK / BCI algebra. If A is an intuitionistic Q- fuzzy sub algebra of X, then the image f(A) is intuitionistic Q- fuzzy sub algebra of X.

Kim [2006] introduced the new class of algebra's related to BCK algebra's and semi groups called KS-semi group and define an ideal of a KS- semi groups and a strong KS-semi groups. He also defined a congruence relation on KS-semi groups and quotient KS-semi group and proved (i) every p-ideal of a KS-semi group X in an ideal but the converse is not true. (ii) Let X be a strong KS-Semi group with a unity 1 and A any non- empty subset of X. If $g \in A$ and $x \leq y$ imply $x \in A$ then A is an ideal of X. (iii) Let X be a KS-semi group. Then an equivalence relation ρ on X is congruence iff is both left and right compatible. (iv) Let f : X→ Y be a homomorphism of KS-semi groups. Then kerlf is an ideal of X.

Abu osman [1987] explained the closure operator on the set of fuzzy relation on S and to show that (i) The closed hull of a fuzzy relation (s, μ) is given by $\acute{\Gamma}(s, \mu) = (s\sigma, \mu)$. (ii) The composition of two closed fuzzy relations need not be closed fuzzy relations.

Kim [2000] introduced the notion of sensible fuzzy R- subgroups in near rings, and he showed that (i) Let S be a s-norm. Every sensible fuzzy R- subgroup of R with respect to S is an anti fuzzy R- subgroup of R. (ii) An onto homomorphic image of a fuzzy right R- subgroup with respect to S is a fuzzy right R- subgroup. (iii) If μ is a fuzzy right R- subgroup of R with respect to s and θ is an endomorphism's of R, then $\mu[\theta]$ is a fuzzy right R- subgroup of R with respect to s..

There is a quite substantial literature on fuzzy group theory. The study of fuzzy groups was started firstly by Rosenfeld [1971] . He used the min operating to define his fuzzy groups and showed how some basic notions of fuzzy group theory should be extended in an elementary manner to develop the theory of fuzzy groups. It was extended by Anthony and Sherwood [1979]. They used the t- norm operating instead of the min to define the t- fuzzy groups. Roventa and Spircu [2001] introduced the fuzzy group operating on fuzzy sets. Sidkey and Misherf [1991] defined t- cyclic fuzzy groups by using t- level sets in the crisp environment. Ray [1993] defined a cyclic fuzzy group of a given fuzzy group family simply by restriction. In this chapter, we give a sufficient condition for a Q- fuzzy subset to be a Q- cyclic fuzzy group. By using this Q- cyclic fuzzy group, we then define a Q- cyclic fuzzy group family and investigate its structure properties with applications.

4.2 Section II: Preliminaries

4.2.1 Definition: Let A and B be fuzzy sets. Then A is a subset of B if $\mu_A(x) \leq \mu_B(x)$ for every $x \in U$ and it is denoted by $A \subseteq B$ or $B \supseteq A$.

4.2.2 Definition: Two fuzzy sets A and B are called equal if $\mu_A(x) = \mu_B(x)$ for every $x \in U$ and it is denoted by A = B

4.2.3 Definition: Let A and B be fuzzy sets. Then the algebraic product of two fuzzy sets A and B is defined by $A \bullet B = \{ (x, \mu_{\tilde{A}}(x)) / x \in U, \mu_{A \bullet B} = \mu_A \bullet \mu_B \}$

4.2.4 Definition: Let A and B be fuzzy sets. Then the Union $A \cup B$ and Intersection $A \cap B$ are respectively defined by the equations.

$A \cup B = \{(x, \mu_{A \cup B}(x)) / x \in U, \mu_{A \cup B}(x) = \max \{(\mu_A(x), \mu_B(x)\}\}$ and

$A \cap B = \{(x, \mu_{A \cap B}(x)) / x \in U, \mu_{A \cap B}(x) = \min \{(\mu_A(x), \mu_B(x)\}\}$.

4.2.5 Remark: These definitions can be generated for countable number of fuzzy sets. If $\tilde{A}_1, \tilde{A}_2$,...., are fuzzy sets with membership functions $\mu_{\tilde{A}1}(x), \mu_{\tilde{A}2}(x)$,...., then the membership functions of $X = \cup \tilde{A}_i$ and $Y = \cap \tilde{A}_i$ are defined as

$\mu_X(x) = \max\{\mu_{\tilde{A}\,1(x),}\mu_{\tilde{A}\,2(x)\ldots}\}$ and $\mu_Y(x) = \min\{\mu_{\tilde{A}1}(x), \mu_{\tilde{A}2}(x),...\}$ respectively where $x \in U$.

4.2.6 Definition: Let f be a non- fuzzy function from X to Y. The image f(Ã) of a fuzzy set Ã on X is defined by means of the extension principle as

$$f(\tilde{A}) = \{ ((y, q) , \mu_{f(\tilde{A})}(y, q)) \ / \ y = f(x) , x \in X \} ,$$

where $\mu_{f(\tilde{A})}(y, q) = \sup_{(x,q)\varepsilon f^{-1}(y,q)} \mu_{\tilde{A}}(x, q)$ if $f^{-1}(y, q) \neq \Phi$; 0, otherwise.

4.2.7 Definition: Let G = < a > be a cyclic group. If $\tilde{A} = \{ (a^n, \mu (a^n)) / n \in z \}$ is a fuzzy group, then Ã is called a cyclic fuzzy group generated by (a, μ(a)) and denoted by < a, μ(a)>.

4.2.8 Definition: (Q-Cyclic fuzzy group**)** Let A = < a, q > be a Q- cyclic group. If $\tilde{A} = \{ ((a^n, q) , \mu(a^n, q)) \ / n \in z \}$ is a Q- fuzzy group, then Ã is called a Q- cyclic fuzzy group generated by < (a, q) , μ (a, q) >.

Aim: In Crisp environment, the notion of cyclic group on a set is well known. We study an extension of this classical notion to the Q- fuzzy sets to define the concept of Q- cyclic fuzzy groups. By using these Q- cyclic fuzzy groups, we then define a Q- cyclic fuzzy group family and investigate its structure properties with applications.

The following are the Properties of Q- fuzzy groups

4.2.1 Proposition: Let A be a Q- fuzzy group of G. Then

(i) $A(x, q) \leq A(e, q)$ for all $x \in G$ and $q \in Q$.

(ii) The subset $G_A = \{ x \in G / A(x, q) = A(e, q)\}$ is a Q- fuzzy group of G.

Proof: Let x be any element of G. Then $A(x, q) = \min \{A(x, q), A(x, q)\} = \min \{A(x, q), A(x^{-1},q)\} \leq A(xx^{-1},q) = A(x, q)$ and (i) is proved.

To verify (ii), $e \in G_A$, then $G_A \neq \Phi$. Now let $x, y \in G_A$ and $q \in Q$.

$A(xy^{-1}, q) \geq \min \{A(x, q) , A(y^{-1}, q)\} = \min \{A(x, q), A(e, q)\} = \min\{A(e, q). A(e, q)\} = A(e, q)$ but from (i) $A(xy^{-1}, q) \leq A(e, q)$ for $x, y \in G$ and $q \in Q$. Therefore $A(xy^{-1},q) = A(e, q)$ which means $(xy^{-1}, q) \in G_A$ and G_A is Q- fuzzy group of G.

4.2.2 Corollary: Let G be a finite group and A be a Q- fuzzy group of G. Consider the subset H of G given by $H = \{x \in G / A(x, q) = A(e, q)\}$ .Then H is a crisp subgroup of G.

Proof: It is obvious.

4.2.3 Proposition: If A is Q- fuzzy group of G, then the set U(A; t) is also Q-fuzzy group for all $q \in Q$, $t \in Im (A)$.

Proof: Let $t \in Im (A) \subset [0, 1]$ and let $x, y \in U (A; t)$, $q \in Q$ Then $A(x, q) \geq t$, $A(y, q) \geq t$. since A is Q- fuzzy group of G. It follows that $A(xy, q) \geq \min \{ A(x, q) , A(y, q)\} \geq t$. Hence $xy \in U(A; t)$. Let $x \in U(A; t)$ and $q \in Q$. Then $A(x^{-1},q) = A(x, q) \geq t$ which implies $x^{-1} \in U(A ; t)$ .Therefore U (A ; t) is Q – fuzzy group of G.

4.2.4 Proposition: If A is Q- fuzzy set in G such that all non- empty level subset U(A; t) is Q- fuzzy group of G, then A is Q- fuzzy group of G.

Proof: Assume that the non-empty level set U(A; t) is Q- fuzzy group of G. If t_0 = min { A(x, q) , A(y, q)} and for x, y $\in$ G, q $\in$ Q, then x, y $\in$U(A; t_0) so A(xy, q) $\geq t_0$ = min {A(x, q) , A(y, q)} which implies that the condition (QFG1) is valid. For $x^{-1} \in$ G and q $\in$ Q, then $x^{-1} \in$ U(A ; t_0) Thus A(x^{-1},q) = t_0 = A(x, q) which gives that the condition (QFG2) is valid and therefore A is Q- fuzzy group of G.

4.2.5 Proposition: A set of necessary and sufficient conditions for a Q- fuzzy set of a group G to be a Q- fuzzy group of G is that A(xy^{-1}, q) $\geq$ min (A(x, q) , A(y ,q)} for all x, y in G and q in Q.

Proof: Let A be a Q- fuzzy group of G. Then

A(xy^{-1},q) $\geq$ min{ A(x, q) , A(y^{-1},q)}= min { A(x, q) , A(y, q)} for x, y $\in$G and q $\in$ Q.

For the converse part, suppose that A be a Q- fuzzy set of the group G of which e is the identity element. Now A(yy^{-1}, q) $\geq$ min {A(y, q) , A(y, q)}. or A(e, q) $\geq$ A(y, q) .

Now A(ey^{-1},q) $\geq$ min { A(e, q) , A(y, q)}

or A(y^{-1},q) $\geq$ A(y, q) also A(xy, q) $\geq$ min {A(x, q) , A(y^{-1},q)} $\geq$ min{A(x, q) , A(y, q)}.

4.2.6 Proposition: If G is a group, then $\tilde{A}^m$ = { ((a^n, q) , $\mu_{\tilde{A}}(a^n, q)^m$) / n $\in$ z } is also a Q –cyclic fuzzy group.

Proof: Let us show that $\tilde{A}^m$ satisfies three conditions (QFG1-QFG3) in definition 2.2.4. We can consider only its membership function because the m th power of $\tilde{A}$ effects just only the membership function of $\tilde{A}^m$.

(QFG1) Since Ã is a Q- fuzzy group and $\mu_{\tilde{A}}(a, q) \in [0,1]$, we have

$$\mu_{\tilde{A}}((a^{n1}, q), (a^{n2}, q))^m \geq \min \{ \mu_{\tilde{A}}(a^{n1}, q), \mu_{\tilde{A}}(a^{n2}, q)\}^m$$

$$= \min \{ \mu_{\tilde{A}}(a^{n1}, q)^m , \mu(a^{n2}, q)^m\}$$

(QFG2) Now $\mu(a^n, q) = \mu_{\tilde{A}}(a^{-n}, q)$ since Ã is a Q- fuzzy group.

Accordingly, we get $(\mu_{\tilde{A}}(a^n, q)^m = (\mu_{\tilde{A}}(a^{-n,} q))^{m.}$

(QFG3) $\mu_{\tilde{A}}(e, q) = 1$ since Ã is Q- fuzzy group, then $(\mu_{\tilde{A}}(e, q))^m = 1$.

Now we give an example of a Q- cyclic fuzzy group.

4.2.7 Example: Let A be Q- cyclic group with 12 elements and generated by (a, q). Let Ã be a Q- fuzzy set of the group A defined as follows. $\mu_{\tilde{A}}(a^0, q) = 1$, $\mu_{\tilde{A}}(a^4, q) = \mu_{\tilde{A}}(a^8, q) = t_1$, $\mu_{\tilde{A}}(a^2, q) = \mu_{\tilde{A}}(a^6, q) = \mu_{\tilde{A}}(a^{16}, q) = t_2$ and $\mu_{\tilde{A}}(x) = t_3$ for all other elements x in A, where $t_1,t_2,t_3 \in [0, 1]$ with $t_1 > t_2 > t_3$. It is clear that Ã is a Q- fuzzy group of A. Thus $\tilde{A} = \{ (a^k, q) , \mu_{\tilde{A}}(a^k, q) / k \in z \}$ is a Q-cyclic fuzzy group generated by $((a, q) , \mu_{\tilde{A}}(a, q))$.

4.2.9 Definition: Let e be the identity element of the group A. We define the identity Q- fuzzy group E = { {e, q) , $\mu_{\tilde{A}}(e, q) / \mu_{\tilde{A}}(e, q) = 1$}.

4.2.8 Corollary: The Q-fuzzy group $\tilde{A}^n$ is a Q- fuzzy subgroup of $\tilde{A}^m$, if $m \leq n$.

Proof: Clearly $\tilde{A}^n$ and $\tilde{A}^m$ are Q- fuzzy groups by (2.2.4). For all $x \in [0,1]$, $(x, q)^m \geq (x, q)^n$ implies that $\tilde{A}^n \subset \tilde{A}^m$ (since $\mu_{\tilde{A}}{}^n(x ,q) \leq \mu_{\tilde{A}}{}^m(x, q)$ for all $x \in G$ and $q \in Q$).

4.2.9 Proposition: If $\tilde{A}^i$ and $\tilde{A}^j$ are Q- cyclic fuzzy groups, then $\tilde{A}^i \cup \tilde{A}^j$ is also Q- cyclic fuzzy group if $i < j$.

Proof: Since $i < j$, then we have $\mu_{\tilde{A}}^i > \mu_{\tilde{A}}^j$

(QFG1) $\mu_{\tilde{A}i \cup \tilde{A}j}(a^n a^m, q) = \max\{\mu_{\tilde{A}}^i(a^n a^m, q), \mu_{\tilde{A}}^j(a^n a^m, q)\}$

$= \max\{\mu_{\tilde{A}}(a^n a^m, q)^i, (\mu(a^n a^m, q))^j\}$

$= (\mu_{\tilde{A}}(a^n a^m, q))^i$

$> \min\{\mu_{\tilde{A}}^i(a^n, q), \mu_{\tilde{A}}^i(a^m, q)\}$

$> \min\{\max\{\mu_{\tilde{A}}^i(a^n, q), \mu_{\tilde{A}}^i(a^m, q)\}, \max\{\mu_{\tilde{A}}^i(a^n, q), \mu_{\tilde{A}}^j(a^m, q)\}\}$

$= \min\{\max\{\mu_{\tilde{A}}^i(a^n, q), \mu_{\tilde{A}}^j(a^n, q)\}, \max\{\mu_{\tilde{A}}^i(a^m, q), \mu_{\tilde{A}}^i(a^m, q)\}\}$

$\geq \min\{\mu_{\tilde{A}i \cup \tilde{A}j}(a^n, q), \mu_{\tilde{A}i \cup \tilde{A}j}(a^m, q)\}$

QFGI is satisfied.

(QFG2) $\mu_{\tilde{A}i \cup \tilde{A}j}(a^{-n}, q) = \max\{\mu_{\tilde{A}i}(a^{-n}, q), \mu_{\tilde{A}i}(a^{-n}, q)\}$

$= \max\{\mu_{\tilde{A}}(a^{-n}, q)^i, \mu_{\tilde{A}}(a^{-n}, q)^j\}$

$= \max\{\mu_{\tilde{A}}(a^n, q)^i, \mu_{\tilde{A}}(a^n, q)^j\}$

$= \max\{\mu_{\tilde{A}i}(a^n, q), \mu_{\tilde{A}j}(a^n, q)\}$

$= \mu_{\tilde{A}i \cup \tilde{A}j}(a^n, q)$

QFG2 is satisfied.

(QFG3) $\mu_{\tilde{A}i \cup \tilde{A}j}(e, q) = \max\{\mu_{\tilde{A}i}(e, q)\ \mu_{\tilde{A}j}(e, q)\}$

$= \max\{1,1\}$ ($\tilde{A}$ is a Q- fuzzy group.

$= 1$

$\tilde{A}^i \cup \tilde{A}^j$ forms A Q-cyclic fuzzy group.

4.2.10 Proposition: If $\tilde{A}_i$ and $\tilde{A}_j$ are Q- cyclic fuzzy groups, then $\tilde{A}_i \cap \tilde{A}_j$ is also a Q- cyclic fuzzy group.

Proof*:* This theorem may be proved similarly to theorem (4.2.9).

4.2.10 Remark: Since a Q- cyclic fuzzy group is an abelian group, it is clear that $\mu(xy, q) = \mu(yx, q)$ for $x, y \in A$, $q \in Q$. Therefore, the Q- cyclic fuzzy groups $\tilde{A}^m$, $\tilde{A}_i \cup \tilde{A}_j$ and $\tilde{A}_i \cap \tilde{A}_j$ are also normal Q- fuzzy groups.

4.2.11 Definition: Let Ã be a Q- cyclic fuzzy group, then the following set of Q- cyclic fuzzy groups { Ã , $Ã^2$, $Ã^3$, .., .$Ã^m$,…., E } is called the Q- cyclic fuzzy group family generated by Ã. It will be denoted by < Ã >.

4.2.11 Proposition: Let < A > = { A , A^1 , A^2,-------------A^m-------E. Then $\cup$ A^P = A and $\cap$ A^P = E where p varies 1 to ∞.

Proof: The proof is immediate from propositions (4.2.9) and (4.2.10).

4.2.12 Proposition: Let Ã be a Q- cyclic fuzzy group. Then $A \supset A^2 \supset A^3 \ldots\ldots \supset A^m \ldots\ldots \supset E$.

Proof: It is known that $\mu_{Ã}$ (a, q) $\in$ [0, 1]. Hence

$\mu_{Ã}(a, q) .\supset \mu_{Ã}(a, q)^2 \supset \mu_{Ã}(a^2, q) \supset (\mu_{Ã}(a^2, q))^{2,} \ldots \mu_{Ã}(a^n, q) \supset (\mu_{Ã}(a^n, q))^2$..

By using the definition of Q-fuzzy subsets, this gives that $Ã \supset Ã^2$. By generalizing it for any natural numbers i and j with i < j, we then obtain

$(\mu_{Ãi}(a, q))^i \geq (\mu_{Ãj}(a, q))^j$, $(\mu_{Ã}(a^2, q))^i \geq (\mu_{Ã}(a^2, q))^j$ …. $(\mu_{Ãi}(a^n, q) \geq (\mu_{Ãj}(a^n, q))^j$.

So $Ã_i \supset Ã_j$ for any natural numbers i and j with $i \leq j$, which means that

$Ã \supset Ã^2 \supset Ã^3 \supset \ldots \supset Ã^m \ldots$

Finally , for n = 1 to ∞, $E = \cap Ã^n$, which is immediate from proposition 4.2.11 since

$$\lim \mu_{Ã}(a, q)^n = \begin{cases} 1 & \text{if } (a, q) = (e, q) \\ 0 & \text{if } (a, q) \neq (e, q). \end{cases}$$

We then obtain the required relations.

4.2.13 Corollary: Let < Ã> = { Ã , $Ã^2$, $Ã^3$ ………… $Ã^m$ ……. E }. Then

$Ã \supset Ã^2 \supset \ldots \supset Ã^m \ldots \supset E$.

Proof: The proof is similar to that of proposition (4.2.12).

4.2.14 Proposition: Let f be a group homomorphism of a Q- cyclic fuzzy group Ã. Then the image of Ã under f is a Q- cyclic fuzzy group.

Proof: It is well known that in the theory of classical groups, the image of any cyclic group is a cyclic group, and a homomorphic image of a fuzzy subgroup is an fuzzy subgroup [Rosenfeld, Proposition 5.8] . From these results and Definition (4.2.6) , it is clearly seen that the image of Ã under *f* is a Q- cyclic fuzzy group.

4.2.15 Proposition: Let { $\tilde{A}^m$, $\tilde{A}^{m-1}$,…, Ã } be a finite Q- cyclic fuzzy group family. Then $\tilde{A}^m \times \tilde{A}^{m-1} \times \ldots \times \tilde{A} = \tilde{A}^m$.

Proof: Using the definition of the product of Q-fuzzy groups and proposition 4.2.13, it is proved easily.

Conclusion: In this section, we extend Q- cyclic fuzzy groups to Q- fuzzy sets to define the concept of Q- cyclic fuzzy groups. We give a sufficient condition for a Q- fuzzy subset to be a Q- cyclic fuzzy group family and investigate its structure properties with applications. One can obtain similar results by using the definition of t- fuzzy groups instead of Q- fuzzy groups.

4.3 Section III: Some Algebraic properties of BCK—algebra and Fuzzy S-algebra

Introduction: Y. Imai and K. Iseki [1966] introduced two classes of abstract algebras; BCK - algebras and BCI – algebras. It is known that the notion of BCI-algebras is a generalization of BCK –algebras. J. Neggers and H.S Kim [1999] introduced the class of d-algebras which is another generalization of BCK-algebras and investigated relations between d-algebras and BCK-algebras. L.A Zadeh [1965] introduced the notion of fuzzy sets and A.Rosenfeld[1971] introduced the notion of fuzzy group. Following the idea of fuzzy groups, O.G Xi [1991] introduced the notion of fuzzy BCK-algebras. After that, Y.B Jun [1992] studied fuzzy BCK-algebras. Recently, the new class of algebraic structure introduced by Kim [2006], called S-semi group which is the combination of BCK-algebras and semi groups. we fuzzify the new class of algebraic structure introduced by Kim [2006]. We have proved some interesting results which are very closer to the results in BCK – algebras. we have proved some results on S-semi groups.

Aim: In this section, we fuzzify the new class of algebraic structure introduced by Kim [2006]. In this fuzzification (called fuzzy S-semi groups), introduced the notions of fuzzy sub S-semi groups and investigate some of their related properties. The purpose leads to development of new notions over fuzzy S-semi groups. Introduced the notions of fuzzy sub algebra, intuitionistic fuzzy sub algebra in d-algebras and investigate some of their results.

The following are established on properties of BCK-algebra and fuzzy sub algebra

4.3.1 Definition: An algebra (X,*,0) is called BCK-algebra if it satisfies the following conditions

1. ((x*y) * (x*z) * (z*y) = 0; 2. (x*(x*y))*y = 0; 3. x*x = 0;
4. x*y=0, y*x = 0 → x=y; 0*x = 0 for all x,y,z in X

Example-: Let X = { 0,1,2,…} be a set and the operations * be defined as follows:
x*y = 0 if $x \leq y$; otherwise. Then (X;*,0) is an infinite d- algebra, but not BCK – Algebra, since (2*(2*0))*0= (2*1)*0 = 1*0 =1 $\neq$ 0.

4.3.1 Proposition: Intersection of two BCK-algebras is BCK-algebra with respect to * and Δ.

Proof: Let (x,*, 0) and (Y,*, 0^1) be two BCK-algebras. Here 0 ∈ X and 0^1 ∈Y.

(i) Let x, y ,z ∈X ∩ Y
implies x, y, z, ∈ X and x, y, z ∈Y.
implies ((x*y)*(x*z))*(z*y) = 0 and ((x Δ y) Δ (x Δ z)) Δ (z Δ y)) = 0.

(ii) Let x, y ∈ X ∩ Y. So x ∈ X and y ∈ Y
Thus (x* (x * y)) * y = 0 and (x Δ (x Δy) = 0

(iii) Let x ∈ X ∩ Y
implies x ∈X and y∈ Y
Therefore x * x = 0 and x Δ x = 0

(iv) Let x ∈X ∩ Y
so that x ∈X and x ∈ Y
Thus 0*x = 0 and 0 Δ y = 0

(v) Let x, y ∈ X ∩ Y.
Thus x, y ∈ X and x, y ∈ Y. Let x*y = 0 = y*x
since ‘X’ is a BCK – algebra x = y
x Δ y = 0 = y Δ x we have x = y (since Y is BCK-algebra).

4.3.2 Proposition: Union of two BCK-algebras is BCK-algebra with respect to. * if one is contained in other.

Proof: X and Y be two BCK-algebras. Here 0 ∈ X and 0^1∈Y.

(i) Let $x, y, z \in X \cup Y$

Implies $x, y, z \in X$ or $x, y, z \in Y$

Thus $((x*y)*(x*z))*(z*y)) = 0$ or $((x*y)*(x*z)*(z*y)) = 0$ (since X and Y BCK-algebra's).

(ii) Let $x, y \in X \cup Y$. So that $x, y \in X$ or $x, y \in Y$
gives that $(x*(x*y))*y = 0$ or $(x*(x*y))*y = 0$
Therefore $(x*(x*y)*y = 0$.

(iii) Let $x \in X \cup Y$
implies $x \in X$ or $x \in Y$
so that $x*x = 0$ or $x*x = 0$
Therefore $x*x = 0$

(iv) Let $x \in X \cup Y$ and $0 \in X$.
So $x \in X$ or $x \in Y$
implies $0*x=0$ or $0*x = 0$
Therefore $0*x = 0$.

(v) Let $x \in X \cup Y$
implies $x, y \in X$ or $x, y \in Y$ since $x \subset y$ or $y \subset x$
$x*y = 0, y*x = 0$ Therefore $x = y$ where $x, y \in X$ and $x, y \in Y$.

4.3.3 Proposition: A subset of BCK-algebra is BCK-algebra.

Proof: (X*, 0) is a BCK-algebra and $A \subset X$. Claim**:** (A, *, 0) is a BCK-algebra.

Let $x, y, z \in A \subset X$. It follows that $x, y, z \in X$.

$((x*y)*(x*z)*(z*y)) = 0$
$(x*(x*y))*y = 0$
$x*x = x*y = 0, y*x = 0$ implies $x = y$
$0*x = 0$ Therefore (A,*,0) is BCK-algebra.

4.3.4 Corollary: Let: $(X,*,0) \rightarrow (Y,\Delta, 0)$ be a homomorphism on BCK-algebra then kernel is a BCK-algebra with respect to *.

4.3.5 Proposition: Homomorphic image of BCK-algebra is BCK-algebra.

Proof: Let (X, *, 0) be a BCK-algebra and $f : X \rightarrow Y$ be one to one homomorphism. $f(0) = 0^1 f(x*y) = f(x) \Delta f(y)$, for all x, y in X but there is no guarantee for $(Y, \Delta, 0^1)$ as a BCK-algebra.

Let $x, y, z \in f(X)$. There exists an element $a, b, c \in X$ such that $f(a) = x, f(b) = y, f(c) = z$. .

$$\begin{aligned} \text{Now } ((x \Delta y) \Delta (x \Delta z)) \Delta (z \Delta y)) &= (f(a) \Delta f(b) \Delta f(a) \Delta f(c)) \Delta f(c) \Delta f(b) \\ &= f((a*b) * (a*c)) * (c*b) \\ &= f(0) \\ &= 0^1 \quad \text{(since f is 1-1)} \end{aligned}$$

(ii) $(x \Delta (x \Delta y))\Delta y$ = $(f(a) \Delta (f(a) \Delta f(b)) \Delta f(b)$

= $f((a*(a*b)*b)) = f(0) = 0^1$.

(iii) $x \Delta x$ = $f(a) \Delta f(b) = f(a*a) = f(0) = 0^1$

(iv) $x \Delta y = 0^1$ = $f(a) \Delta f(b) = 0^1 = f(a*b) = f(0) = a*b = 0$

$y \Delta x = 0^1$ = $f(b) \Delta f(a) = = 0^1 = f(b*a) = f(0) = b*a = 0$.

X is a BCK-algebra, and so a = b. Therefore f(a) = f(b) implies x = y.

(v) $0^1 \Delta x$ = $f(0) \Delta f(a) = f(0*a) = f(0) = 0^1$. Thus $(f(X), \Delta, 0^1)$ is a BCK-algebra.

4.3.2 Definition: A fuzzy set A in d-algebra X is called a fuzzy sub algebra of X if it satisfies A(x*y) > min {A(x),A(y)} for all x, y in X and B be two fuzzy sets in X. The Cartesian product (A×B) (x, y) = min {A(x), A(y)} for all in x, y in X.

Example:: Let X= {0,1,2} be a set given by the following cayley table:

*	0	1	2
0	0	0	0
1	2	0	2
2	1	1	0

Then (X,*,0) is a d-algebra, but not a BCK-algebra, since $(2*(2*2))*2 = (2*0)*2 = 1*2=2 \neq 0$. We define a fuzzy set $\mu : X \to [0,1]$ by $\mu(0) = 0.7$, $\mu(x) = 0.02$,where for all $x \neq 0$. It is easy to see that μ is a fuzzy sub algebra of X.

Example: Let X= {0,1,2,….}be a set and the operations * be defined as follows: x*y = 0 if $x \leq y$; x-y if y < x.; Then (X;*,0) is an infinite d-algebra. If we define a fuzzy set $\mu ; X \to [0,1]$ by $\mu(0) = t_1$, $\mu(x) = t_2$ for all $x \neq o$, where $t_1 > t_2$. Then μ is a fuzzy sub algebra of X.

4.3.6 Proposition: Union of fuzzy sub algebra is a fuzzy sub algebra.

Proof: Let A and B be two fuzzy sub algebra of X.

$(A\cup B)(x*y) = \max\{A(x*y), B(x*y)\} \geq \max\{\min\{A(x), A(y)\}, \min\{B(x),B(y)\}\}$

$\geq \min\{\max\{A(x), B(x)\}, \max\{A(y), B(y)\}\} \geq \min\{(A\cup B)(x), (A\cup B)(y)\}$.

4.3.7 Proposition: Intersection of two fuzzy sub algebra is fuzzy sub algebra.

Proof: Let A and B be two given fuzzy sub algebra's.

$(A\cap B)(x*y) = \min\{A(x*y), B(x*y)\} \geq \min\{\min\{A(x), A(y)\}, \min\{A(y), B(y)\}\}$

$\geq \min\{(A\cap B)(x), (A\cap B)(y)\}$.

4.3.8 Proposition: α - cut of fuzzy sub algebra is fuzzy sub algebra.

Proof: Let A_α be the α - cut fuzzy sub algebra on 'A'. $A_\alpha = \{x\in X / A_\alpha(x) \geq \alpha\}$

Claim: A_α is a fuzzy sub algebra. $x \in X$, $y \in Y$, we have $x*y \in X$ and $A(x*y) \geq \alpha$

$A_\alpha(x*y) \geq \alpha \geq \min\{\alpha,\alpha\} \geq \min\{A_\alpha(x), A_\alpha(y)\}$.

4.3.9 Proposition: Product of two fuzzy sub algebra is fuzzy sub algebra.

Proof: Let A and B be two fuzzy sub algebra's on X. Define $(x_1,y_1) * (x_2,y_2) = (x_1*x_2, y_1*y_2)$ for all x_1, x_2, y_1, y_2 in X.

$(A \times B)((x_1,y_1) * (x_2,y_2) = (A \times B)(x_1 * x_2, y_1 * y_2) = \min\{A(x_1 * x_2), B(y_1 * y_2)\}$

$\geq \min\{\min\{A(x_1), A(x_2)\}, \min\{B(y_1), B(y_2)\}\} \geq \min\{\min\{A(x_1),B(y_1)\}, \min\{A(x_2), B(y_2)\}\}$

$\geq \min\{(A \times B)(x_1,x_2), (A \times B)(y_1, y_2)\}$.

4.4 Section IV- Intuitionistic Fuzzy sub algebra and S- semi groups

4.4.1 Definition : An intuitionistic fuzzy set $A = < t_A, f_A >$ in X is called an intuitionistic fuzzy sub algebra of X if it satisfies the following conditions: $t_A(x*y) > \min\{t_A(x), t_A(y)\}$;

$f_A(x*y) < \text{mix}\{f_A(x), f_A(y)\}$ for all x,y in X.

Example: Let G = {0,a,b,c,d} be a BCK- algebra with the following cayley table;
Let $A = < \mu_A,\lambda_A>$ be an intuitionistic fuzzy set in G defined by $\mu_A(d) = 0.06$, $\mu_A(x) = 0.7$, $\lambda_A(x) = 0.5$, $\lambda_A(d) = 0.06$ for all $x \neq d$. Then A is an intuitionistic fuzzy sub algebra of G.

*	0	a	b	c	d
0	0	0	0	0	0
A	a	0	a	0	0
B	b	b	0	0	0
C	c	c	a	0	0
D	d	c	d	c	0

4.4.1 Proposition: Union of two intuitionistic fuzzy sub algebra is an intuitionstic fuzzy sub algebra.

Proof: Let A and B be two intuitionistic fuzzy sub algebra's on a set X

(i). $(t_A\cup t_B)(x*y) = \max\{t_A(x*y), t_B(x*y)\} \geq \max\{\min\{t_A(x), t_A(y)\}, \min\{t_B(x), t_B(y)\}\}$

$\geq \min\{t_A\cup t_B)(x), (t_A\cup t_B)(y)\}$

(ii). $(f_A \cup f_B)(x*y) = \max \{ f_A(x*y), f_A(x*y)\}$

$$\leq \max \{ \max \{f_A(x), f_A(y)\}, \max \{f_B(x), f_B(y)\}\}$$

$$\leq \max \{ \max \{ f_A(x), f_B(x)\}, \max \{ f_A(y), f_B(y)\}\}$$

$$\leq \max \{(f_A\cup f_B)(x), (f_A\cup f_B)(y)\}.$$

4.4.2 Proposition: The intersection of two intuitionistic fuzzy sub algebra's is also an intuitionistic fuzzy sub algebra.

Proof: The proof is obvious.

4.4.3 Proposition: α-cut of an intuitionistic fuzzy sub algebra is an intuitionistic fuzzy sub algebra.

Proof : Let A_α be the α -cut of X. $A_\alpha = \{x \in X/ A(x) \geq \alpha\}$.

$x\in X$, $y\in Y$ implies $x*y \in x$, $A(x*y) \geq \alpha$ So $t_A(x) \geq \alpha$, $t_A(y) \geq \alpha$ and $f_A(x) \leq \alpha$, $f_B(y) \leq \alpha$.

(i) $t_{A\alpha}(x*y) \geq \alpha \geq \min \{\alpha, \alpha\} \geq \min \{ t_{A\alpha}(x), t_{A\alpha}(y)\}$

(ii) $f_{A\alpha}(x*y) \geq \alpha = \max \{\alpha, \alpha\} = \max \{ f_{A\alpha}(x), f_{A\alpha}(y)\}$.

4.4.4 Proposition: Product of intuitionistic fuzzy sub algebra is also an intuitionistic fuzzy sub algebra.

Proof: (i) $(t_A \times t_B)((x_1, x_2)*(x_2, y_2)) = (t_A\times t_B)(x_1* x_2, y_1, y_2)$

$$= \min \{ t_A(x_1* x_2), t_B(y_1* y_2)\}$$

$$\geq \min \{\min \{ t_A(x_1), t_A(x_2)\}, \min \{ t_B(y_1), t_B(y_2)\}\}$$

$$\geq \min \{\min \{ t_A(x_1), t_B(y_1)\}, \min \{t_A(x_2), t_B(y_2)\}\}$$

$$\geq \min \{(t_A\times t_B)(x_1, x_2), (t_A\times t_B)(y_1, y_2)\}.$$

$(f_A \times f_B)((x_1, x_2)*(x_2, y_2)) = (f_A\times f_B)((x_1* x_2), (y_1, y_2))$

$$= \min \{ f_A(x_1* x_2), f_B(y_1* y_2)\}$$

$$\leq \min \{ \max \{f_A(x_1), f_A(x_2)\}, \max\{f_B(y_1), f_B(y_2)\}\}$$

$$= \max \{ \min \{f_A(x_1), f_B(y_1)\}, \min \{f_A(x_2), f_B(y_2)\}\}$$

$$\leq \max \{ (f_A \times f_B)(x_1,y_1), (f_A \times f_B)(x_2,y_2)\}.$$

4.4.2 Definition: A S-semi group is a non-empty set X with two binary operations * and a constant 0 satisfying the following axioms (s1) (X,*,0) is a d-algebra. (s2) (X,$\cdot$) is a semi group.

(s3) $x \cdot (y*z) = (x \cdot y) * (x \cdot z)$ and $(x*y) \cdot z = (x \cdot z) * (y \cdot z)$ for all x, y, z in X

Example: Let X= [0,a] is subset of [0,1], a being a fixed number, and the operations * be defined as follows: x*y = min(x, max(x, y) – min(x, y)), for all x, y € X.

Then (X;*,0) is an infinite d- algebra.

4.4.3 Remark: X denotes the S-semi group unless or otherwise specified.

• For the sake of convenience, we shall write the implication x.y by xy.

4.4.4 Definition: Let X be a non-empty set. A fuzzy subset of X is a function $\mu : X \rightarrow [0, 1]$.

Let μ be a fuzzy subset of X. for a fixed $0 \leq t \leq 1$. The set $\mu_t = \{x \in X / \mu(x) \geq t\}$ is called level set of μ.

4.4.5 Definition: A non-empty subset A of X with binary operations * and is called sub S-semi grou*p* of X if it satisfies the following conditions $x*y \in A$ and $xy \in A$ for all $x, y \in X$.

4.4.6 Definition: Let X be a sub S-semi group if it satisfies the following conditions.

(i) $A(x*y) \geq \min \{A(x), A(y)\}$

(ii) $A(xy) \geq \min \{A(x), A(y)\}$ for all x, y in X.

4.4.5 Proposition: A fuzzy set μ of X is a fuzzy sub S-semi group if and only if the upper level set μ_t is either empty or a sub S-semi group of , for every $0 \leq t \leq 1$.

Proof: Let μ is a fuzzy sub S-semi group of X and $\mu_t = 0$. Then for any $x, y \in \mu_t$, we have

$$\mu(x*y) \geq \min \{\mu(x), \mu(y)\}$$

$$\geq \min \{t, t\}$$

$$\geq t .$$ Thus $x*y \in \mu_{t.}$

Also $\mu(xy) \geq \min \{\mu(x),\mu(y)\} = t$ Thus $xy \in \mu_t$.

It gives that μ_t is a sub S-semi group of X.

Conversely, let $t = \min \{\mu(x) , \mu(y)\}$ for all x, y in X.. Since $\mu_t = 0$ is a S-semi group of X, gives $\mu(x*y) \geq t = \min \{\mu(x), \mu(y)\}$

and $\mu(xy) \geq t = \min \{\mu(x) , \mu(y)\}$

μ is a fuzzy sub S-semi group of X.