CHAPTER III

Q-FUZZY SUBGROUPS OF β-FUZZY CONGRUENCE RELATIONS ON A GROUP

Introduction: The concept of fuzzy sets was first introduced by Zadeh in 1965 and since then there has been a tremendous interest in the subject due to diverse applications ranging from engineering and computer science to social behavior studies. The concept of fuzzy relation on a set was defined by Zadeh [1965] and other authors like Rosenfeld [1971] , Tamura et.al. [1971] , and Yeh and Bang [1975] considered it further. The notion of fuzzy congruence on a group was introduced by Kuroki [1992] and that the universal algebra was studied by Filep and Maurer [1989] , and Murali [1991].

The study of fuzzy algebraic structures was started with the introduction of the concept of fuzzy subgroups by Rosenfeld in 1971. Several mathematicians have followed the Rosenfeld approach in investigating the fuzzy subgroup theory. Fuzzy normal subgroups were studied by Wu [1981]. Dib [1988] , Kumar et.al. [1992] and Mukherjee [1984] . The concept of fuzzy quotient group was studied by Kuroki [1992]. In this study, we define some new special fuzzy equivalence relations and derive some simple consequences. Then using those relations we define suitable Q-fuzzy subgroupoids and Q- fuzzy quotient subgroup of G / H differently.

3.2 Section II - Preliminaries

In this section, we shall formulate the preliminary definitions, and results that that are required in more contents of this section are contained in the literature. Let X be a nonempty set and I be the unit interval. A fuzzy binary relation on X is a fuzzy subset A on X×X. By a fuzzy relation, we mean a fuzzy binary relation given by A: $X \times X \rightarrow I$. All fuzzy subsets considered here are assumed to take values in I.

3.2.1 Definition: (i) A fuzzy relation A on X is said to be reflexive if A(x, x) = 1 for all $x \in X$ and said to be symmetric if A (x, y) = A(y, x) for all x, y in X. (ii) If A_1 andA_2 are two relations on X, then their max- product composition denoted by $A_1 \circ A_2$ is defined as $A_1 \circ A_2$ (x, y) = max { A_1(x, z) ,A_2(z, y)}. (iii) If $A_1 = A_2 = A$ and $A \circ A \leq A$, then the fuzzy relation A is called transitive.

3.2.2 Definition: A fuzzy binary relation A in X is called similarity relation if A is reflexive, symmetric and transitive.

Example: Let G = $\{1,w,w^2\}$ be the group with respect to the usual multiplication, where w denotes the cube root of unity. Define $\lambda,\mu : G \rightarrow [0,1]$ by

$\lambda(x) =$ 1 if x =1; 0.6 if x = w; 0.5 if x = w^2

and μ (x) = 0.5 if x =1; 0.4 if x =w; 0.3

if x =w^2. It can be found that for every x έ G, $R_{\mu\cap\lambda}(x,x) = (\mu\cap\lambda)(xx^{-1}) = (\mu\cap\lambda)(1) = 0.5$. Hence $R_{\mu\cap\lambda}$ is not reflexive and not a similarity relation on the group G.

3.2.3 Definition: Let S be a semi group. A fuzzy binary relation A on S is called fuzzy left (right) compatible if and only if $A(x, y) \leq A(tx, ty)$ for all x, y, $t \in S$ ($A(x, y) \leq A(xt, yt)$ for all x, y, $t \in S$.

3.2.4 Definition: A fuzzy binary relation A on a semi group S is called fuzzy compatible if and only if min{ A(a, b), A(c, d)} $\leq$ A(ac, bd) for all a, b, c, $d \in S$.

3.2.5 Definition: Fuzzy compatible similarity relation on a semi group S is called fuzzy congruence.

3.2.6 Definition: A Q- fuzzy subgroup A_H of G is called a Q- fuzzy normal subgroup of G if $A_H(xy, q) = A_H(yx, q)$ for all x, y of G and $q \in Q$.

3.3 Section III: β_q – Fuzzy relation and Fuzzy congruence

In this section, we shall define some special fuzzy relation and give some its results. We need to define a special relation β_q as follows.

3.3.1 Definition: Let G be a group with identity e and A_H be a Q-fuzzy subgroup of G. A fuzzy relation β_q can be defined on G by $\beta_q(a, b) = \min \{ A_H(a, q), A_H(b, q)\}$, if $(a, q) \neq (b, q)$ $A_H(e, q)$ if $(a, q) = (b, q)$. Now we can show some properties of β_q.

In the chapter, β_q- fuzzy congruence by using special fuzzy equivalence relation of Q- fuzzy subgroups which is defined in this study and we define suitable Q-fuzzy subgroupoids and Q- fuzzy quotient subgroup of finite group G / H differently then we investigate some basic properties.

The following propositions are proved

3.3.1 Proposition: Let G be a group with identity e and A_H be a Q- fuzzy subgroup of a group G, Then the relation β_q defined on G is Q-similarity relation on G.

Proof: β_q is reflexive, for each $a \in G$ and $q \in Q$, $\beta_q(a, a) = A_H(e, q) = 1$

β_q is symmetric. $\beta_q(a, b) = \min \{A_H(a, q), A_H(b, q)\}$

$= \min \{ A_H(b, q), A_H(a, q)\}$

$= \beta_q(b. a)$, for $a, b \in G$.

β_q is transitive.

$$\beta_q{}^o\beta_q(a, c) = \max_{b\varepsilon G} \{ \beta_q(a, b), \beta_q(b, c)\}$$

$$= \max_{b\,\varepsilon G} \{ \min\{ A_H(a, q), A_H(b, q)\} ,\min\{ A_H(b, q), A_H(c, q)\}\}$$

$$\leq \max_{b\varepsilon G} \{ \min\{A_H(a, q), A_H(b, q)\} \max_{b\varepsilon G} \{ \min\{ A_H(b, q), A_H(c, q)\}\}$$

$$\leq \max_{b\varepsilon G} \{ A_H(a, q)\} \max_{b\varepsilon G}\{A_H(c, q)\}$$

$$=AH(a,q)AH(c,q) \leq \min \{ A_H(a, q), A_H(c, q)\}$$
$$= \beta_q(a, c), \text{ for all } a,c \in G.$$

Therefore β_q is a Q- similarity relation.

3.3.2 Corollary: $\beta_q(x^{-1}, y^{-1}) = \beta_q(x, y)$ for all $x, y \in G$, $q \in Q$.

Proof: A_H is a Q- fuzzy subgroup of G. It gives that

$$\beta_q(x^{-1}, y^{-1}) = \min\{A_H(x^{-1}, q), A_H(y^{-1}, q)\} = \min\{ A_H(x, q), A_H(y, q)\}= \beta_q(x, y).$$

3.3.3 Proposition: The fuzzy relation β_q defined on G is Q- fuzzy compatible.

Proof: By using the definition of Q- fuzzy compatible and the definition of β_q

$$\beta_q(ac, bd) = \min \{ A_H(ac, q), A_H(bd, q)\}$$

$$\geq \min \{ \min\{A_H(a, q), A_H(c, q)\},\min\{ A_H(b, q), A_H(d, q)\}\}$$

$$= \min \{ \min\{A_H(a, q), A_H(b, q),A_H(c, q), A_H(d, q)\}\}$$

$$= \min\{ \min\{A_H(a, q), A_H(b, q)\}, \min\{A_H(c, q), A_H(d, q)\}\}$$

$= \min \{ \beta_q(a, b), \beta_q(c, d)\}$. This completes the proof.

3.3.4 Proposition: The fuzzy relation β_q defined on G is a Q- fuzzy congruence.

Proof: β_q is Q- fuzzy compatible is proved in Proposition (3.3.3). Therefore β_q is a Q- fuzzy congruence.

3.3.2 Definition: If a Q- fuzzy set is a Q-fuzzy subgroup of G / H, then it is called Q-fuzzy quotient subgroup. Similarly, if it is a Q- fuzzy normal subgroup of G / H, then it is called Q- fuzzy quotient normal subgroup.

By using the Q- fuzzy congruence β_q, we define a special function N as follows.

3.3.3 Definition: Let G be group and A_H be Q-fuzzy normal subgroup of G. $N : G / H \times Q \to [0, 1]$ can be defined by $N(xH, q) = \beta_q(x, h)$ for all $h \in H$ and $q \in Q$.

Now some algebraic properties of N are investigated.

3.3.5 Proposition: The defined fuzzy set N is a Q- fuzzy quotient subgroup of G / H.

Proof: We have to show that the N is a Q- fuzzy subgroup of G / H. A_H is a Q- fuzzy subgroup of G. Using this, for every $xH, yH \in G / H$, we get

$$N(xHyH, q) = \beta_q(xy, h) = \min\{A_H(xy, q), A_H(h, q)\}$$

$$= A_H(xy, q)$$

$$\geq \min\{A_H(x, q), A_H(y, q)\}$$

$$= \min\{\min\{A_H(x, q), A_H(h, q)\}, \min\{A_H(y, q), A_H(h, q)\}\}$$

$$= \min\{\beta_q(x, h), \beta_q(y, h)\}$$

$$= \min\{N(xH, q), N(yH, q)\}.$$

and $$N(x^{-1}H, q) = \beta_q(x^{-1}, h)$$

$$= \min\{A_H(x^{-1}, q), A_H(h, q)\}$$

$$\geq \min\{A_H(x, q), A_H(h, q)\}$$

$= \beta_q(x, h)$

$= N(xH, q)$. Thus N is a Q- fuzzy quotient subgroup of G / H.

3.3.6 Proposition: The defined fuzzy set N is a Q- fuzzy quotient normal subgroup of G / H.

Proof: Here we have to prove that the N is a Q- fuzzy normal subgroup of G / H. Since A_H is Q- fuzzy normal subgroup of G, it gives that

$N(xHyH, q) = \beta_q(xy, h) = \min\{ A_H(xy, q), A_H(h, q)\}$

$\geq \min\{ A_H(yx, q), A_H(h, q)\}$

$= \beta_q(yx, q)$

$= N(yHxH, q)$. Hence N is a Q- fuzzy normal subgroup of G / H.

3.3.7 Proposition: If N is a Q- fuzzy quotient subgroupoid of finite group G / H, then N is a Q-fuzzy subgroup.

Proof: Let $xH \in G / H$. Since G / H is finite, xH has finite order, say r. Then $(xH)^r = x^rH = H$, where H is identity of G/H. Thus $(xH)^{-1} = x^{-1}H = x^{r-1}H$. Now using the definition of a Q- fuzzy subgroupoid repeatedly, it follows that

$N(x^{-1}H, q) = N(x^{r-1}H, q) = \beta_q(x^{r-1}, h)$

$= \min\{ A_H(x^{r-2}x, q), A_H(h, q)\}$

$= A_H(x^{r-2}x, q)$

$\geq \min\{ A_H(x^{r-2},q), A_H(x,q)\}$

$\geq A_H(x,q)$

$= \min\{ A_H(x, q), A_H(h, q)\}$

$= \beta_q(x, h) = N(xH, q)$.

Interchanging xH with $x^{-1}H$, then $N(xH, q) \geq N(x^{-1}H, q)$. Hence N is a Q- fuzzy quotient subgroup.

3.3.8 Proposition: Let N be a Q- fuzzy quotient subgroup of a group G / H and let $xH \in G / H$. Then $N(xHyH, q) = N(yH, q)$, for all $yH \in G / H \leftrightarrow N(xH, q) = N(H, q)$.

Proof: Suppose that $N(xHyH, q) = N(yH, q)$ for all $yH \in G / H$. Then by choosing $yH = H$, we obtain $N(xH, q) = N(H, q)$.

Conversely, suppose that $N(xH, q) = N(H, q)$. Since N is a Q- fuzzy subgroup of G / H and A_H be Q- fuzzy subgroup of G, it implies that

$$
\begin{aligned}
N(xHyH, q) &\geq \min\{ N(xH, q), N(yH, q)\} \\
&= \min\{ N(H, q), N(yH, q)\} \\
&= \min\{ \beta_q(e, h), \beta_q(y, h)\} \\
&= \min\{ \min\{ N_H(h, q)\}, \min\{N_H(y, q), N_H(h, q)\}\} \\
&= \{N_H(h, q), N_H(y, q)\} = \beta_q(y, h) = N(yH, q).
\end{aligned}
$$

Interchanging xHyH with yH, we get $N(yH, q) \geq N(xHyH, q)$.

3.3.9 Proposition: Let N and R be two Q- fuzzy quotient subgroups of G / H. Then $N \cap R$ is a Q- fuzzy quotient normal subgroup of G / H.

Proof: For every $xH, yH \in G / H$ and $q \in Q$, the observation is that

$$
\begin{aligned}
N \cap R(xHyH, q) &= \min\{ N(xHyH, q), R(xHyH, q)\} \\
&\geq \min\{ \min\{N(xH, q), N(yH, q)\}, \min\{R(xH, q), R(yH, q)\}\} \\
&= \min\{ \min\{N(xH, q), R(xH, q)\}, \min\{ N(yH, q), R(yH, q)\}\} \\
&= \min\{ N \cap R(xH, q), N \cap R(yH, q)\}
\end{aligned}
$$

and

$N \cap R\,(x^{-1}, q) = \min\{ N(x^{-1}H, q), R(x^{-1}H, q)\}$

$= \min\{ N(xH, q), R(xH, q)\}$

$= N \cap R\,(xH, q).$

Interchanging now xH with $x^{-1}H$, it makes that $N \cap R\,(xH, q) \leq N \cap R\,(x^{-1}H, q)$.

Hence $N \cap R$ is a Q-fuzzy subgroup of G / H.

$N \cap R(xHyH, q) = \min\{ N(xHyH, q), R(xHyH, q)\}$

$= \min\{ N(yHxH, q), R(yHxH, q)\}$

$\leq N \cap R\,(yHxH, q)$. Hence $N \cap R$ is Q- fuzzy normal subgroup of G / H.

By using the Q-fuzzy normal quotient subgroup N, Q- fuzzy relation μ_N is defined as follows.

3.3.4 Definition: For all (xH, yH) $\in$ G / H × G / H , the Q-fuzzy relation μ_N on G / H is defined by $\mu_N(xH, yH) = N(xHy^{-1}H, q)$ where $q \in Q$.

3.3.10 Proposition: The Q-fuzzy relation μ_N is a Q-fuzzy congruence on G / H.

Proof: Let xH, yH be any element of G / H. Then μ_N is fuzzy reflexive, since $\mu_N(xH, xH) = N(xHx^{-1}H, q) = N(H, q) = 1$. μ_N is a fuzzy symmetric, since

$$\mu_N(xH, yH) = N(xHy^{-1}H, q)$$

$$= N((yx^{-1})^{-1}H, q)$$

$$= N(yx^{-1}H, q)$$

$$= N(yHx^{-1}H, q)$$

$$= \mu_N(yH, xH).$$

Let xH, yH be elements of G / H and N_H be a Q- fuzzy normal subgroup of G.

Then N_H is Q-fuzzy transitive, since

$$\mu_N{}^o\mu_N(xH, yH) = \max \{ \mu_N(xH, zH), \mu_N(zH, yH)\}$$

$$= \max \{N(xHz^{-1}H, q)\ N(zHy^{-1}H, q)\}$$

$$= \max \{ N(xz^{-1}H, q)\ N(zy^{-1}H, q)\}$$

$$= \max \{ \beta_q(xz^{-1}, h)\ \beta_q(zy^{-1}, h)\}$$

$$= \max\{ \min\{ N_H(xz^{-1}, q), N_H(h, q)\} \min\{ N_H(zy^{-1}, q), N_H(h, q)\}\}$$

$$\leq \max \{ \min\{ \min\{N_H(xz^{-1}, q)\ N_H(zy^{-1}, q), N_H(h, q)\}\}\}$$

$$\leq \max \{\min\{ N_H(xy^{-1}, q), N_H(h, q)\}\}$$

$$= \min\{ N_H(xy^{-1}, q), N_H(h, q)\}$$

$$= \beta_q (xy^{-1}, h) = N(xHy^{-1}H, q)$$

$$= \mu_N (xH, yH).$$

Thus μ_N is Q- fuzzy compatible, since

$$\min\{\mu_N (xH. yH), \mu_N (zH, wH)\} = \min\{ N(xHy^{-1}H, q), N(zHw^{-1}H, q)\}$$

$$= \min \{ N(xy^{-1}H, q), N(zw^{-1}H, q)\}$$

$$= \min \{ \beta_q(xy^{-1}, h), \beta_q(zw^{-1}, h)\}$$

$$= \min\{ \min\{N_H(xy^{-1}, q), N_H(h, q)\}, \min\{N_H(zw^{-1}, q), N_H(h, q)\}\}$$

$$= \min\{ N_H(xy^{-1}, q), N_H(zw^{-1}, q)\}$$

$$= \min\{ N_H(y^{-1}x, q), N_H(zw^{-1}, q)\}$$

Since N_H is a Q- fuzzy normal subgroup of G

$$\leq N_H (y^{-1}xzw^{-1}, q) = N_H (xzw^{-1}y^{-1}, q).$$

Since N_H is a Q- fuzzy normal subgroup of G

$$= \min\{ N_H(xz(yw)^{-1}, q), N_H(h, q)\}$$

$$= \beta_q (xz(yw)^{-1}, h)$$

$$= N (xzH(yw)^{-1}H, q)$$

$$= \mu_N (xzH, ywH),$$

So it is Q- fuzzy congruence on G / H. This completes the proof.

Conclusion: W.M.Wu [1981] and A.Rosenfeld [1971] introduced the concept of fuzzy normal subgroups and fuzzy groups. We investigate the concept of special fuzzy relations of Q- fuzzy group and derive some simple consequences.

3.4 Section IV: S-Product of S-anti-Fuzzy right R-subgroup of near rings

Introduction: Fuzzy set theory is a useful tool to describe situations in which the data are imprecise or vague. Fuzzy sets handle such situation by attributing a degree to which a certain object belongs to a set. B. Schweitzer and A.Sklar [1963] introduce the notions of Triangular norm (t-norm) and Triangular co-norm (S-norm) are the most general families of binary operations that satisfy the requirement of the conjunction and disjunction operators respectively. First, Abu. Osman [1987] introduced the notion of fuzzy subgroup with respect to t-norm.

S. Abou. Zaid [1991] also introduced the concept of R-subgroups of a near-rings and Kyunghokim [2007] introduced the concept of fuzzy R- subgroups of a near-ring. Then J. Zhan [2005] introduced the notion of fuzzy hyper ideals in hyper near-rings with respect to t-norm. Recently , Y.U. Cho et,al [2005] introduced the notion of fuzzy sub algebras with respect to S-norm of BCK algebras and M. Akram [2006] introduced the notion of sensible fuzzy ideals.

In this section, we redefine anti-fuzzy right R- subgroups of a near-ring 'R' with respect to a S-norm and investigate it is related properties.

3.4.2 Preliminaries: A semi ring S is a system consisting of a non-empty set S together with two binary operations on S called addition and multiplication such that

(i) S together with addition is a semi group.

(ii) S together with multiplication is a semi group.

(iii) $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$ for all $a, b, c \in S$.

A semi ring S is said to be additively commutative if $a + b = b + a$ for all $a,b \in S$. A zero element of a semi ring S is an element 0 such that $0.\, x = x.\, 0 = 0$ and $0 + x = x + 0 = x$ for all $x \in S$. By a near-ring we mean a non-empty set 'R' with two binary operations + and $\cdot$ satisfies the following axioms

(i) (R.+) is a group.

(ii) $(R, \cdot)$ is a semi group.

(iii) $(b + c)a = ba + ca$ for all $a, b, c \in R$.

Precisely speaking it is a right near-ring because it satisfies the right distribution law.

Note that $x0 = 0$ and $x(-y) = -(xy)$ but in general $0x \neq 0$ for some $x \in R$. A two sided R- subgroup of a near- ring 'R' is a subset N of R such that

(i) (N, +) is a subgroup of (R, +).

(ii) $RN \subset N$

(iii) $NR \subset N$.

If N satisfies (i) and (ii), then it is called a right R subgroup of R.

Some fuzzy logic concepts are given.

A fuzzy set μ in a set R is a function μ: R → [0, 1]. Let Im (μ) denote the image set of μ. Let μ be a fuzzy set in R. For t ∈ [0, 1], the set L(μ; α) = { x ∈ R / μ(x) ≤ α } is called a lower level subset of μ. Let R be a near-ring and let μ be a fuzzy set in R. We say that μ is a fuzzy sub near-ring of R if for all x, y ∈ R, (FS1) μ(x - y) ≥ min {μ(x, μ(y) } and (FS2) μ (xy) ≥ min { μ(x) , μ(y) }. If a fuzzy set μ in a near-ring R satisfies the property (FS1), then $\mu(0) \geq \mu(x)$ for all x∈ R.

3.4.1Definition: By a s-norm S, we mean a function S: [0,1]→ [0,1] satisfying the following conditions ; (S1) S (x , 0) = x; (S2) S (x, y) ≤ S(x, z) if y ≤ z; (S3) S(x, y) = S(y, x)

(S4) S(x, S(y, z)) = S(S(x, y), z), for all x, y, z ∈ [0, 1].

Replacing 0 by 1 in condition S1 we obtain the concept of t- norm T.

3.4.2 Definition: For an S-norm, S(x, y) ≥ max{x, y} holds for all x, y ∈ [0, 1].

3.4.3 Definition: Let S be a s-norm. A fuzzy set μ in R is said to be sensible with respect to S if Im (μ) ⊂ Δs, where Δs = { s(α, α) = α / α ∈ [0, 1] }.

3.4.4 Definition: Let (R, +, ·) be a near-ring. A fuzzy set μ in R is called an anti fuzzy right (resp. left) R-subgroup of R if (AF1) μ (x - y) ≤ max { μ(x) , μ(y) }, for all x, y ∈ R; (AF2) μ (xr) ≤ μ(x) for all r, x ∈ R.

3.4.5 Definition: Let (R,+, ·) be a near-ring. A fuzzy set μ in R is called a fuzzy right (resp. left) R-subgroup of R if (FR1) μ is a fuzzy subgroup of (R, +);

(FR2) μ (xr) ≥ μ (x) (resp. μ (rx) ≥ μ (x)) , for all r, x ∈ R.

Example: Let R= {a, b, c, d} be set with two binary operations as follows:

+	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	b	a
d	d	c	a	b

Then (R,+, .) is a near ring. a fuzzy subset μ: R→ [0,1] is defined by by μ(c) =μ (d) < μ(b) < μ(a). Then μ is a fuzzy right R- subgroup of R.

3.4.6 Definition: Let S be a s- norm. A function μ: R→ [0, 1] is called a fuzzy right (resp. left) R- subgroup of R with respect to S if

(C1) $\mu(x - y) \leq S(\mu(x), \mu(y))$

(C2) $\mu(xr) \leq \mu(x)$ (resp. $\mu(rx) \leq \mu(x)$ for all $r, x \in R$.

If a fuzzy R-subgroup μ of R with respect to S is sensible, we say that μ is a sensible fuzzy R- subgroup of R with respect to S.

3.4.7 Example: Let K be the set natural numbers including 0 and K is a R-subgroup with usual addition and multiplication.

3.4.8 Definition: A fuzzy subset $\mu: R \rightarrow [0,1\}$ by $\mu(x) = 0$ if x is even; = 1, otherwise.

Let S_m: $[0, 1] \rightarrow [0, 1]$ by a function defined by $S_m(\alpha, \beta) = \min\{x + y, 1\}$ for all $x, y \in [0, 1]$. Then S_m is a t-norm. Here μ is sensible R-fuzzy subgroup of R.

In this section, we introduce the notion of S-anti-fuzzy right R-subgroups of near-rings and its basic properties are investigated. We also study the homomorphic image and pre image of S-anti-fuzzy right R-subgroups. Using S-norm, we introduce the notion on sensible anti-fuzzy right R-subgroups in near- rings and some related properties of near-rings 'R' are discussed.

The following are the properties of anti-fuzzy R subgroups.

3.4.1 Proposition: Let S be a s-norm. Then every sensible S-anti fuzzy right R- subgroups μ of R is anti-fuzzy R- subgroups of R.

Proof: Assume that μ is a sensible S- anti fuzzy right R-subgroups of R. Then (AF1) $\mu(x - y) \leq S(\mu(x), \mu(y))$ and (AF2) $\mu(xr) \leq \mu(x)$ for all $x, y \in S$. Since 'μ' is sensible, we have $\max\{\mu(x), \mu(y)\} = S(\min\{\mu(x), \mu(y)\}, \min\{\mu(x), \mu(y)\})$

$$\geq S(\mu(x), \mu(y)) \geq \max\{\mu(x), \mu(y)\}$$

and so $S(\mu(x), \mu(y)) = \max\{\mu(x), \mu(y)\}$.

It follows that $\mu(x-y) \leq S(\mu(x), \mu(y)) = \max\{\mu(x), \mu(y)\}$ for all x, y in R.

Clearly $\mu(xr) \leq \mu(x)$ for all r,x in R. So μ is an anti-fuzzy R- subgroup of R.

3.4.2 Proposition: If μ is S-anti fuzzy right R-subgroups of a near ring R and θ is an endomorphism of R, then $\mu[\theta]$ is S- anti fuzzy right R- subgroups of R.

Proof: For any x, y $\in$ R, we have

(i) $\mu_{[\theta]}(x - y) = \mu(\theta(x-y))$
$= \mu(\theta(x) - \theta(y))$
$\leq S(\mu_{[\theta]}(x), \mu_{[\theta]}(y))$

(ii) $\mu_{[\theta]}(xr) = \mu(\theta(xr))$
$= \mu(\theta(x) r)$
$\leq \mu(\theta(x))$
$\leq \mu_{[\theta]}(x)$. Hence $\mu[\theta]$ is a S-anti fuzzy right R-subgroups of R.

3.4.9 Definition: Let f be a mapping defined on R. If ψ is a fuzzy subset in f(R), then the fuzzy subset $\mu = \psi \circ f$ in R . $\mu(x) = \psi(f(x))$ for all x in R is called the pre image of 'ψ' under f.

3.4.3 Proposition: An onto homomorphic pre image of a S- anti fuzzy right R- subgroups of a near- ring is S-anti fuzzy right R- subgroups.

Proof: Let f: $R \rightarrow R^1$ be an onto homomorphism of near- ring and let ψ be an S- anti fuzzy right R-subgroups of R and μ the pre image of ψ under f. Then it follows that

(i) $\mu(x-y) = \psi(f(x-y))$
$= \psi(f(x)-f(y))$
$\leq S(\psi(f(x), \psi(f(y))$
$= S(\mu(x), \mu(y))$

(ii) $\mu(xr) = \psi(f(xr))$
$= \psi(f(x) r)$
$\leq \psi(f(x))$
$= \mu(x)$.

Hence μ is a S- anti fuzzy right R-subgroups of R.

3.4.4 Proposition: An onto homomorphic image of a anti fuzzy right R- subgroups with inf property is anti-fuzzy right R- subgroups.

Proof: Let $f: R \to R^1$ be an onto homomorphism of near-ring and let μ be an S-anti fuzzy right R-subgroup of R with inf property. Given $x, y \in R$, $x_o \in f^{-1}(x^1)$, and $y_o \in f^{-1}(y^1)$ be such that

$$\mu(xo) = \inf_{h \in f^{-1}(x^1)} \mu(h), \qquad \mu(y_o) = \inf_{h \in f^{-1}(y^1)} \mu(h) \quad \text{respectively.}$$

Then it can deduce that

$$\mu^f(x^1-y^1) = \inf_{z \in f^{-1}(x^1-y^1)} [\mu(z)]$$

$$\leq \max \{ \mu(x_o), \mu(y_o) \}$$

$$= \max \{ \inf_{h \in f^{-1}(x^1)} \mu(h), \inf_{h \in f^{-1}(y^1)} \mu(h) \}$$

$$= \max \{ \mu^f(x^1), \mu^f(y^1) \}$$

$$\mu^f(xr) = \inf_{z \in f^{-1}(x^1 r^1)} \mu(z) \leq \mu(y_0)$$

$$= \inf_{h \in f^{-1}(y^1)} \mu(h) = \mu f(y^1)$$

Hence μ^f is anti fuzzy right R- subgroups of R.

The above proposition can be further strengthened; we first give the following definitions.

3.4.10 Definition: A S-norm S on [0, 1] is called a continuous function from $[0, 1] \times [0, 1] \to [0, 1]$with respect to the usual topology. We observe that the function max is always a continuous S- norm

3.4.5 Proposition: Let f: $R \to R^1$ be a homomorphism of near-rings. If μ be S- anti fuzzy right R-subgroups of R^1, then μ^f is S- anti fuzzy right R- subgroup of R.

Proof: Suppose μ is S- anti fuzzy right R- subgroups of R^1.

Then

(i) For all x, y $\in$ R, it gives that

$$\mu^f(x-y) = \mu f(x-y)$$

$$\leq S(\mu f(x), \mu f(y))$$

$$\leq S(\mu^f(x), \mu^f(y))$$

(ii) For all x,y $\in$ R, it makes

$$\mu^f(xr) = \mu f(xr)$$

$$= \mu(f(x), r)$$

$$\leq \mu(f(x))$$

$$\leq \mu^f(x)$$

Hence μ^f is a S- anti fuzzy right R- subgroup of R.

3.4.6 Proposition: Let f: $R \to R^1$ be a homomorphism of near-rings. If μ^f is a S- anti fuzzy right R- subgroups of R, then μ is S- anti fuzzy right R- subgroup R^1.

Proof: Let $x^1, y^1 \in R^1$, there exists x, y$\in$ R, such that $f(x) = x^1$ and $f(y) = y^1$,

It follows that $\mu(x^1 - y^1)$ $= \mu (f(x) - f(y))$

$$= \mu (f(x - y))$$

$$= \mu^f (x - y)$$

$$\leq S(\mu^f(x), \mu^f(y))$$

$$= S(\mu(f(x), \mu f(y))$$

$$= S(\mu(x^1), \mu(y^1))$$

(iii) Let $x^1, y^1 \in R$, there exists $x,r \varepsilon R$ such that $f(x) = x^1$, $f(y) = y^1$, We have

$$\mu(x^1r^1) = \mu(f(x), f(y))$$

$$= \mu(f(xr))$$

$$\leq \mu^f(x) \leq \mu(f(x) \leq \mu(x^1).$$

3.4.7 Proposition: Let S be a continuous S- norm and let f be a homomorphism on a near-ring R. If μ is S-anti fuzzy right R-subgroups of R, then μ^f is S-anti fuzzy right R-subgroups of f(R).

Proof: Let $A_1 = f^{-1}(y_1)$, $A_2 = f^{-1}(y_2)$ and $A_{12} = f^{-1}(y_1-y_2)$, where $y_1-y_2 \in f(R)$. Consider the set $A_1 - A_2 = \{x \in R / x = a_1-a_2$ for some $a_1 \in A_1 , a_2 \in A_2\}$. If $x \in A_1 - A_2$, then $x = x_1 - x_2$ for some $x_1 \in A_1$ and $x_2 \in A_2$ so that $f(x) = f(x_1 - x_2) = f(x_1) - f(x_2) = y_1 - y_2$.

So $x \in f^{-1}(y_1-y_2) = A_{12}$. Thus $A_1 - A_2 \subset A_{12}$.

It follows that

$$\mu^f(y_1-y_2) = \inf \{ \mu(x) / x \in f^{-1}(x_1-x_2) \}$$

$$= \inf \{ \mu(x) / x \in A_{12}\}$$

$$\leq \inf \{ \mu(x) / x \in A_1-A_2\}$$

$$\leq \inf \{ \mu(x_1-x_2) / x_1 \in A_1 , x_2 \in A_2\}$$

$$\leq \inf \{ S(\mu(x_1) , \mu(x_2)) / x_1 \in A_1, x_2 \in A_2\}$$

Since S is continuous for every $\varepsilon > 0$, we see that $\inf \{ \mu(x_1)/ x_1 \in A_1\}-x_1^* \leq \delta$ and

$\inf \{ \mu(x_2) / x_2 \in A_2\}- x_2^* \leq \delta$.

Then $S(\inf \{\mu(x_1) / x_1 \in A_1\}, \inf \{ \mu(x_2) / x_2 \in A_2\}) -S(x_1^*,x_2^*) \leq \varepsilon$.

Choose $a_1\in A_1$,and $a_2 \in A_2$, such that

$\inf \{\mu(x_1) / x_1 \in A_1 \} – \mu(a_1) \leq \delta$ and

$\inf \{\mu(x_2) / x_2 \in A_2\}- \mu(a_2) \leq \delta$. Then

$S\{\inf \{\mu(x_1) / x_1 \in A_1\} , \inf \{ \mu(x_2) / x_2 \in A_2\}- S\{\mu(a_1), \mu(a_2) \} \leq \varepsilon$.

Thus we have

(i) $\mu^f(y_1-y_2) \leq \inf \{ S(\mu(x_1), \mu(x_2)) / x_1 \in A_1, x\in A_2\}$

$$= S(\inf \{ \mu(x_1) / x_1 \in A_1\} , \inf \{ \mu(x_2) / x_2 \in A_2\})$$

$$= S\{\mu^f(y_1), \mu^f(y_2)\}.$$

(ii) Similarly, we can prove that $\mu^f (xr) \leq \mu^f (x)$.

Hence μ^f is a S- anti fuzzy right R- subgroups of f(R).

3.4.8 Lemma: Let T be a t- norm. Then t- co norm S is S (x, y) = 1 - T (1 - x, 1 - y).

Proof: straight forward.

3.4.9 Proposition: A fuzzy subset μ of R is a T- anti fuzzy right R- subgroups if and only if μ^c is a S anti fuzzy right R- subgroup of R.

Proof: Let 'μ' be a 'T' –anti fuzzy right R- subgroups of R. for all x, y ∈R. We have

(i) $\mu^c(x-y) = 1- \mu(x-y) \leq 1- T(\mu(x), \mu(y)) = 1- T(1- \mu^c(x), 1- \mu^c(y))$

$= S(\mu^c(x), \mu^c(y))$

(ii) $\mu^c(xr) = 1- \mu(xr) \leq 1- \mu(x) = \mu^c(x)$

μ^c is a anti fuzzy right R- subgroups of R.

3.4.11 Definition: A fuzzy relation on any set X is a fuzzy set $\mu: X \times X \rightarrow [0, 1]$.

3.4.12 Definition: Let S be a s- norm. If 'μ' is a fuzzy relation on a set R and χ be fuzzy set in R, Then μ is a S- fuzzy relation on χ if $\mu_\chi(x, y) \geq S(\chi(x), \chi(y))$ for all x, y ∈ R.

3.4.13 Definition: Let S be a s- norm. let μ and χ be a fuzzy subset of R. Then direct S- product of μ and χ is defined as $(\mu\times \chi)(x, y) = S(\mu(x), \chi(y))$, for all x, y ∈R.

3.4.10 Lemma: Let S be a s- norm. let μ and χ be a fuzzy set of R. Then

(i) $\mu \times \chi$ is a S-fuzzy relation on S.
(ii) $L(\mu \times \chi; t) = L(\mu; t) \times L(\chi; t)$ for all t ∈ [0,1].

Proof: It is obvious.

3.4.14 Definition: Let S be a s- norm. let μ be a fuzzy subset of R. Then μ is called strongest S- fuzzy relation on R if $\mu_\chi(x, y) \geq S(\chi(x), \chi(y))$ for all x, y ∈R.

3.4.11 Proposition: Let S be a s- norm. let μ and χ be a S- anti fuzzy right R- subgroup of R. Then $\mu \times \chi$ is a anti fuzzy right R- subgroup of R.

Proof: (i) $(\mu \times \chi)(x-y) = (\mu \times \chi)((x_1,x_2) - (y_1,y_2))$

$= (\mu \times \chi) ((x_1-y_1) , (x_2-y_2))$

$= S(\mu(x_1-y_1) , \chi(x_2-y_2))$

$\leq S (S(\mu(x_1), \mu(y_1)) , S(\chi(x_2) , \chi(y_2)))$

$= S (S(\mu(x_1) , \chi(x_2)) , S(\mu(y_1) , \chi(y_2))$

$= S ((\mu \times \chi)(x_1,x_2) , (\mu \times \chi)(y_1,y_2))$

$= S ((\mu \times \chi)(x) , (\mu \times \chi)(y))$

(iii) $(\mu \times \chi) (xr) = (\mu \times \chi) ((x_1,x_2)(r_1,r_2))$

$= (\mu \times \chi) (x_1r_1, x_2r_2)$

$= S(\mu(x_1) , \chi(x_2))$

$= (\mu \times \chi)(x_1,x_2)$

$= (\mu \times \chi) (x)$.

3.4.12 Proposition: Let μ and χ be sensible S- anti fuzzy right R- subgroups of a near- ring R. Then $\mu \times \chi$ is a sensible S- anti fuzzy right R- subgroup of R×R.

Proof: By proposition 3.4.11, we have $\mu \times \chi$ is S- anti fuzzy right R- subgroup of R × R.

Let $x = (x_1, x_2)$ be any element of S×S. Then

$S((\mu \times \chi)(x) , (\mu \times \chi)(x)) = S((\mu \times \chi)(x_1,x_2) , (\mu \times \chi)(x_1,x_2))$

$= S((S(\mu(x_1) , \chi(x_2)) , S(\mu(x_1) , \chi(x_2))$

$= S(S(\mu(x_1), \mu(x_1)) , S(\chi(x_2) , \chi(x_2)))$

$= S(\mu(x_1) , \chi(x_2))$

$= (\mu \times \chi) (x_1,x_2) = (\mu \times \chi)(x)$.

3.4.13 Remark: If $\mu \times \chi$ is a sensible S- anti fuzzy right R- subgroup of R×R, then $\mu \times \chi$ need not be sensible S- anti fuzzy right R- subgroup of R.