CHAPTER II

A NEW STRUCTURE AND CONSTRUCTION OF Q-FUZZY GROUPS

Introduction: In Q-fuzzy left R- subgroups of near rings with respect to T- norms Y.U.Cho, Y.B.Jun [2005], they showed that if A is an intuitionistic fuzzy right R-subgroup of a near ring R then the set $R_A = \{ x \in R \,/\, \mu_A(x) = \mu_A(0),\ \gamma_A(x) = \gamma_A(0)\}$ is a right R- subgroup of R. If A = < R; μ_A, γ_A > be an intuitionistic fuzzy right R-subgroup of a near ring R, then $R_A(\alpha, \beta)$ is a right R-subgroup of R for every $(\alpha,\beta) \in Im(\mu_A)$ and $Im(\gamma_A)$ with $(\alpha + \beta) \leq 1$. If also A = < R; μ_A, γ_A > be an intuitionistic fuzzy sets in a near ring R such that $R_A(\alpha,\beta)$ is a right R- subgroup of R for every $(\alpha,\beta) \in Im(\mu_A) \times Im(\gamma_A)$ with $\alpha + \beta \leq 1$, then A is an intuitionistic fuzzy right R- subgroup of R.

Consider a chain of right R- subgroups $G_0 \subset G_1 \subset G_2 \subset, \ldots, \subset G_n = R$ (a near ring), where $\subset$ denotes proper inclusion. Then there exists an intuitionistic fuzzy right R-subgroup of R whose upper and lower level right R-subgroups are exactly the right R- subgroups in the above chain. Further A be an intuitionistic fuzzy right R- subgroups of a near ring R, and let μ_{A+} & γ_{A+} be fuzzy sets in R defined by $\mu_{A+}(x) = \mu_A(x) + 1 - \mu_A(0)$ that is a normal intuitionistic fuzzy right R- subgroup of R containing A.

Hence they concluded that the notion of normal intuitionistic fuzzy R- subgroups in near rings was introduced, and related properties were investigated. Characterization of intuitionistic fuzzy R- subgroup was given. Using a collection of right R- subgroups, an intuitionistic fuzzy right R- subgroup was established. Using a chain of right R- subgroups, an intuitionistic fuzzy right R- subgroup was also found.

Section I: Previous results

K.H Kim and Y.B.Jun [2001] established the notion of a normal fuzz R-subgroup in a near ring and related properties discussed.

K.H Kim and Y.B.Jun [2000] discussed about S-norm on [0, 1]. If μ is sensible fuzzy R- subgroup of R with respect to S, then $\mu(0) \leq \mu(x)$ for all $x \in R$, and every sensible fuzzy R- subgroup of R with respect to S is an anti fuzzy R- subgroup of R. If an onto homomorphic image of fuzzy right (resp left) R- subgroup with respect to S is a fuzzy right (resp left) R- subgroup with respect to S, and μ is a fuzzy right (resp left) R- subgroup of R with respect to S, and θ is an endomorphism of R, then $\mu[\theta]$ is a fuzzy right (resp left) R- subgroup of R with respect to S.

Osmankazanci, sultan yamark and serife yilmaz, [2007] identified that if $\{A_i\}_{i\varepsilon A}$ is a family of intuitionistic Q- fuzzy R- subgroups of R, then $\cap A_i$ is an intuitionistic Q- fuzzy R- subgroup of R. If $A= (\mu_A, \gamma_A)$ is an intuitionistic Q- fuzzy R- subgroup of R then (1) so is $^oA = (\mu_A, \mu_A^c)$ (2) A is intuitionistic Q- fuzzy R- subgroup of R if and only if ^oA and $\bullet A$ are intuitionistic Q- fuzzy R- subgroup of R. (3) $U(\mu_A; t)$ and $L(\lambda_A; t)$ are R- subgroups of R for all $q \varepsilon Q$. (4) Finally if $\theta : R \to S$ be an epimorphism, $B = (\mu_B, \lambda_B)$ is an intuitionistic Q- fuzzy set in S, and $\theta^{-1}(B)= (\mu_{\theta-1}(B) , \lambda_{\theta-1}(B))$ is an intuitionistic Q-fuzzy R-subgroup of R, then B is an intuitionistic Q- fuzzy R- subgroup of S. Hence they concluded that the notion of Q- fuzzification of R- subgroups in a near rings and characterizations of intuitionistic Q- fuzzy subgroups are given.

The notion of Q-fuzzification of left R- subgroups is introduced in a near-ring and investigated some related properties. Characterization of Q-fuzzy left R-subgroups with respect to a t-norm are given.

B.Davvaz, W.A.Dudek and Y.B.Yun [2005] established the notion of intuitionistic fuzzy sets introduced by Atanassov as a generalization of the notion of fuzzy sets in the work titled A new Structure and Construction of Q- fuzzy groups. They consider the intuitionistic fuzzification of the concept of sub hyper quasi groups in hyper-quasi groups as discussed on some properties of such sub hyper quasi groups. Also they concluded that some natural equivalent relations on the set of all intuitionistic fuzzy sub hyper- quasi groups of hyper quasi group were investigated.

B.Davvaz, W.A.Dudek and Y.B.Yun [2006] explained the concept of an intuitionistic fuzzy set to Hv- modules. They also introduced the notion of intuitionistic fuzzy Hv- sub modules of an Hv-submodule and discussed with some properties.

F.H.Rho, K.H.Kim, J.G.Lu [2006] showed that if A in X is an intuitionistic fuzzy Q- sub algebra of X, then $\mu_A(0, q) \geq \mu_A(x, q)$ and $\lambda_A(0, q) \leq \lambda_A(x, q)$. For an intuitionistic Q-fuzzy set B in X, then B is an intuitionistic Q- fuzzy sub algebra of X. If A is an intuitionistic Q- fuzzy sub algebra of X, then $X_A(\alpha, \beta)$ is a sub algebra of X with $(\alpha + \beta) \leq 1$. They discussed that any sub algebra of X can be realized as both μ-level sub algebra and γ- level sub algebra of some intuitionstic Q- fuzzy sub algebra of X. Clearly they showed that let f: X → Y be a homomorphism of BCK / BCI algebras. If B is an intuitionstic Q- fuzzy sub algebra of Y, then preimage $f^{-1}(B)$ of B under f is intuitionstic Q- fuzzy sub algebra of X. Hence they concluded that the intuitionistic Q- fuzzification of the concept of sub algebra in BCK/BCI algebra.

L.A.Zadeh [1965] discussed on a fuzzy set that is a class of objects with a continuum of grades of membership. Such a set is characterized by membership function which assigns to each object a grade of membership ranging between zero and one.

The notions of inclusion, union, intersection, complement, relation, convexity etc, are extended to such sets and various notions in the context of fuzzy sets. He concluded that a separation theorem for convex fuzzy sets is proved without requiring that the fuzzy sets be disjoint.

We fuzzify the new class of algebraic structures introduced by K. H. Kim [2006]. In this fuzzification, we introduce the notion of Q- fuzzy groups (QFG) and investigate some of their related properties. Some properties on group theory in Q- fuzzy groups are obtained. This fuzzification leads to development of new notions over fuzzy groups. Characterizations of Q- fuzzy groups (QFCG) and normal Q- fuzzy groups (QFNG) are given.

In Lattice Valued Q-fuzzy left R- sub modules of near rings with respect to T-norms, a technique of generating of Q- fuzzy R- sub module by a given arbitrary Q- fuzzy set was provided. It is shown that (i) the sum of two Q- fuzzy R- sub module of a module M is the Q- fuzzy R- sub module generated by their union and (ii) the set of all Q-fuzzy sub module of a given module forms a complete lattice. Consequently the collection of all Q-fuzzy R- sub module, having the same values at zero of M of the lattice of Q- fuzzy R- sub module of M. Interrelationship of these finite range sub lattices was established.

Finally it was shown that the lattice of all Q- fuzzy R- sub module of M can be embedded into a lattice of Q- fuzzy R- sub module of M that denotes as Q-fuzzy R- sub module and R is the commutative near ring with unity. Characterization of Q-fuzzy left R- sub modules with respect to t- norm was also given.

2.2 Section II: Q –fuzzy group

The notion of fuzzy sets was first introduced by Zadeh [1965]. Fuzzy set theory has been developed in many directions by many scholars and has evoked great interest among mathematicians working in different fields of mathematics such as topological spaces, functional analysis, loop, and group, ring, near- ring, vector spaces, and automation. There have been wide ranging applications of the theory of fuzzy sets from the design of robots and computer simulation to Engineering and water resources planning. Rosenfeld [1971] introduced the fuzzy sets in the realm of group theory.

Recently, Davvaz [2006] considered intuitionistic fuzzification of the concept of the H_v-submodules in a H_v-module and Dudex [2005] considerer the intuitionistic fuzzification of the concept of sub hyperquasigroups in a hyper quasigroup. The notion of an intuitionistic Q- fuzzy R- subgroups of a near rings is given by Osman kazanci, sultan yamark and serife yilmaz [2007].

The notion of intuitionistic Q- fuzzy semi primality in a semi group is given by Kim [2006]. Also Rho[2006] considered the intuitionistic Q-fuzzification of BCK / BCI algebra's. In this chapter, some characterization of Q-fuzzy sets and then proved some results on Q- fuzzy groups are made.

2.2.1 Definition: A mapping $\mu: X\rightarrow [0, 1]$, where X is an arbitrary non-empty set is called a fuzzy set in X.

2.2.2 Definition: Let G be any group. A mapping $\mu: G\rightarrow [0, 1]$ is a fuzzy group if (FG1) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$ (FG2) $\mu(x^{-1}) = \mu(x)$ for all $x,y \in G$.

Example-: Let Z be the additive group of all integers. For any integer n, nZ denote the set of all integers multiplies of n.

(i,e) n Z = { 0, ± n, ±2n, ±3n……}. we have Z > 2Z > 4Z > 8Z > 16Z. Define $\mu : Z\rightarrow [0,1]$ by $\mu(x) = 1$, if x έ 16Z; = 0.7, if x έ 8Z -16Z; = 0.5 if x έ 4Z-8Z; = 0.2 if x έ 2Z- 4Z; = 0 if x έ Z-2Z . It can be easily verified that μ is Fuzzy group of Z.

2.2.3 Definition: Let Q and G a set and a group respectively. A mapping $\mu : G \times Q\rightarrow [0,1]$ is called Q – fuzzy set in G . For any Q- fuzzy set μ in G and $t \in [0,1]$ we define the set $U(\mu; t) =\{ x \in G / \mu(x, q) \geq t, q\in Q\}$ which is called an upper cut of μ and can be used to the characterization of μ.

2.2.4 Definition: A Q- fuzzy set A is defined Q-fuzzy group of G if (QFG1) $A(xy,q) \geq \min \{ A(x,q), A(y,q)\}$ (QFG2) $A(x^{-1},q) = A(x,q)$ (QFG3) $A(e,q) = 1$ for all $x,y \in G$ and $q \in Q$.

Example-1: Let G={a,b,c,d} be a set with + operations as follows.

+	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	b	a
d	d	c	a	b

Then (G,+) is a group. Let Q= {1,2,3,4} and let A be Q- fuzzy set in R defined by $\mu_A(a,0) = \mu_A(a,1) = \mu_A(a,2) = \mu_A(a,3)= \mu_A(a,4) = 1$,
$\mu_A(b,0) = \mu_A(b,1) = \mu_A(b,2) = \mu_A(b,3)= \mu_A(b,4) = 2/3$,
$\mu_A(c,0) = \mu_A(c,1) = \mu_A(c,2) = \mu_A(c,3) = \mu_A(c,4) = 1/3$,
$\mu_A(d,0) = \mu_A(d,1) = \mu_A(d,2) = \mu_A(d,3) = \mu_A(d,4) = 1/3$. We can check that A is Q- fuzzy groups of R.

In this chapter, the new classes of algebraic structures introduced by K.H.Kim, [2006] are defuzzified. In this fuzzification, the notion of Q- fuzzy groups (QFG) and investigates some of their related properties. It leads to development of new notions over fuzzy groups. Characterisation of Q- fuzzy groups (QFCG) and normal Q- fuzzy groups (QFNG) are given.

The following results on the properties of Q-fuzzy group are obtained.

2.2.1 Proposition: Let 'A' be a Q- fuzzy group of G. Then $A(x,q) \leq A(e,q)$ for all $x \in G$ and $q \in Q$. The subset $G_A = \{ x \in G / A(x,q) = A(e,q)\}$ is a Q- fuzzy group of G.

Proof: Let x be any element of G. Then $A(x,q) = \min \{A(x,q), A(x,q)\} = \min \{A(x,q), A(x^{-1},q)\} \leq A(xx^{-1},q)=A(e,q)$ implies (i). To verify (ii), it follows that $e \in G_A$, and $G_A \neq \Phi$.

Now let $x,y \in G_A$ and $q \in Q$. $A(xy^{-1},q) \geq \min \{A(x,q), A(y^{-1},q)\} = \min \{A(x,q), A(e,q)\}= \min\{A(e,q). A(e,q)\} = A(e,q)$ but from (i) $A(xy^{-1},q) \leq A(e,q)$ for $x,y \in G$ and $q \in Q$. So $A(xy^{-1},q) = A(e,q)$ which means $(xy^{-1},q) \in G_A$ and G_A is Q- fuzzy group of G.

2.2.2 Corollary: Let G be a finite group and A be a Q- fuzzy group of G. consider the subset H of G given by $H = \{x \in G / A(x,q) = A(e,q)\}$ Then H is a crisp subgroup of G.

Proof: It is obivious.

2.2.3 Proposition: Let A and B be two Q- fuzzy groups of a group G. Then $(A \cap B)$ is Q-fuzzy group of G.

Proof: Let $x, y \in G$ and $q \in Q$.

(QFG1) $(A \cap B)(xy, q) = \min\{A(xy, q), B(xy, q)\}$

$\geq \min\{\min\{A(x, q), A(y, q)\}, \min\{B(x, q), B(y, q)\}\}$

$= \min\{\min\{A(x, q), B(x, q)\}, \min\{A(y, q), B(y, q)\}\}$

$= \min\{(A\cap B)(x, q), (A\cap B)(y, q)\}$.

(QFG2) $(A \cap B)(x^{-1},q) = \min\{A(x^{-1},q), B(x^{-1},q)\}$

$= \min\{A(x, q), B(x, q)\}$

$= (A\cap B)(x, q)$.

2.2.4 Proposition: If A is a Q- fuzzy group of G, then A^C is also Q- fuzzy group of G.

Proof: For any $x, y \in G$, and $q \in Q$, it follows that

(QFG1) $A^C(xy, q) = 1 - A(xy, q)$

$\leq 1 - \min\{A(x, q), A(y, q)\}$

$= \max\{1 - A(x, q), 1 - A(y, q)\}$

$= \max\{A^C(x, q), A^C(y, q)\}$

(QFG2) $A^C(x^{-1}, q) = 1 - A(x^{-1}, q)$

$= 1 - A(x, q) = A^C(x, q)$.

2.2.5 Proposition: If A is Q- fuzzy group of G, then the set $U(A; t)$ is also Q-fuzzy group for all $q \in Q$, $t \in Im(A)$.

Proof: Let $t \in Im(A) \subseteq [0, 1]$; $x, y \in U(A; t)$, and $q \in Q$. Then $A(x, q) \geq t$, $A(y, q) \geq t$. Since A is Q- fuzzy group of G, $A(xy, q) \geq \min\{A(x, q), A(y, q)\} \geq t$ implies $xy \in U(A; t)$. Let $x \in U(A; t)$ and $q \in Q$ implies $A(x^{-1},q) = A(x, q) \geq t$ which gives $x^{-1} \in U(A; t)$. Thus $U(A; t)$ is Q – fuzzy group of G.

2.2.5 Definition: Let θ be a mapping from X to Y. If A and B are Q- fuzzy sets in X and Y respectively, then the inverse image of B under θ denoted by $\theta^{-1}(B)$ is Q- fuzzy set in X defined by $\theta^{-1}(B) = \mu_{\theta}^{-1}(B)$ where $\mu_{\theta^{-1}(B)}(x, q) = \mu_B\, \theta(x, q)$ and $\mu_{\theta^{-1}(B)}(x^{-1}, q) = \mu_B(\theta(x, q)$ for all $x \in Y$, $q \in Q$ and the image of A under θ denoted by $\theta(A)$, where $\mu_{\theta(A)}(y, q) = \{ V_{x\varepsilon\theta^{-1}(y)}\ \mu_A(x, q)$, if $\theta^{-1}(y) \neq 0$; 0, otherwise for all $y \in Y$, $q \in Q$.

2.2.6 Proposition: Let G and G^1 be two groups and $\theta: G \rightarrow G^1$ a homomorphisms. If B is Q – fuzzy group of G^1, then the pre image $\theta^{-1}(B)$ is Q- fuzzy group of G.

Proof: Assume that B is Q- fuzzy group of G^1. Let $x, y \in G$, and $q \in Q$.

(QFG1) $\mu_{\theta^{-1}(B)}(xy, q) = \mu_B(\theta(xy, q))$

$= \mu_B(\theta(x)\,\theta(y), q)$

$\geq \min\{ \mu_B(\theta(x), q), \mu_B(\theta(y), q)\}$

$= \min\{ \mu_{\theta^{-1}(B)}(x, q), \mu_{\theta^{-1}(B)}(y, q)\}$.

(QFG2) $\mu_{\theta^{-1}(B)}(x^{-1},q) = \mu_B(\theta(x^{-1},q)) = \mu_B(\theta x^{-1},q) = \mu_B(\theta x, q)$

$= \mu_{\theta^{-1}(B)}(x, q)$ therefore $\theta^{-1}(B)$ is Q- fuzzy group of G.

2.2.7 Proposition: Let $\theta: G \rightarrow G^1$ be an epimorphisms and B be Q–fuzzy set in G^1. If $\theta^{-1}(B)$ is Q- fuzzy group of G, then B is Q- fuzzy group of G^1.

Proof: Let $x, y \in G^1$, and $q \in Q$. Then there exists $a,b \in G$ such that $\theta(a) = x$, $\theta(b) = y$. It follows that

(QFG1) $\mu_B(xy, q) = \mu_B(\theta(a)\theta(b), q)$

$= \mu_B(\theta(ab), q)$

$= \mu_{\theta^{-1}(B)}(ab, q)$

$\geq \min\{ \mu_{\theta^{-1}(B)}(a, q), \mu_{\theta^{-1}(B)}(b, q)\}$

$\geq \min\{ \mu_B(\theta(a), q), \mu_B(\theta(b), q)\}$

$\geq \min\{ \mu_B(x, q), \mu_B(y, q)\}$.

(QFG2) $\mu_B(x^{-1},q) = \mu(\theta(a)^{-1}, q)$

$= \mu_B(\theta(a^{-1}), q)$

$= \mu_{\theta\text{-}1(B)}(a^{-1}, q)$

$= \mu_{\theta\text{-}1(B)}(a, q)$

$= \mu_B(\theta(a), q) = \mu_B(x, q)$.

2.2.8 Proposition: If $\{A_i\}_{i\varepsilon A}$ is a family of Q- fuzzy groups of G, then $\cap A_i$ is Q-fuzzy group of G where $\cap A_i = \{((x, q), \Lambda\, \mu_{Ai}(x, q)) / x \in G, q \in Q\}$, where $i \in A$.

Proof: Let $x, y \in G$, $q \in Q$. Then for $i \in A$, it follows that

(QFG1) $(\cap\, \mu_{Ai})(xy, q) = \Lambda\, \mu_{Ai}(xy, q) \geq \Lambda \min\{\mu_{Ai}(x, q), \mu_{Ai}(y, q)\}$

$= \min\{(\Lambda\, \mu_{Ai}(x, q)), (\Lambda \mu_{Ai}(y, q))\}$

$= \min\{(\cap \mu_{Ai})(x, q), (\cap\, \mu_{Ai})(y, q)\}$

(QFG2) Let $x \in G$ and $q \in Q$. For $i \in A$, it gives that

$(\cap\, \mu_{Ai})(x^{-1},q) = \Lambda \mu_{Ai}(x^{-1},q) \geq \Lambda \mu_{Ai}(x, q) = (\cap \mu_{Ai})(x, q)$

Hence $\cap A_i$ is Q- fuzzy group of G.

2.2.9 Proposition: If A is Q-fuzzy set in G such that all non- empty level subset U(A; t) is Q- fuzzy group of G, then A is Q-fuzzy group of G.

Proof: Assume that the non-empty level set U (A; t) is Q- fuzzy group of G.

If $t_0 = \min\{A(x, q), A(y, q)\}$ and for $x, y \in G$, $q \in Q$, then $x,y \in U(A; t_0)$ implies $A(xy, q) \geq t_0 = \min\{A(x, q), A(y, q)\}$ which gives that the condition (QFG1) is valid.

For $x^{-1} \in G$ and $q \in Q$, then $x^{-1} \in U(A; t_0)$ so $A(x^{-1},q) = t_0 = A(x, q)$ which implies that the condition (QFG2) is valid and therefore A is Q- fuzzy group of G.

2.2.10 Proposition: A set of necessary and sufficient conditions for a Q- fuzzy set of a group G to be a Q- fuzzy group of G is that $A(xy^{-1}, q) \geq \min (A(x, q), A(y, q)\}$ for all x, y in G and q in Q.

Proof: Let A be a Q- fuzzy group of G. Then $A(xy^{-1},q) \geq \min\{ A(x, q) , A(y^{-1}, q)\}$ $= \min \{ A(x, q) , A(y, q)\}$ for $x, y \in G$ and $q \in Q$.

For the converse, suppose that A be a Q-fuzzy set of the group G of which e is the identity element. Now $A(yy^{-1},q) \geq \min \{A(y, q) , A(y, q)\}$ or $A(e, q) \geq A(y, q)$.

Then $A(ey^{-1},q) \geq \min \{ A(e, q) , A(y, q)\}$or $A(y^{-1}, q) \geq A(y, q)$.

Also $A(xy, q) \geq \min \{A(x, q) , A(y^{-1}, q)\} \geq \min\{A(x, q) , A(y, q)\}$.

2.3 Section III – Characterization of Q-fuzzy normal group

The investigation in this section is to define Q- fuzzy characteristic group (QFCG) and to study their properties. For this, the notions of μ_A^{θ} is defined and it will be useful in the discussion.

2.3.1 Definition: Let μ_A be a Q- fuzzy set of G. Let $\theta: G \times Q \rightarrow G$ be a map and define the map $\mu_A^{\theta}: G\times Q \rightarrow [0, 1]$ by $\mu_A^{\theta}(x, q) = \mu_A(\theta(x, q))$.

2.3.2 Definition: Q-fuzzy group A of G is called Q-fuzzy characteristic of G if $\mu_A^{\theta} = \mu_A$.

2.3.3 Definition: Any Q-fuzzy group A of G is said to be normal if there exists $x \in G$ and $q \in Q$ such that A (x, q) = 1.

Note that if μ is normal Q-fuzzy group of G, then A(e, q) = 1 and hence A is normal if and only if A(e, q) = 1.

2.3.4 Definition: Let A be a Q- fuzzy group of G. Then A is called Q- fuzzy normal group (QFNG) if for all x, y $\in$ G, A(xy, q) = A(yx, q), q$\in$Q. Alternatively, a Q-fuzzy group A is said to be Q- fuzzy normal if A(x, q) = A(yxy^{-1}, q) for x,y$\in$G and q $\in$ Q. The notation [x, y] stands for the expression $x^{-1}y^{-1}xy$.

Example -1: Let G = {a,b,c,d,e,f} , where a = (1), b = (12) (36) (45), c = (13) (25) (46), d = (14) (26) (35), e = (156) (234) and f = (165) (243).If A is fuzzy subset of G defined by A(a) = 1, A(b) = A(c) = A(d) = t_1 and A(e) = A(f) = t_2 such that $1 \geq t_2 \geq t_1$, where t_1 and t_2 in [0,1]. Then A is fuzzy group of G under Min and G = M (M is the normaliser of A) . Hence A is normal.

Example -2 : Let G = {a,b,c,d,e,f} , where a = (1), b = (12) (36) (45), c = (13) (25) (46), d = (14) (26) (35), e = (156) (234) and f = (165) (243).If A is fuzzy subset of G defined by A(a) = 1, A(b) = A(c) = A(d) = t_1 and A(e) = A(f) = t_2 such that $1 \geq t_2 \geq t_1$, where t_1 and t_2 in [0,1]. Then A is fuzzy group of G under Min and its normaliser M = {a,b}. Note that if $t_2 \neq 0$, then A is not contained M.

The following propositions are proved:

2.3.1 Proposition: If A is Q- fuzzy group of G and θ is a homomorphism of G, then the Q-fuzzy set A^θ = {< (x, q), μ_A^θ (x, q)>; x$\in$G, q $\in$ Q} is Q-fuzzy group of G.

Proof: Let x, y $\in$ G. Then (QFG1) $\mu_A^\theta(xy, q) = \mu_A(\theta(xy, q)) = \mu_A(\theta(x)\theta(y), q)$

$\geq \min\{\mu_A(\theta(x), q), \mu_A(\theta(y), q)\} = \min\{\mu_A^\theta(x, q), \mu_A^\theta(y, q)\}$

(QFG2) $\mu_A^\theta(x^{-1}, q) = \mu_A(\theta(x^{-1}, q)) = \mu_A(\theta(x^{-1}), q) = \mu_A^\theta(x, q)$.

Thus A^θ is Q- fuzzy group of G.

2.3.2 Proposition: Let A be Q–fuzzy group of G. Let A^+ be a Q- fuzzy set in G defined by A^+(x, q) = A (x, q) + 1 – A (e, q) for all x $\in$ G. Then A^+ is normal Q-fuzzy group of G which contains A.

Proof: For any x, y $\in$ G, it follows that A^+ (x, q) = A (x, q) + 1- A (e, q) = 1.

(QFG1) $A^+(xy, q) = A(xy, q) + 1 - A(e, q)$

$\geq \min\{A(x, q), A(y, q)\} + 1 - A(e, q)$

$= \min\{A(x, q) + 1 - A(e, q), A(y, q) + 1 - A(x, q)\}$

$= \min\{A^+(x, q), A^+(y, q)\}$.

(QFG2) $A^+(x^{-1},q)$ $= A(x^{-1}, q) + 1 – A(x, q)$

$= A(x, q) + 1 – A(e, q)$

$= A^+(x, q)$ hence A^+ is normal Q- fuzzy group of G.

Clearly $A \subseteq A^+$.

2.3.3 Proposition: Let A be a QFNG of a group G. Then A ([x, y], q) = A (e, q) for all x, y ∈ G.

Proof: Since A is QFNG of G, itgives that $A(x, q) = A(yxy^{-1}, q)$, for all x,y ∈ G, q ∈ Q by replacing x by x^{-1} and y by y^{-1}, it omplies $A(y^{-1}, q) = A(x^{-1}y^{-1}xy, q)$ or $A(x^{-1}y^{-1}xyy^{-1}, q) = A(y^{-1}, q)$ or $A([x, y]y^{-1}, q) = A(y^{-1}, q)$ or A ([x, y], q) = A(e, q).

2.3.4 Proposition: If A is QFCG of a group G, then A is QFNG of G.

Proof: Let x, y ∈ G. Consider the map $\theta : G \rightarrow G$ given by $\theta(x, q) = a^{-1}xa$, for all a ∈ G, and q∈ Q. clearly θ is an automorphism of G.

Now $\mu_A(xy, q) = \mu_A{}^{\theta}(xy, q)$

$= \mu_A(\theta(xy , q))$

$= \mu(a^{-1}xya, q) = \mu_A(a^{-1}yxa, q) = \mu_A(yx, q)$ .Therefore A is QFNG of G.

Conclusion: Group theory has vast and potential applications in many core areas like physics, chemistry, communications, coding theory, computer science etc. The concept of Q-fuzzy groups and their properties are studied. A result on classical groups with the help of Q-fuzzy group theory is obtained.

2.4 Section IV: Q-fuzzy R-subgroups of near-rings with respect to T-norm

The aim is to study the notion of Q- fuzzification of left R- subgroups in a near-ring and investigate some related properties. Characterization of Q- fuzzy left R-subgroups with respect to a t-norm are given.

The theory of fuzzy sets which was introduced by Zadeh [1965] is applied to many mathematical branches. Abou-zoid [1991] , introduced the notion of a fuzzy sub near-ring and studied fuzzy ideals of near-ring. This concept discussed by many researchers among cho, Jun, Kim [2005]. In osman kazanci [2007], considered the intuitionistic fuzzification of a right (resp left) R- subgroup in a near-ring. Also [cho 2005] the notion of normal intuitionistic fuzzy R- subgroup in a near-ring is introduced and related properties are investigated. The notion of intuitionistic Q- fuzzy semi primality in a semi group is given by Kim [2000]. We introduce the notion of Q- fuzzification of left R- subgroups in a near ring and investigate some related properties. Characterization of Q- fuzzy left R- subgroups are given.

2.4.1 Preliminaries.

2.4.1 Definition: A non empty set with two binary operations + and . is called a near-ring if it satisfies the following axioms

(i) (R,+) is a group.

(ii) (R,.) is a semi group.

(iii) x.(y+z) = x.y + x.z for all $x,y,z \in R$.

Precisely speaking it is a left near-ring.

A R – subgroup of a near- ring ‘R’ is a subset H of R such that

(i) (H , +) is a subgroup of (R, +).

(ii) $RH \subseteq H$

(iii) $HR \subseteq H$.

If H satisfies (i) and (ii), then it is called left R- subgroup of R. If H satisfies (i) and (iii), then it is called a right R- subgroup of R. A map $f: R \rightarrow S$ is called homomorphism if f(x + y) = f(x) + f (y) for all x,y in R.

2.4.2 Definition: Let R be a near ring. A fuzzy set μ in R is called fuzzy subnear ring in R if (i) $\mu(x-y) \geq \min\{\mu(x), \mu(y)\}$ (ii) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$ for all x, y in R.

Example-2: Let R = {a, b, c, d} be a set with two binary operations as follows:

+	a	b	c	D
a	a	b	c	D
b	b	a	d	C
c	c	d	b	A
d	d	c	a	B

•	a	b	c	D
a	a	a	a	A
b	a	a	a	A
c	a	a	a	A
d	a	b	c	D

Then (R,+,•) is a near ring. let Q = {1,2,3,4} and A be Q- fuzzy set in R defined by

$\mu_A(a,0) = \mu_A(a,1) = \mu_A(a,2) = \mu_A(a,3) = \mu_A(a,4) = 1.$

$\mu_A(b,0) = \mu_A(b,0) = \mu_A(b,0) = \mu_A(b,0) = \mu_A(b,0) = 2/3$

$\mu_A(c,0) = \mu_A(c,0) = \mu_A(c,0) = \mu_A(c,0) = \mu_A(c,0) = 1/3$

$\mu_A(d,0) = \mu_A(d,0) = \mu_A(d,0) = \mu_A(d,0) = \mu_A(d,0) = 1/3$. We can choose that A is both Q- fuzzy sub near ring and Q- fuzzy R- subgroup of R.

2.4.3 Definition: A Q-fuzzy set μ is called a fuzzy left R- subgroup of R over Q if μ satisfies (i) $\mu(x-y, q) \geq T\{\mu(x, q), \mu(y, q)\}$ (ii) $\mu(rx, q) \geq \mu(x, q)$.

2.4.4 Definition: (T- norm) A triangular form is a function $T: [0,1] \times [0,1] \rightarrow [0,1]$ that satisfies the following conditions for all x,y,z in [0,1]

1. $T(x,1) = x$
2. $T(x,y) = T(y,x)$
3. $T(x, T(y,z)) = T(T(x,y), z)$
4. $T(x,y) \leq T(x,z)$ when $y \leq z$

The following results are established on the properties of Q-fuzzy left R-subgroup

2.4.1 Proposition: Let T be a t- norm. Then every imaginable Q- fuzzy left R- subgroup μ of a near ring S is a fuzzy left R-subgroup of S.

Proof: Assume μ is imaginable Q- fuzzy left R- subgroup of S.

It follows that $\mu(x - y, q) \geq T\{\mu(x, q), \mu(y, q)\}$ and $\mu(rx, q) \geq \mu(x, q)$ for all x,y in S.

Since μ is imaginable, it gives

$\min \{ \mu(x, q), \mu(y, q)\} = T \{ \min \{ \mu(x, q), \mu(y, q)\}, \min \{ \mu(x, q), \mu(y, q) \} \}$

$\leq T (\mu(x, q), \mu(y, q))$

$\leq \min \{\mu(x, q), \mu(y, q) \}$.

and so $T(\mu(x, q), \mu(y, q)) = \min \{ \mu(x, q), \mu(y, q) \}$.

Finally $\mu(x-y, q) \geq T(\mu(x, q), \mu(y, q)) = \min \{ \mu(x, q), \mu(y, q) \}$ for all $x,y \in S$.

Hence μ is a fuzzy left R- subgroup of S.

2.4.2 Proposition: If μ is a Q-fuzzy left R- subgroups of a near ring S and Θ is an endomorphism of S, then $\mu[\Theta]$ is a Q- fuzzy left R- subgroup of S.

Proof: For any $x, y \in S$, it implies

(i) $\mu[\Theta] (x-y , q) = \mu (\Theta(x - y , q))$

$= \mu (\Theta(x, q), \Theta(y, q))$

$\geq T (\mu(\Theta(x, q)), \mu(\Theta(y, q)))$

$= T (\mu[\Theta] (x, q), \mu[\Theta] (y, q))$.

(ii) $\mu[\Theta] (rx , q) = \mu(\Theta (rx, q))$

$\geq \mu (\Theta(x, q))$

$\geq \mu [\Theta] (x, q)$.

Hence $\mu[\Theta]$ is a Q-fuzzy left R- subgroup of R.

2.4.3 Proposition: An onto homomorphisms of a Q- fuzzy left R-subgroup of near ring S is Q- fuzzy left R-subgroup.

Proof: Let $f: S \rightarrow S^1$ be an onto homomorphism of near rings and ξ be a Q-fuzzy left R- subgroup of S^1 and μ be the pre image of ξ under f. Then it gives that

(i) $\mu(x - y , q) = \xi (f(x-y , q))$

$= \xi (f(x, q) , f(y, q))$

$\geq T (\xi (f(x, q)) , \xi(f(y, q)))$

$\geq T (\mu(x, q) , \mu(y, q))$.

(ii) $\mu(rx, q) = \xi (f (rx, q))$

$\geq \xi (f(x, q))$

$\geq \mu(x, q)$.

2.4.4 Proposition: An onto homomorphic image of a fuzzy left R- subgroup with the sup property is a fuzzy left R- subgroup.

Proof: Let $f: S \to S^1$ be an onto homomorphism of near rings and let μ be a sup property of fuzzy left R- subgroups of S.

Let $x^1, y^1 \in S^1$, and $x_0 \in f^{-1}(x^1)$, $y_0 \in f^{-1}(y^1)$ be such that

$\mu(x_0, q) = \sup [\mu(h, q), \mu(y_0, q)]$ where $(h, q) \in f^{-1}(x^1)$

$= \sup [\mu(h, q)]$ where $(h, q) \in f^{-1}(y^1)$ respectively.

(i) $\mu^f (x^1-y^1, q) = \sup [\mu(z, q)]$ where $(z, q) \in f^{-1}(x^1-y^1, q)$

$\geq \min \{ \mu(x_0, q) , \mu(y_0, q)\}$

$$= \min \{ \sup_{(h, q) \in f^{-1}(x^1, q)} [\mu(h , q)] , \sup_{(h, q) \in f^{-1}(y^1, q)} [\mu(h, q)]\}$$

$= \min \{ \mu^f(x^1, q) , \mu^f(y^1, q) \}$

(ii) $$\mu^f(rx, q) = \sup_{(z, q) \in f^{-1}(r^1x^1, q)} [\mu(z, q)]$$

$\geq \mu(y_0 , q)$

$$= \sup_{(h, q) \in f^{-1}(y^1, q)} [\mu(h, q)]$$

$= \mu^f(y^1, q)$.

Hence μ^f is a fuzzy left R- subgroup of S^1.

2.4.5 Proposition: Let T be a continous t-norm and f be a homomorphism on a near ring S. If μ is Q-fuzzy left R- subgroup of S, then μ^f is a Q- fuzzy left R- subgroup of f(S).

Proof: Let $A_1 = f^{-1}(y_1, q)$, $A_2 = f^{-1}(y_2, q)$ & $A_{12} = f^{-1}(y_1\text{-}y_2 , q)$ where $y_1,y_2 \in f(S)$, $q\in Q$. Consider the set $A_1\text{-} A_2 = \{ x\in S / (x, q) = (a_1, q) - (a_2, q) \}$ for some $(a_1, q) \in A_1$ and $(a_2, q) \in A_2$. If $(x, q) \in A_1\text{-}A_2$, then $(x, q) = (x_1, q) - (x_2, q)$ for some $(x_1, q) \in A_1$ and $(x_2, q) \in A_2$, then $f (x, q) = f(x_1, q) - f(x_2, q) = y_1 - y_2$.

$(x, q) \in f^{-1}((y_1, q) - (y_2, q)) = f^{-1}(y_1\text{-}y_2, q) = A_{12}$. Thus $A_1 - A_2 \subseteq A_{12}$.

It follows that

$$\begin{aligned}
\text{(i) } \mu^f(y_1\text{-}y_2, q) &= \sup \{ \mu(x, q)/ (x, q) \in f^{-1}((y_1, q)\text{-} (y_2, q))\} \\
&= \sup\{ \mu(x, q) / (x, q) \in A_{12} \} \\
&\geq \sup \{ \mu(x, q)/ (x, q) \in A_1\text{-}A_2\} \\
&\geq \sup \{ \mu((x_1, q)\text{-} (x_2, q)) / (x_1, q) \in A_1 \text{ and } (x_2, q) \in A_2\} \\
&\geq \sup \{ T(\mu(x_1, q) , \mu(x_2, q))/ (x_1, q) \in A_1 \text{ and } (x_2, q) \in A_2\}.
\end{aligned}$$

Since 'T' is continous, and for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\sup \{ \mu(x_1, q) / (x_1, q) \in A_1\} - (x_1^*, q) \} \leq \delta \text{ and}$$

$$\sup \{ \mu(x_2, q) / (x_2, q) \in A_2\} - (x_2^*, q) \} \leq \delta.$$

$$T\{\sup\{\mu(x_1, q) / (x_1, q) \in A_1\} , \sup \{ \mu(x_2, q) / (x_2, q) \in A_2 \} - T ((x_1^*, q), (x_2^*, q)) \leq \varepsilon.$$

Choose $(a_1, q) \in A_1$ and $(a_2, q) \in A_2$ such that

$$\sup \{ \mu(x_1, q) / (x_1. q) \in A_1 \} - \mu(a_1, q) \leq \delta \text{ and}$$

$$\sup \{ \mu(x_2, q) / (x_2, q) \in A_2\} - \mu(a_2, q) \leq \delta.$$

Then it gives

$T\{\sup\{ \mu(x_1, q) / (x_1, q) \in A_1\}, \sup \{ \mu(x_2, q) / (x_2, q) \in A_2 \}\} - T(\mu(a_1, q), \mu(a_2, q)) \leq \varepsilon$.

consequently, $\mu^f(y_1\text{-}y_2, q) \geq \sup\{ T(\mu(x_1, q), \mu(x_2, q)) / (x_1, q) \in A_1, (x_2, q) \in A_2\}$

$$\geq T(\sup\{\mu(x_1, q) / (x_1, q) \in A_1\}, \sup\{\mu(x_2, q) / (x_2, q) \in A_2\})$$

$$\geq T(\mu^f(y_1, q), \mu^f(y_2, q)\}.$$

Similarly $\mu^f(rx, q) \geq \mu^f(y, q)$ veified. Hence 'μ^f' is a Q- fuzzy left R- subgroup of f(S).

Conclusion: Osman kazanci, Sultanyamark and Serifeyilmaz [2007] introduced the intutionistic Q- fuzzy R-subgroups of near rings. In this chapter, we investigate the notion of Q- fuzzy left R- subgroup of near ring with respect to t-norm and characterization of them.

2.5 Section V: Lattice valued Q-fuzzy left R-submodules of near-rings with respect to T-norm

Introduction: In the Trajectory of stupendous growth of fuzzy set theory. Fuzzy algebra has become an important area of research. A.Rosenfeld [1971] used the concept of fuzzy set theory due to Zadeh [1965] in abstract algebra and opened up a new insight in the field of mathematical science. Since then the study of fuzzy algebraic structures has been pursued in many directions such as semi groups, groups, rings, semi rings, near rings, lattices and so on.

Fuzzy sub modules of a module M over a near ring R were first introduced by Negoita and Ralescu [1975]. Consequently Pan studied fuzzy finitely generated modules and fuzzy quotient modules. Pan, Katsaras and Liu [1987] introduced the concept of fuzzy sub modules and fuzzy vector spaces respectively. More recently in the notion of set products is discussed in details and in the lattice theoretical aspect of fuzzy subgroups and fuzzy normal subgroups are explored. The formation of a lattice of sub modules of module is well known feature in classical algebra. However, the same has not been explored in fuzzy setting. In order to initiate such studies, the concept of fuzzy sub module generated by an arbitrary fuzzy set is formulated in this chapter.

Using this concept, we construct various type of lattices of fuzzy sub modules and establish an embedding of the lattice of all submoduls of a module M into the lattice of fuzzy sub modules M. The notion of an intuitionistic Q- fuzzy R- subgroups of near rings is given. The notion of intuionistic Q- fuzzy semi primality in a semi group is given Kim [2006] . Also [Rho.2006] considered the intuionistic Q- fuzzy BCK / BCI algebra. In this chapter, we make some characterization of Q- fuzzy left R- sub module and then proved some results on Q- fuzzy lattice of R- modules.

2.5.1 Preliminaries

2.5.1 Definition: Let μ be a Q- fuzzy subset of a set S and $t \in [0, 1]$. Then the set $\mu_t = \{ x \in S \ / \ \mu(x,q) \geq t \}, q \in Q$ is called a level subset of μ.

2.5.2 Definition: A Q–fuzzy subset μ of M is called a Q–fuzzy R- sub module of M if the following conditions are satisfied,

(i) $\mu(x+y, q) \geq T(\mu(x, q), \mu(y, q)\}$ for all $x,y \in M$ and

(ii) $\mu(rx, q) \geq \mu(x, q)$ for all $r \in M$ and $q \in Q$.

2.5.3 Definition: Let μ be a Q- fuzzy subset on M. Define a Q-fuzzy subset $< \mu >$ of M as follows: $< \mu > (x, q) = \sup \{ K / x \in < \mu_k >, x \in M$. $< \mu >$ is called the Q- fuzzy subset of M generated by μ. Here $< \mu_k >$ is the sub module of M generated by the level subset.

For a Q- fuzzy left R–module of μ of M the level subset $\mu_t = \{x \in M / \mu(x, q) \geq t\}$, $t \in Im(\mu)$ are sub modules of M called the Q- level sub modules of M.

A Technique of generating of Q- fuzzy R- sub module by a given arbitrary Q- fuzzy set is provided. It is shown that (i) The sum of two Q- fuzzy R- sub module of a module M is the Q- fuzzy R- sub module generated by their union and (ii) The set of all Q- fuzzy sub module of a given module forms a complete lattice. Consequently it is established that the collection of all Q- fuzzy R- sub module, having the same values at zero, of M of the lattice of Q- fuzzy R- sub module of M. Interrelationship of these finite range sub lattices is established.

Finally it is shown that the lattice of all Q- fuzzy R- sub module of M can be embedded into a lattice of Q- fuzzy R- sub module of M. Through out this section , M denote as Q- fuzzy R- sub module where R is the commutative near ring with unity. Characterization of Q- fuzzy left R- sub modules with respect to t- norm are also given.

The following results are found.

2.5.1 Lemma: A Q-fuzzy subset μ of M is a Q- fuzzy R- sub module of M if and only if each level subset μ_t, $t \in Im(\mu)$, is a sub module of M.

2.5.2 Proposition: Let μ be a Q fuzzy R- sub module of M. Then the Q- fuzzy subset $< \mu >$ is a Q- fuzzy R- sub module of M generated by. More over $< \mu >$ is the smallest Q- fuzzy R- sub module containing μ

Proof: Let $x, y \in M$ and let $\mu(x, q) = t_1$, $\mu(y, q) = t_2$ and $\mu(x+y, q) = t$

Let it possible $t = < \mu >(x+y , q)$

$$\leq T\{ <\mu> (x, q) , <\mu> (y, q) \}. \text{ So } T\{t_1,t_2\} = t_1 \text{ (say)}$$

Then $t_1 = <\mu> (x, q) = \sup \{ k / x \in <\mu_k> \} \geq t$. Therefore there exist k_1, such that $x \in <\mu_{k1}>$. Also $t_2 = <\mu>(y, q) = \sup \{k / y \in <\mu_k> \}$ $t_1 \geq t$. So there exists $k_2 \geq t$ satisfying $y \in <\mu_k>$. Without loss of generality, assume that k_1 , k_2 with $<\mu_{k1}> \subseteq <\mu_{k2}>$. Then $x, y \in <\mu_k>$ implies $(x + y)$ which is a contradiction since $k_2 \geq t$. Thus $t \geq t_1$.

Consequently, $\mu(x + y, q) \geq T \{<\mu>(x, q) , <\mu> (y, q)\}$ --- (1).

Now let $t_3 = <\mu> (rx, q) \leq <\mu> (x, q) = t_1$ Then $t_1 = <\mu> (x, q) = \sup \{k / x \in <\mu_k>\} \geq t_3$, (if possible). Thus there exists k such that $x \in <\mu_k>$ and $t_1 \geq k \geq t_3$ so that $rx \in <\mu_k> \subseteq \{\mu_{t1}\}$ which is a contradiction. So $t_3 = <\mu> (rx, q) \geq <\mu> (x, q) = t_1$ --- (2).

Consequently conditions (1) and (2) yield that $<\mu>$ is a Q- fuzzy R- sub module of M.

Finally, the aim is to show that $<\mu>$ is the smallest Q- fuzzy R- sub module containing μ.

Assume that θ is a Q- fuzzy R- sub module of M such that $\mu \subseteq \theta$ and claim that $<\mu> \subseteq \theta$. Let $t = <\mu> (x, q) \geq \theta (x,q)$ for some $x \in m$, $q \in Q$ (if possible). Let $\varepsilon > 0$ be given. Then $t = \mu_t = \sup \{k / x \in <\mu_k> \}$. Thus there exists K such that $x \in <\mu_k>$ and $t-\varepsilon \leq k \leq t$ so that $x \in <\mu_k> \subseteq <\mu_{t-\varepsilon}>$ for all $\varepsilon > 0$.

Now $x = \acute{\alpha}_1 x_1 + \acute{\alpha}_2 x_2 + \ldots + \acute{\alpha}_n x_n$, $\acute{\alpha}_i \in R$, x_i belongs to $t-\varepsilon$. $X_i \in \mu_{t-\varepsilon}$ implies $\mu (x_i, q) \geq t-\varepsilon$, that is $\theta (x_i, q) \geq t-\varepsilon$ for all $\varepsilon > 0$.

Thus $\theta (x, q) \geq T \{ \theta(x_1, q) , \theta (x_2, q) ,\ldots, \theta(x_n, q)\} \geq t-\varepsilon$ for $\varepsilon > 0$.

Hence $\theta (x, q) = t$ which is a contradiction to our supposition.

2.5.4 Definition: Let μ and θ be Q- fuzzy R- sub modules of an R- sub module M. Then the sum of μ and θ denoted by $\mu + \theta$ is defined as

$$(\mu + \theta) (x, q) = \sup_{(x_i, q) = (a + b , q)} [T (\mu (a, q) , \mu (b, q))] \text{ for all } x_i \in M.$$

Clearly $\mu + \theta$ is a Q- fuzzy subset of M.

2.5.3 Proposition: Let μ and θ be a Q- fuzzy R- sub modules of M. Then the sum $\mu + \theta$ is Q- fuzzy sub module of M.

2.5.4 Proposition: Let μ and θ be a Q- fuzzy R- sub modules of M such that $\mu(0, q) = \theta(0, q)$. Then $\mu \subseteq \mu + \theta$, $\theta \subseteq \mu + \theta$.

Proof: Let $x \in M$. Then

$$(\mu + \theta)(x, q) = \sup_{(x,q) = (a+b,q)} \{T(\mu(a, q), \theta(b, q))\}$$

$$\geq T(\mu(x, q), \theta(x, q))$$

$$= T(\mu(x, q), \mu(0, q))$$

$$= \mu(x, q).$$

Similarly $(\mu + \theta)(x, q) \geq \theta(x, q)$.

2.5.5 Lemma: (2.5.4) is not true if $\mu(0, q) \neq \theta(0, q) < \mu(0, q)$ so μ does not contain $\mu + \theta$.

2.5.6 Proposition: Let μ and θ be a Q- fuzzy R- sub module of M such that $\mu(0, q) = \theta(0, q)$. Then $\mu + \theta = \langle \mu + \theta \rangle$.

Proof: Let $x \in M$.

$$t_1 = (\mu + \theta)(x, q) = \sup_{(x,q) = (a+b,q)} \{T(\mu(a, q), \mu(b, q)\}$$

Let $\varepsilon > 0$ be given.

Then $t_{1-\varepsilon} \leq T(\mu(a, q), \theta(b, q))$ for some $a, b \in M$ such that $x = a + b$, so that $t_{1-\varepsilon} \leq \mu(a, q)$ and $t_{2-\varepsilon} < \theta(b, q)$ but $\mu, \theta \subset \langle \mu \cup \theta \rangle \subset \langle \mu \cup \theta \rangle$.

Therefore

$$t_{1-\varepsilon} \leq T\{\langle \mu \cup \theta \rangle(a, q), \langle \mu \cup \theta \rangle(b, q)\}$$

$$\leq \langle \mu \cup \theta \rangle(a+b, q)$$

$$= \langle \mu \cup \theta \rangle(x, q), \text{ for all } \varepsilon > 0.$$

Hence $t_1 \leq < \mu + \theta > (x, q) = t$ (say).

Let it possible $(\mu + \theta)(x, q) = t_1 \leq < \mu \cup \theta > (x, e) = \sup \{ k / x \in < (\mu \cup \theta) > \}$.

Thus there exists k such that $x \in < (\mu + \theta) k >$ and $t_1 \leq k \leq t$.

Then $(\mu \cup \theta)_t \subset (\mu \cup \theta)_{t1}$. so that $<(\mu \cup \theta)_t > \subset < (\mu \cup \theta)_k > \subset < (\mu \cup \theta)_{t1} >$ implying $x \in < (\mu + \theta)_{t1} >$ which is a contradiction. Hence $t_1 = (\mu + \theta)(x, q) = < \mu \cup \theta > (x, q) = t$.

2.5.7 Corolarry: For any Q- fuzzy R- sub module μ of M, $\mu + \mu = \mu$.

2.5.8 Lemma: (2.5.4) is not true if $\mu (0, q) \neq \theta (0, q)$, for let $\mu (0, q) > \theta (0, q)$ implies $< \mu \cup \theta > (0, q) = \mu(0, q) > (\mu \cup \theta) (0, q)$ so $< \mu \cup \theta > > (\mu \cup \theta) (0, q)$ and so $\mu + \theta \neq < \mu \cup \theta >$.

2.5.5 Definition: Q-fuzzy R-sub module μ of a near ring is said to be normal if $\mu(0, q)=1$.

2.6 Section VI: Properties of Q- fuzzy R- sub modules of near rings

2.6.1 Proposition: Let μ be a Q- fuzzy R- sub module of a near ring and let μ* be a Q-fuzzy set in R defined by $\mu^*(x, q) = \mu(x, q) +1- \mu(0, q)$ for all $x \in R$. Then μ* is a normal Q- fuzzy R- sub module of R containing μ.

Proof: For any $x,y \in R$ and $q \in Q$, it follows that

$$\mu^*(x + y, q) = \mu(x+y, q) +1 - \mu(0, q)$$

$$\geq T(\mu(x, q)+1- \mu(0, q) , \mu(y, q)+1- \mu(0, q))$$

$$= T (\mu^*(x) , \mu^*(y)).$$

$$\mu^*(rx, q) = \mu(rx, q) +1 - \mu(0, q)$$

$$= \mu(x, q) +1- \mu(0, q)$$

$$= \mu(x, q).$$

2.6.2 Proposition: Let T be a t- norm. Then every imaginable Q-fuzzy left R-sub module μ of a near ring R is a fuzzy left R-sub module of R.

Proof: Assume μ is imaginable Q- fuzzy left R- sub module of R. Then it gives

$\mu(x + y, q) \geq T\{\mu(x, q), \mu(y, q)\}$ and $\mu(rx, q) \geq \mu(x, q)$ for all x, y in R.

Since μ is imaginable, it implies that

$$\min\{\mu(x, q), \mu(y, q)\} = T\{\min\{\mu(x, q), \mu(y, q)\}, \min\{\mu(x, q), \mu(y, q)\}\}$$
$$\leq T(\mu(x, q), \mu(y, q))$$
$$\leq \min\{\mu(x, q), \mu(y, q)\}.$$

So $T(\mu(x, q), \mu(y, q)) = \min\{\mu(x, q), \mu(y, q)\}$. It follows that

$$\mu(x+y, q) \geq T(\mu(x, q), \mu(y, q)) = \min\{\mu(x, q), \mu(y, q)\} \text{ for all } x,y \in R.$$

Hence μ is a fuzzy left R- sub module of R.

2.6.3 Proposition: If μ is a Q- fuzzy left R- sub module of a near ring R and Θ is an endomorphism of R, then $\mu_{[\Theta]}$ is a Q- fuzzy left R- sub module of 'R'.

Proof: For any x, y $\in$ S, it gives that

(i) $\mu_{[\Theta]}(x+y, q) = \mu(\Theta(x+y, q))$
$$= \mu(\Theta(x, q), \Theta(y, q))$$
$$\geq T(\mu(\Theta(x, q)), \mu(\Theta(y, q)))$$
$$= T\{\mu_{[\Theta]}(x, q), \mu_{[\Theta]}(y, q)\}$$

(ii) $\mu_{[\Theta]}(rx, q) = \mu(\Theta(rx, q)$
$$\geq \mu(\Theta(x, q))$$
$$\geq \mu_{[\Theta]}(x, q).$$ Hence $\mu_{[\Theta]}$ is a Q- fuzzy left R- sub module of R.

2.6.4 Proposition: An onto homomorphism of a Q- fuzzy left R- sub module of near ring R is Q- fuzzy left R- sub module.

Proof: Let $f : R \rightarrow R^1$ be an onto homomorphism of near rings and let ξ be a Q- fuzzy left R- subgroup of R^1 and 'μ' be the pre image of ξ under 'f. Then we have

$$\text{(i) } \mu(x + y, q) = \xi (f(x + y, q))$$
$$= \xi (f(x, q) , f(y, q))$$
$$\geq T (\xi (f(x, q)) , \xi(f(y, q)))$$
$$\geq T (\mu(x, q) , \mu(y, q)).$$
$$\text{(ii) } \mu(rx,q) = \xi (f (rx, q))$$
$$\geq \xi (f(x, q))$$
$$\geq \mu(x, q).$$

2.6.5 Proposition: An onto homomorphism image of a fuzzy left R- sub module with the sup property is a fuzzy left R- sub module.

Proof: Let f: $R \to R^1$ be onto homomorphism of near rings and let μ be a sup property of fuzzy left R- sub module of R.

Let $x^1, y^1 \in R^1$, and $x_0 \in f^{-1}(x^1)$, $y_0 \in f^{-1}(y^1)$ be such that

$$\mu(x_0, q) = \sup_{(h, q) \in f^{-1}(x^1)} [\mu(h, q)]; \qquad \mu(y_0, q) = \sup_{(h, q) \in f^{-1}(y^1)} [\mu(h, q)] \text{ respectively.}$$

It deduces that

$$\text{(i) } \mu^f (x^1 + y^1, q) = \sup_{(z,q) \in f^{-1}(x^1+y^1, q)} [\mu(z, q)]$$

$$\geq \min \{ \mu(x_0, q) , \mu(y_0, q)$$

$$= \min \{ \sup_{(h, q) \in f^{-1}(x^1,q)} [\mu(h ,q)] , \sup_{(h,q) \in f^{-1}(y^1,q)} [\mu(h,q)] \}$$
$$= \min \{ \mu^f(x^1, q) , \mu^f(y^1, q) \}.$$

$$\text{(ii) } \mu^f(rx, q) = \sup_{(z, q) \in f^{-1}(r^1x^1, q)} [\mu(z, q)]$$
$$\geq \mu(y_0 , q)$$

$$= \sup_{(h, q) \in f^{-1}(y^1, q)} [\mu(h, q)]$$

$$= \mu^f(y^1, q).$$

Hence μ^f is a fuzzy left R- sub module of R^1.

2.6.6 Proposition: Let T be a continuous t-norm and let f be a homomorphism on a near ring R. If μ is Q- fuzzy left R- sub module of R, then μ^f is a Q- fuzzy left R-sub module of f(R).

Proof: Let $A_1 = f^{-1}(y_1, q)$, $A_2 = f^{-1}(y_2, q)$ and $A_{12} = f^{-1}(y_1-y_2, q)$ where $y_1, y_2 \in f(R)$, $q \in Q$. Consider the set $A_1-A_2 = \{x \in S / (x, q) = (a_1, q)-(a_2, q)\}$ for some $(a_1, q) \in A_1$ and $(a_2, q) \in A_2$.

If $(x, q) \in A_1-A_2$, then $(x, q) = (x_1, q) - (x_2, q)$ for some $(x_1, q) \in A_1$ and $(x_2, q) \in A_2$ so that $f(x, q) = f(x_1, q) - f(x_2, q) = y_1 - y_2$ implies $(x, q) \in f^{-1}((y_1, q) - (y_2, q)) = f^{-1}(y_1-y_2, q) = A_{12}$. Thus $A_1 - A_2 \subset A_{12}$. It follows that

(i) $\mu^f(y_1+y_2, q) = \sup \{ \mu(x, q) / (x, q) \in f^{-1}((y_1, q)-(y_2, q))\}$

$= \sup\{ \mu(x, q) / (x, q) \in A_{12} \}$

$\geq \sup \{ \mu(x, q) / (x, q) \in A_1-A_2\}$

$\geq \sup \{ \mu((x_1, q)-(x_2, q)) / (x_1, q) \in A_1 \text{ and } (x_2, q) \in A_2\}$

$\geq \sup \{ T(\mu(x_1, q), \mu(x_2, q)) / (x_1, q) \in A_1 \text{ and } (x_2, q) \in A_2\}$.

Since T is continuous, and for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$\sup \{ \mu(x_1, q) / (x_1, q) \in A_1\} - (x_1^*, q) \} \leq \delta$ and

$\sup \{ \mu(x_2,q) / (x_2,q) \in A_2\} - (x_2^*,q) \} \leq \delta$.

$T\{\sup\{\mu(x_1, q) / (x_1, q) \in A_1\}, \sup \{ \mu(x_2, q) / (x_2, q) \in A_2 \} - T((x_1^*, q), (x_2^*, q) \leq \varepsilon$.

Choose $(a_1, q) \in A_1$ and $(a_2, q) \in A_2$ such that

$\sup \{ \mu(x_1, q) / (x_1. q) \in A_1\} - \mu(a_1, q) \leq \delta$, and

$\sup \{ \mu(x_2, q) / (x_2, q) \in A_2\} - \mu(a_2, q)) \leq \delta$.

Then it implies that

$T\{\sup\{ \mu(x_1, q) / (x_1, q) \in A_1\}, \sup \{ \mu(x_2, q) / (x_2, q) \in A_2 \}\} - T(\mu(a_1, q), \mu(a_2, q)) \leq \varepsilon$.

Consequently, $\mu^f(y_1-y_2, q) \geq \sup\{ T(\mu(x_1, q), \mu(x_2, q)) / (x_1, q) \in A_1, (x_2, q) \in A_2\}$

$\geq T (\sup\{\mu(x_1, q) / (x_1, q) \in A_1\}, \sup\{\mu(x_2, q) / (x_2, q) \in A_2\})$

$\geq T (\mu^f(y_1, q), \mu^f(y_2, q))$.

Similarly $\mu^f(rx, q) \geq \mu^f(y, q)$. Hence μ^f is a Q- fuzzy left R- sub module of 'f(R)'.

2.7 Section VII: Complete, modular and sub lattices of R-submodules

Introduction: Let L be the set of all Q- fuzzy R- sub modules of M. Then clearly the intersection of an arbitrary family of Q- fuzzy R- sub modules of M is a fuzzy sub module and proposition (2.5.4) shows that the existence of the least Q-fuzzy sub module containing the union of an arbitrary family of Q- fuzzy R- sub modules of M. These facts give rise to the following.

2.7.1 Proposition: L is a lattice under the usual ordering of Q- fuzzy set inclusion. More over L is a complete lattice,

Next, let t be an arbitrary but fixed real number in [0, 1] and let us denote by L_t, the subset of all Q- fuzzy R- sub modules θ of M such that θ (0, q) = t .

2.7.2 Proposition: L_t is a complete lattice of L.

It is known that L_t denotes the set of all Q- fuzzy R- sub modules θ of M such that $I_m \theta = \{ \theta(x, q) / x \in M \}$ is finite (that is has finite range).

2.7.3 Proposition: L_t is a sub lattice of L where let L_t denote the set of all Q-fuzzy R-sub modules θ of M such that θ (0, q) = t and $I_m\theta$ is finite

2.7.4 Proposition: Lf_t is a sub lattice of L.

Proof: Since $Lf_t = L_f \cap L_t$ and intersection of sub lattices is a sub lattice, L_{ft} is a sublattice of L. Finally, it is intended to demonstrate an embedding of well known lattice of all sub modules of M into the lattice L of all Q- fuzzy R- sub modules defined as follows $L_2 = \{ \theta / \theta \in M$ such that $I_m\theta = (r,t)$, $r \leq t \}$

2.7.5 Proposition: L_2 is a sublattice of L_{ft} and so is L.

Proof: It is obivious.

2.7.6 Proposition: L(M) as the lattice of all sub modules of M can be embedded in L_2

Proof: Let $A \in L(M)$. Define $\theta : M \times Q \rightarrow [0, 1]$ by $\theta(x, q) = 1$ if $x \in A$; $= 0$ if $x \notin A$. Then θ is a Q –fuzzy R- sub module of M (2. 5. 6) such that $I_m\theta\ (0, t)$, $0 \leq t \leq 1$. So $\theta \in L_2$. Define $\sigma: L(M) \times Q \rightarrow L_2$, such that $\sigma(A, q) = \theta$. Then σ is clearly well defined. Let $\sigma(A, q) = \sigma(B, q)$. Then $\theta = \mu$ implies $A = B$. so σ is one- one.

Let $A, B \in L(M)$. Let $\sigma(A, q) = \theta$, $\sigma(B, q) = \mu$.

Then $\sigma(A, q) + \sigma(B, q) = \theta+\mu = < \theta U\mu >$ by

$= \sigma(A + B , q)$ and $\sigma(A \cap B , q) = \theta \cap \mu = \sigma(A, q) \cap \sigma(B, q)$.

Thus σ is a lattice homomorphism. Hence σ is lattice embedding.

2.7.7 Remark: The above result justifies our modification of Pan's definition [1987] of Q- fuzzy R- sub modules wherein he assumes that the Q-fuzzy R-submodules necessarily assumes value 1 at zero of given module. Since Q-fuzzy R-submodules are Q-fuzzy normal subgroups.

2.7.8 Proposition: L_t is a modular lattice of L for each $t \in [0, 1]$.

2.7.9 Corollary: The set L (M) of all sub modules of M forms a modular lattice.

Proof: Let L_χ denote the set of characteristic functions of all the sub modules of M. It is easy to verify that L_χ is a sub lattice of L_t for t =1 . More over L (M) is isomorphic to L_χ under the map $A \rightarrow \chi_A$ are $\chi_{A+B} = \chi_A + \chi_{B.}$since sub lattice of a modular lattice is a modular lattice. L_t is modular and hence, L (M) is modular.

Conclusion: N. Ajmal and K.V Thomas [1994] introduced the concept of lattice of fuzzy subgroups and Osman Kazanci, Sultan Yamark and Serife Yimaz [2007] introduced notion of the intuitionist Q- fuzzy R-subgroups of near rings. we investigate the notion of Lattice valued Q- fuzzy left R- sub modules of near ring with respect to t-norm and characterization of them.