CHAPTER I
PRELIMINARIES OF VARIOUS FUZZY GROUPS

Introduction: Demirci [2001] showed that the concept of smooth groups and smooth homomorphism are introduced and their basic properties are investigated. Dib [1994] explained that the concept of fuzzy space is introduced. This concept corresponds to the concept of the universal set in the ordinary case. The algebra of fuzzy spaces and fuzzy subspaces are studied. Using the concept of fuzzy space and fuzzy binary operation, a new approach can be considered as a generalization and reformulation of the Rosenfeld theory of fuzzy groups. Therefore it is an active tool to develop the theory of fuzzy groups.

Dobrista and yahhyaeva [2002] showed that the Classification is suggested for the early notion of homomorphisms of fuzzy groups conditions are considered for fulfillment of various properties of homomorphisms of ordinary groups as well as properties specific to systems with fuzzy operation correctness is discussed of the introduction notion of preservation of fuzzy operation. The theory of fuzzy sets has developed in many directions and is finding applications in a wide variety of fields. Rosenfeld in 1971 used this concept to develop the theory of fuzzy groups .In this section we have given independent proof of several theorems on fuzzy groups. We discuss about Fuzzy groups and investigate some of their structures on the concept of Fuzzy Group family. Ray [1999] discussed that in the short communication, some properties of the product of two fuzzy subsets and fuzzy subgroups.

Rovento [2001] proved that the crisp environment the notions of normal subgroup and group operating on a set are well known due to many applications.

Section 1: Fuzzy group, Fuzzy normal subgroup, and Fuzzy subgroupoid

Rosenfeld [1971] defined fuzzy subgroupoids and proved that a homomorphic image of a fuzzy subgroupoid with the sup propert was a fuzzy groupoid and hence that a homomorphic image of a fuzzy subgroup with sup property was a fuzzy subgroup. This theorem needs the sup property, but we can show the theorem without sup property. Moreover Mukherjee and Battacharya [1991] showed that if [Ậ] is a fuzzy subgroup of a finite group G such that all the level subgroups of G are normal subgroups then [Ậ] is a fuzzy normal subgroup. They can also obtained the theorem without finites using the transfer principle which is a fundamental tool developed here.

There is a result that if A is fuzzy subgroup of G, then gAg^{-1} is also fuzzy subgroup of G for all g in G and $\cap$ gAg^{-1} is a normal subgroup of G (under a t- norm as A). If A and B be two fuzzy subgroups of G under the t- norms T_1 and T_2 respectively, then $A\cap B$ is a fuzzy subgroups under any t- norm T such that $T_1,T_2 \geq T$. The intersection of any two normal fuzzy subgroups of G is also a normal fuzzy subgroup of G under any t- norm weaker than the t- norms of the two fuzzy subgroups. Mukherjee and Battacharya [1991] also explained let f: $G \rightarrow H$ be a group homomorphism if A is a normal fuzzy subgroups of H, then $f^{-1}(A)$ is a normal fuzzy subgroup of G and if f is an epimorphiism then f(A) is a normal fuzzy subgroup of H. Also author derived that $G^t = \{ x \in G / A(x) \geq t \}$ is a subgroup of G. The normaliser of a fuzzy subgroup of G is a subgroup of G. Ray's results will be generalized. P-level subset and p-level subgroups are introduced in the thesis, and then we study of their algebraic properties.

We conclude that the concept of a normal fuzzy subgroup and proved some properties of this new concept. The theory of fuzzy sets applications in a many directions and is finding applications in a wide variety of fields. Rosenfeld in 1971 used this concept to develop the theory of fuzzy groups. we have given Independent proof of Several theorems on pseudo fuzzy cossets of fuzzy normal subgroups

ORGANIZATION OF THE THESIS

In Chapter I, basic concepts, previous works, needed definitions, organization of the thesis, all new results in various chapters, and contributions to the field of Q-Fuzzy group, S-semi group, S-product of anti fuzzy right R-subgroup of near rings, β-fuzzy congruence relations on a Q-Fuzzy groups, fuzzy M-group from group theory, some new theorems from fuzzy groups, max[i] interval valued anti fuzzy left h-ideals in hemi rings, fuzzy α-cut and L-fuzzy number in fuzzy group, algebraic properties of intuitionistic fuzzy groups, both homomorphic image and pre image of Pseudo fuzzy cossets in fuzzy groups, and their applications in decision making approach to optimize the product mix in assignment level under fuzzy group parameters. All the contributions mentioned above are newly introduced in each chapter.

In Chapter II, we fuzzify the new class of algebraic structures introduced by K.H.Kim [2006]. In this fuzzification, we introduce the notion of Q- fuzzy groups (QFG) and investigate some of their related properties. Some properties on group theory in Q- fuzzy groups are obtained. This fuzzification leads to development of new notions over fuzzy groups. Characterizations of Q- fuzzy groups (QFCG) and normal Q- fuzzy groups (QFNG) are given.

In Lattice Valued Q- fuzzy left R- sub modules of near rings with respect to T-norms, a technique of generating of Q- fuzzy R- sub module by a given arbitrary Q- fuzzy set was provided. It is shown that (i) The sum of two Q- fuzzy R- sub module of a module M is the Q- fuzzy R- sub module generated by their union and (ii) The set of all Q-fuzzy sub module of a given module forms a complete lattice. Consequently the collection of all Q-fuzzy R- sub module, having the same values at zero of M of the lattice of Q- fuzzy R- sub module of M. Interrelationship of these finite range sub lattices was established. Finally it was shown that the lattice of all Q- fuzzy R- sub module of M can be embedded into a lattice of Q- fuzzy R- sub module of M where M denote as Q-fuzzy R- sub module and R is the commutative near ring with unity. Characterization of Q-fuzzy left R- sub modules with respect to t- norm was also given.

In chapter III, the purpose is to define the β_q- fuzzy congruence by using special fuzzy equivalence relation of Q- fuzzy subgroups which is defined in this study and we define suitable Q-fuzzy subgroupoids and Q- fuzzy quotient subgroup of finite group G / H differently then we investigate some basic properties.

In chapter IV, we give a sufficient condition for a Q-Fuzzy subset to be a Q-cyclic fuzzy groups. By using this Q-Fuzzy cyclic group, we define a Q-Fuzzy cyclic group family, and fuzzify the new class of algebraic structure introduced by Kim [2006]. In this fuzzification (called fuzzy S-semi groups), introduced the notions of fuzzy sub S-semi groups and investigate some of their related properties. The purpose of leads to development of new notions over fuzzy S-semi groups. Introduced the notions of fuzzy sub algebra, intuitionistic fuzzy sub algebra in d-algebras and investigate some of their results.

In Chapter V, the theory of fuzzy sets has developed in many directions and is finding application in a wide variety of fields. Rosenfeld [1971] used this concept to develop the theory of fuzzy groups; we have given independent proof of several theorems on M- fuzzy groups. We discuss about M- fuzzy groups and investigate some of their structures on the concept of M- fuzzy group family.

In chapter VI, the concept of max ^{i}- interval valued anti fuzzy left h- ideals in a hemi rings and extension principle of interval valued fuzzy set are introduced. Some of their properties and structural characteristics, some theorems for homomorphic image are investigated and its inverse image on max ^{i} – interval valued anti fuzzy left h- ideals of a hemi rings is verified. Relationship between anti fuzzy left h- ideals in a hemi ring and fuzzy left h- ideals is also given. Using lower level set, a characterization of interval value anti fuzzy left h ideals is given.

In chapter VII, the intuitionistic fuzzification of the concept of groups and investigate some properties of such groups. The purpose leads to development of new notions over intuitionistic fuzzy groups (IFG) intuitionistic normal groups (ING) and takes some characterizations of them. We give some properties on α-cuts fuzzy groups.

In chapter VIII, the theory of fuzzy sets has developed in many directions and is finding applications in a wide variety of fields. Rosenfeld in 1971 used this concept to develop the theory of fuzzy groups. We have given independent proof of several theorems on fuzzy groups. We discuss about Fuzzy groups and investigate some of their structures on the concept of Fuzzy Group family. We have given Independent proof of several theorems on pseudo fuzzy cossets of fuzzy normal subgroups

In chapter IX, the improved algorithm benefits from the advantage of reaching optimal solution. In the previous researchers all inputs were considered as crisp values. The assumption is not in real cases. This work considers product mix problem as a group decision making problem in which all inputs are fuzzy. A new algorithm for optimizing product mix under fuzzy parameters is developed. For this method, ordering methods are used in order to make decision in a fuzzy group decision making environment.

Section 2: A New structure and Construction of Q-Fuzzy groups

Introduction: In Q-fuzzy left R-subgroups of near rings with respect to T- norms, Y. U. Cho, Y. B. Jun [2007] showed that if A is an intuitionistic fuzzy right R-subgroup of a near ring R then the set $R_A = \{ x \in R / \mu_A(x) = \mu_A(0), \gamma_A(x) = \gamma_A(0)\}$ is a right R- subgroup of R.

K.H Kim and Y. B. Jun [2001] established the notion of a normal fuzzy R-subgroup in a near ring and related properties discussed. They discussed about an S-norm on [0, 1]. If μ is sensible fuzzy R- subgroup of R with respect to S, then $\mu(0) \leq \mu(x)$ for all $x \in R$, and every sensible fuzzy R- subgroup of R with respect to S is an anti fuzzy R- subgroup of R.

Osmankazanci, sultan yamark and serife yilmaz [2007] identified that if $\{A_i\}_{i\varepsilon A}$ is a family of intuitionistic Q- fuzzy R- subgroups of R, then $\cap A_i$ is an intuitionistic Q- fuzzy R- subgroup of R. If $A= (\mu_A,\gamma_A)$ is an intuitionistic Q- fuzzy R- subgroup of R then if $\theta : R\rightarrow S$ be an epimorphism, $B = (\mu_B, \lambda_B)$ is an intuitionistic Q- fuzzy set in S, and $\theta^{-1}(B)= (\mu_{\theta\text{-}1}(B) , \lambda_{\theta\text{-}1}(B))$ is an intuitionistic Q-fuzzy R-subgroup of R, then B is an intuitionistic Q-fuzzy R-subgroup of S.

The notion of Q-fuzzification of left R- subgroups is introduced in a near-ring and investigated some related properties. Characterization of Q-fuzzy left R-subgroups with respect to a t-norm are given.

In the work titled a new Structure and Construction of Q- fuzzy groups, B. Davvaz, W. A. Dudek and Y. B. Yun, [2005] established the notion of intuitionistic fuzzy sets introduced by Atanassov as a generalization of the notion of fuzzy sets. They explained the concept of an intuitionistic fuzzy set to Hv- modules, they also introduced the notion of intuitionistic fuzzy Hv-sub modules of an Hv-sub modules and discussed with some properties.

F. H. Rho, K. H. Kim, and J. G. Lu [2006] showed that if A in X is an intuitionistic fuzzy Q- sub algebra of X, then $\mu_A(0, q) \geq \mu_A(x, q)$ and $\lambda_A(0, q) \leq \lambda_A(x, q)$, and they concluded that the intuitionistic Q-fuzzification of the concept of sub algebra in BCK/BCI algebra.

In Lattice Valued Q- fuzzy left R- sub modules of near rings with respect to T-norms, a technique of generating of Q- fuzzy R- sub module by a given arbitrary Q-fuzzy set was provided. It is shown that (i) The sum of two Q- fuzzy R- sub module of a module M is the Q- fuzzy R- sub module generated by their union and (ii) The set of all Q-fuzzy sub module of a given module forms a complete lattice. Consequently the collection of all Q-fuzzy R-sub module, having the same values at zero of M of the lattice of Q- fuzzy R- sub module of M. Interrelationship of these finite range sub lattices was established. Finally it was shown that the lattice of all Q- fuzzy R- sub module of M can be embedded into a lattice of Q- fuzzy R- sub module of M where M denote as Q-fuzzy R- sub module and R is the commutative near ring with unity. Characterization of Q-fuzzy left R- sub modules with respect to t- norm was also given.

1.2.1 Definition: A mapping $\mu: X \rightarrow [0, 1]$, where X is an arbitrary non-empty set, is called a fuzzy set in X.

1.2.2 Definition: Let G be any group. A mapping $\mu: G \rightarrow [0, 1]$ is a fuzzy group if (FG1) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$ (FG2) $\mu(x^{-1}) = \mu(x)$ for all $x, y \in G$.

1.2.3 Definition: Let Q and G be a set and a group respectively. A mapping $\mu: G \times Q \rightarrow [0, 1]$ is called Q – fuzzy set in G. For any Q- fuzzy set μ in G and $t \in [0, 1]$. The set $U(\mu; t) = \{x \in G / \mu(x, q) \geq t, q \in Q\}$ which is called an upper cut of 'μ' and can be used to the characterization of μ.

1.2.4 Definition: A Q-fuzzy set A is called Q- fuzzy group of G if (QFG1) $A(xy, q) \geq \min \{A(x, q), A(y, q)\}$ (QFG2) $A(x^{-1},q) = A(x, q)$, (QFG3) $A(e, q) = 1$ for all $x, y \in G$ and $q \in Q$.

1.2.5 Definition: Let 'θ' be a mapping from X to Y. If A and B are Q- fuzzy sets in X and Y respectively, then the inverse image of B under θ denoted by $\theta^{-1}(B)$ is Q- fuzzy set in X defined by $\theta^{-1}(B) = \mu_{\theta}{}^{-1}(B)$ where $\mu_{\theta}{}^{-1}{}_{(B)}(x, q) = \mu_B\theta(x, q)$ and $\mu_{\theta}{}^{-1}{}_{(B)}(x^{-1}, q) = \mu_B(\theta(x, q)$ for all $x \in Y$, $q \in Q$ and the image of A under θ denoted by θ(A), where $\mu_{\theta(A)}(y, q) = \{ V_{x\varepsilon\theta^{-1}(y)} \; \mu_A(x, q)$, if $\theta^{-1}(y) \neq 0$; 0, otherwise for all $y \in Y$, and $q \in Q$.

1.2.6 Definition: Let μ_A be a Q- fuzzy set of G. Let $\theta : G \times Q \rightarrow G$ be a map and define the map $\mu_A{}^{\theta}: G \times Q \rightarrow [0,1]$ by $\mu_A{}^{\theta}(x, q) = \mu_A(\theta(x, q))$.

1.2.7 Definition: Q-fuzzy group A of G is called Q-fuzzy characteristics of G if $\mu_A{}^{\theta} = \mu_A$.

1.2.8 Definition: Any Q-fuzzy group A of G is said to be normal if there exists $x \in G$ and $q \in Q$ such that $A(x, q) = 1$.

Note that if μ is normal Q- fuzzy group of G, then $A(e, q) = 1$ and hence A is normal if and only if $A(e, q) = 1$.

1.2.9 Definition: Let A be a Q- fuzzy group of G. Then A is called Q- fuzzy normal group (QFNG) if for all $x, y \in G$, $A(xy, q) = A(yx, q)$, $q \in Q$. Alternatively, a Q- fuzzy group A is said to be Q- fuzzy normal if $A(x, q) = A(yxy^{-1},q)$ for $x, y \in G$ and $q \in Q$. the notation [x, y] stands for the expression $x^{-1}y^{-1}xy$.

1.2.10 Definition: A non empty set with two binary operations + and . is called a near-ring if it satisfies the following axioms

(i) (R, +) is a group.
(ii) (R, .) is a semi group.
(iii) x. (y + z) = x. y + x. z for all x, y, z $\in$ R.

Precisely speaking it is a left near-ring. Because it satisfies the left distributive law,

1.2.11 Definition: R – subgroup of a near- ring 'R' is a subset H of R such that

(i) (H , +) is a subgroup of (R, +).

(ii) RH $\subseteq$ H

(iii) HR $\subseteq$ H.

If H satisfies (i) and (ii), then it is called left R- subgroup of R. If H satisfies (i) and (iii), then it is called a right R- subgroup of 'R'. A map f: R $\rightarrow$ S is called homomorphism if f (x + y) = f(x) + f (y) for all x, y in R.

1.2.12 Definition: Let R be a near ring. A fuzzy set μ in R is called fuzzy sub near ring in R if (i) $\mu(x-y) \geq \text{Min}\ \{\mu(x), \mu(y)\}$ (ii) $\mu(xy) \geq \text{Min}\{\mu(x), \mu(y)\}$ for all x, y in R.

1.2.13 Definition: A Q-fuzzy set μ is called a fuzzy left R- subgroup of R over Q if μ satisfies (i) $\mu(x\text{-}y, q) \geq T\ \{\mu(x, q), \mu(y, q)\}$ (ii) $\mu(rx, q) \geq \mu(x\ q)$.

1.2.14 Definition: (T- norm) A triangular form is a function T: [0, 1] × [0,1] $\rightarrow$ [0,1] that satisfies the following conditions for all x, y, z in [0,1]

1. T(x,1) = x; 2. T(x, y) = T(y, x)
3. T(x, T(y, z)) = T(T(x, y), z); 4. T(x, y) $\leq$ T(x, z) when y $\leq$ z. .

1.2.15 Definition: Let μ be a Q- fuzzy subset of a set S and t $\in$ [0,1], Then the set

$\mu_t = \{ x \in S\ /\ \mu(x, q) \geq t \}$, q $\in$ Q is called a level subset of μ.

1.2.16 Definition: A 'Q' – fuzzy subset μ of M is called a Q – fuzzy R- sub module of M if the following conditions are satisfied,

(i) $\mu(x+y, q) \geq T(\mu(x, q), \mu(y, q)\}$ for all x, y $\in$ M and

(ii) $\mu(rx, q) \geq \mu(x, q)$ for all r $\in$ M and q $\in$ Q.

For a Q- fuzzy left R – module of μ of M the level subset $\mu_t = \{ x \in M \ / \ \mu (x, q) \geq t \}$, $t \in I_m (\mu)$ are sub modules of M called the Q- level sub modules of M.

1.2.17 Definition: Let μ be a Q- fuzzy subset on M. Define a Q-fuzzy subset $< \mu >$ of M as follows: $< \mu > (x, q) = \sup \{ K / x \in < \mu_k >, x \in M$. $< \mu >$ is called the Q- fuzzy subset of M generated by μ. Here $< \mu_k >$ is the sub module of M generated by the level subset μ_k.

1.2.18 Definition: Let μ and θ be Q- fuzzy R- sub modules of an R- sub module M. Then the sum of μ and θ is denoted by $\mu + \theta$ is defined as

$(\mu + \theta) (x, q) = \sup \quad [T (\mu (a, q) , \mu (b, q))]$ for all $x_i \in M$.

$(x_i, q) = (a + b ,q)$.

Clearly $\mu + \theta$ is a Q- fuzzy subset of M.

1.2.19 Definition: Q-fuzzy R-sub module μ of a near ring is said to be normal if $\mu(0, q) = 1$.

In Crisp environment, the notion of cyclic group on a set is well known. We study an extension of this classical notion to the Q- fuzzy sets to define the concept of Q- cyclic fuzzy groups. By using these Q- cyclic fuzzy groups, we then define a Q- cyclic fuzzy group family and investigate its structure properties with applications.

We study extensions of these classical notions to the larger universe of fuzzy sets. We obtain a characterization of operations of fuzzy group on a fuzzy set in terms of homomorphisms of crisp groups. Ray [1999] studied some results of the product of fuzzy sets and fuzzy subgroups.

We fuzzify the new class of algebraic structures introduced by K. H .Kim [2006]. In this fuzzification, we introduce the notion of Q-fuzzy groups (QFG) and investigate some of their related properties. Some properties on group theory in Q- fuzzy groups are obtained. This fuzzification leads to development of new notions over fuzzy groups. Characterizations of Q- fuzzy groups (QFCG) and normal Q- fuzzy groups (QFNG) are given.

The following results on the properties of Q-fuzzy group are obtained.

1. Let A be a Q- fuzzy group of G. Then $A(x, q) \leq A(e, q)$ for all $x \in G$ and $q \in Q$. (ii) The subset $G_A = \{ x \in G / A(x, q) = A(e, q)\}$ is a Q- fuzzy group of G. Let A and B be two Q- fuzzy groups of a group G. Then $(A \cap B)$ is Q-fuzzy group of G. If 'A' is a Q- fuzzy group of G, then A^C is also Q- fuzzy group of G.

2. If A is Q- fuzzy group of G, then the set U(A; t) is also Q-fuzzy group for all $q \in Q$, $t \in Im (A)$.

3. Let G and G^1 be two groups and $\theta: G \rightarrow G^1$ a homomorphism. If B is Q fuzzy group of G^1, then the pre image $\theta^{-1}(B)$ is Q- fuzzy group of G.

4. Let $\theta: G \rightarrow G^1$ be an epimorphism and B is Q – fuzzy set in G^1. If $\theta^{-1}(B)$ is Q- fuzzy group of G, then B is Q- fuzzy group of G^1.

5. If $\{A_i\}_{i \varepsilon A}$ is a family of Q-fuzzy groups of G, then $\cap A_i$ is Q-fuzzy group of G where $\cap A_i = \{((x, q) \ \Lambda \ \mu_{Ai}(x, q)) / x \in G , q \in Q\}$, where $i \in A$.

6. If A is Q- fuzzy set in G such that all non- empty level subset U (A; t) is Q-fuzzy group of G, then A be Q- fuzzy group of G.

7. A set of necessary and sufficient conditions for a Q- fuzzy set of a group G to be a Q- fuzzy group of G is that $A(xy^{-1}, q) \geq \min (A(x, q), A(y, q)\}$ for all x, y in G and q in Q.

8. If A is Q- fuzzy group of G and θ is a homomorphism of G, then the Q-fuzzy set A^θ of G given by $A^\theta = \{< (x, q), \mu_A{}^\theta (x, q)>; x \in G, q \in Q\}$is Q-fuzzy group of G.

9. Let A be Q–fuzzy group of G. Let A^+ be a Q- fuzzy set in G defined by $A^+ (x, q) = A(x, q) + 1 - A(e,q)$ for all $x \varepsilon G$. Then A^+ is normal Q- fuzzy group of G which contains A.

10. Let A be a QFNG of a group G. Then for all $x, y \in G$, $A ([x, y], q) = A(e, q)$.

11. If A is QFCG of a group G, then A is QFNG of G.

12. Let T be a t- norm. Then every imaginable Q- fuzzy left R- subgroup μ of a near ring S is a fuzzy left R-subgroup of S.

13 If μ is a Q-fuzzy left R- subgroups of a near ring S and Θ is an endomorphism of S, then μ[Θ] is a Q- fuzzy left R- subgroup of S.

14. An onto homomorphism of a Q- fuzzy left R-subgroup of near ring S is Q- fuzzy left R- subgroup.

15. An onto homomorphic image of a fuzzy left R- subgroup with the sup property is a fuzzy left R- subgroup.

16. Let T be a continuous t-norm and f be a homomorphism on a near ring S. If μ is Q-fuzzy left R- subgroup of S, then μ^f is a Q- fuzzy left R- subgroup of f(S).

17. Let μ be a Q fuzzy R- sub module of M. Then the Q- fuzzy subset<μ> is a Q- fuzzy R- sub module of M generated by. More over <μ> is the smallest Q- fuzzy R- sub module containing μ.

18. Let μ and θ be a Q- fuzzy R- sub modules of M such that μ (0,q) = θ (0,q). Then $\mu \subseteq \mu + \theta$, and $\theta \subseteq \mu + \theta$.

19. Let μ and θ be a Q- fuzzy R- sub module of M such that Let T be a t- norm. Then every imaginable Q- fuzzy left R- sub module μ of a near ring ' R' is a fuzzy left R-sub module of R. μ (0,q) = θ (0,q) implies that $\mu + \theta = < \mu + \theta >$.

20. Let μ be a Q- fuzzy R- sub module of a near ring and let μ* be a Q- fuzzy set in R defined by μ*(x, q) = μ(x, q) +1- μ(0,q) for all $x \in R$. Then μ* is a normal Q- fuzzy R- sub module of R containing μ.

21. If μ is a Q- fuzzy left R- sub module of a near ring R and Θ is an endomorphism of R, then $\mu_{[\Theta]}$ is a Q- fuzzy left R- sub module of R.

22. An onto homomorphism of a Q- fuzzy left R- sub module of near ring R is Q- fuzzy left R- sub module.

23. Let T be a continuous t-norm and let f be a homomorphism on a near ring R. If μ is Q- fuzzy left R- sub module of R, then μ^f is a Q- fuzzy left R-sub module of f(R).

24. L is a lattice under the usual ordering of Q- fuzzy set inclusion. More over L is a complete lattice, and L_t is a complete lattice of L.

25. L_t is a sub lattice of L. Let L_t denote the set of all Q- fuzzy R- sub modules θ of M such that θ (0,q) = t and $I_m\theta$ is finite

26. L(M), the lattice of all sub modules of M can be embedded in L_2.

Section 3: Q- Fuzzy subgroups of β-Fuzzy Congruence relation on a group

Introduction The concept of fuzzy sets was first introduced by Zadeh in [1965] and since then there has been a tremendous interest in the subject due to diverse applications ranging from engineering and computer science to social behavior studies. The concept of fuzzy relation on a set was defined by Zadeh [1965] and other authors like Rosenfeld [1971] , The notion of fuzzy congruence on a group was introduced by Kuroki [1992] and that the universal algebra was studied. The study of fuzzy algebraic structures was started with the introduction of the concept of fuzzy subgroups by Rosenfeld in [1971]. Several mathematicians have followed the Rosenfeld approach in investigating the fuzzy subgroup theory. Fuzzy normal subgroups were studied by Wu [1981] and Dib [1998] , Kumar et.al. [1995] and Mukherjee [1985] . The concept of fuzzy quotient group was studied.

1.3.1 Definition: (i) A fuzzy relation A on X is said to be reflexive if A(x, x) = 1 for all $x \in X$ and said to be symmetric if A(x, y) = A(y, x) for all x, y in X.(ii) If A_1 andA_2 are two relations on X, then their max- product composition denoted by A_1º A_2 is defined as A_1º A_2 (x, y) = max { A_1(x, z) ,A_2(z, y)}.(iii) If $A_1 = A_2 = A$ say and Aº A ≤ A, then the fuzzy relation A is called transitive.

1.3.2 Definition: A fuzzy binary relation A in X is called similarity relation if A is reflexive, symmetric and transitive.

1.3.3 Definition: Let S be a semi group. A fuzzy binary relation A on S is called fuzzy left (right) compatible if and only if $A(x, y) \leq A(tx, ty)$ for all $x, y, t \in S$ ($A(x, y) \leq A(xt, yt)$ for all $x, y, t \in S$).

1.3.4 Definition: A fuzzy binary relation A on a semi group S is called fuzzy compatible if and only if $\min\{A(a, b), A(c, d)\} \leq A(ac, bd)$ for all $a, b, c, d \in S$.

1.3.5 Definition: Fuzzy compatible similarity relation on a semi group S is called fuzzy congruence.

1.3.6 Definition: Let G be a group with identity e and A_H be a Q- fuzzy subgroup of G. A fuzzy relation β_q can be defined on G by $\beta_q(a, b) = \min \; A_H(a, q) , A_H(b, q)\}$, if $(a, q) \neq (b, q)$; $A_H(e, q)$ if $(a, q) = (b, q)$.

1.3.7 Definition: If a Q- fuzzy set is a Q- fuzzy subgroup of G / H, then it is called Q- fuzzy quotient subgroup. Similarly, if it is a Q- fuzzy normal subgroup of G / H, then it is called Q- fuzzy quotient normal sub group. By using the Q- fuzzy congruence β_q, we define a special function N as follows.

1.3.8 Definition: Let G be group and A_H be Q-fuzzy normal subgroup of G. $N : G / H\times Q \rightarrow [0, 1]$ can be defined by $N(xH, q) = \beta_q(x, h)$ for all $h\in H$ and $q \in Q$.

1.3.9 Definition: For all $(xH, yH) \in G/H\times G/H$, the Q- fuzzy relation μ_N on G / H is defined by $\mu_N(xH, yH) = N(xHy^{-1}H,q)$ where $q \in Q$.

1.3.10 Definition: By a s- norm S , we mean a function $S: [0, 1]\rightarrow [0, 1]$ satisfying the following conditions ;

(S1) $S(x , 0) = x$

(S2) $S(x, y) \leq S(x, z)$ if $y \leq z$

(S3) $S(x, y) = S(y, x)$

(S4) $S(x, S(y, z)) = S(S(x, y),z)$, for all $x, y, z \in [0,1]$.

Replacing 0 by 1 in condition S1 we obtain the concept of t- norm T.

1.3.11 Definition: For a S-norm, then the following statement holds $S(x, y) \geq \max\{x, y\}$, for all $x, y \in [0, 1]$.

1.3.12 Definition: Let S be a s-norm. A fuzzy set μ in R is said to be sensible with respect to S if $I_m(\mu) \subset \Delta s$, where $\Delta s = \{ s(\alpha, \alpha) = \alpha / \alpha \in [0,1] \}$.

1.3.13 Definition: Let (R,+, ·) be a near-ring. A fuzzy set μ in R is called an anti fuzzy right (resp. left) R- subgroup of R if

(AF1) $\mu(x-y) \leq \max \{ \mu(x) , \mu(y) \}$, for all $x, y \in R$.

(AF2) $\mu(xr) \leq \mu(x)$ for all $r, x \in R$.

1.3.14 Definition: Let (R,+, ·) be a near-ring. A fuzzy set μ in R is called a fuzzy right (resp. left) R-subgroup of R if

(FR1) μ is a fuzzy subgroup of (R,+).

(FR2) $\mu(xr) \geq \mu(x)$ (resp. $\mu(rx) \geq \mu(x)$) , for all $r, x \in R$.

1.3.15 Definition: Let S be a s- norm. A function $\mu : R \rightarrow [0,1]$ is called a fuzzy right (resp. left) R- subgroup of R with respect to S if

(C1) $\mu(x-y) \leq S(\mu(x), \mu(y) \}$

(C2) $\mu(xr) \leq \mu(x)$ (resp. $\mu(rx) \leq \mu(x)$ for all $r, x \in R$. If a fuzzy R-subgroup μ of R

With respect to S as sensible, then μ is a sensible fuzzy R- subgroup of R with respect to S.

1.3.16 Example: Let K be the set natural numbers including 0 and K is a R-subgroup with usual addition and multiplication.

1.3.17 Definition; A fuzzy subset $\mu: R \rightarrow [0,1\}$ by $\mu(x) = 0$ if x is even; = 1, otherwise.

and let $S_m : [0, 1] \rightarrow [0, 1]$ by a function defined by $S_m(\alpha, \beta) = \min \{x + y , 1 \}$ for all $x, y \in [0,1]$. Then S_m is a t-norm, we know that μ is sensible R-fuzzy subgroup of R.

1.3.18 Definition: Let f be a mapping defined on Rand ψ be a fuzzy subset in f(R). Then the fuzzy subset $\mu = (\psi of)$ defined by $\mu(x)=\psi(f(x))$ for all x in R is called the pre image of ψ under f.

1.3.19 Definition: A s- norm S on [0, 1] is called a continuous function from $[0, 1] \times [0, 1] \rightarrow [0, 1]$with respect to the usual topology. We observe that the function max is always a continuous S- norm.

1.3.20 Definition: A fuzzy relation on any set X is a fuzzy set $\mu: X \times X \rightarrow [0, 1]$.

1.3.21 Definition: Let S be a s- norm. If μ is a fuzzy relation on a set R and χ be fuzzy set in R, Then μ is a S- fuzzy relation on χ if $\mu_\chi(x, y) \geq S(\chi(x), \chi(y))$ for all $x, y \in R$.

1.3.21 Definition: Let S be a s- norm. let μ and χ be a fuzzy subset of R. Then direct S- product of μ and χ is defined as $(\mu \times \chi)(x, y) = S(\mu(x), \chi(y))$, for all $x, y \in R$.

1.3.22 Definition: Let S be a s- norm . let μ be a fuzzy subset of R . Then μ is called strongest S- fuzzy relation on R if $\mu_\chi(x, y) \geq S(\chi(x), \chi(y))$ for all $x, y \in R$.

We introduced the notion of Q- fuzzy subgroups. In this study, we define some new special fuzzy equivalence relations and derive some simple consequences. Then using those relations we define suitable Q-fuzzy subgroupoids and Q- fuzzy quotient subgroup of G / H differently.

The following results are obtained in chapter III:

27. Let G be a group with identity e and A_H be a Q- fuzzy subgroup of a group G, Then the relation β_q defined on G is Q-similarity relation on G. Further The fuzzy relation β_q defined on G is Q- fuzzy compatible.

28. The fuzzy relation β_q defined on G is a Q- fuzzy congruence.

29: The defined fuzzy set N is a Q- fuzzy quotient subgroup of G/H.

30: The defined fuzzy set N is a Q- fuzzy quotient normal subgroup of G/H.

31: If N is a Q- fuzzy quotient subgroupoid of finite group G/H, then N is a Q- fuzzy subgroup.

32: Let N be a Q- fuzzy quotient subgroup of a group G/H and let xH $\in$ G / H. Then N(xHyH, q) = N(yH, q), for all yH$\in$ G/H $\leftrightarrow$ N(xH, q) = N(H, q).

33: Let N and R be two Q- fuzzy quotient subgroups of G/H, Then N∩R is a Q- fuzzy quotient normal subgroup of G / H, and the Q-fuzzy relation μ_N is a Q- fuzzy congruence on G / H.

S-anti fuzzy right R- subgroups

Introduction: B. Schweizar and A.Sklar [1963] introduce the notions of Triangular norm (t-norm) and Triangular co-norm (S-norm) are the most general families of binary operations that satisfy the requirement of the conjunction and disjunction operators respectively. First,Abu. Osman [1987] introduced the notion of fuzzy subgroup with respect to t-norm. S. Abou. Zaid [1991] also introduced the concept of R-subgroups of a near-rings and Kyunghokim [2007] introduced the concept of fuzzy R- subgroups of a near-ring. Then J. Zhan [2005] introduced the notion of fuzzy hyper ideals in hyper near-rings with respect to t-norm. Recently, Y.U.Cho et,al [2005] introduced the notion of fuzzy sub algebras with respect to S-norm of BCK algebras and M.Akram [2006] introduced the notion of sensible fuzzy ideal.

We redefine anti-fuzzy right R- subgroups of a near-ring R with respect to a S-norm and investigate it is related properties. Also, we review several results described and using S-norm.

The following results are also obtained in chapter III

34: Let S be a s-norm. Then every sensible S-anti fuzzy right R- subgroups μ of R is anti- fuzzy R- subgroups of R.

35: If μ is a S- anti fuzzy right R-subgroups of a near ring R and θ is an endomorphism of R, then μ[θ] is a S- anti fuzzy right R- subgroups of R.

36: An onto homomorphic pre image of a S- anti fuzzy right R- subgroups of a near- ring is S- anti fuzzy right R- subgroups.

37: An onto homomorphic image of a anti fuzzy right R- subgroups with the inf property is a anti-fuzzy right R- subgroups.

38: Let f: $R \rightarrow R^1$ be a homomorphism of near-rings. If μ is a S- anti fuzzy right R- subgroups of R^1, then μ^f is S- anti fuzzy right R- subgroup of R.

39: Let f; $R \rightarrow R^1$ be a homomorphism of near-rings. If μ^f is a S- anti fuzzy right R- subgroups of R, then μ is S- anti fuzzy right R- subgroup R'. right R- subgroup of R^1.

40: Let S be a continuous S- norm and let f be a homomorphism on a near-ring R. If μ is a S- anti fuzzy right R- subgroups of R, then μ^f is a S- anti fuzzy right R- subgroups of f(R).

41: A fuzzy subset μ of R is a T- anti fuzzy right R- subgroups if and only if μ^c is a S anti fuzzy right R- subgroup of R.

The theorems are got about S-Product of S-anti Fuzzy right R-subgroups in Chapter III

42: Let S be a s- norm. let μ and χ be a S- anti fuzzy right R- subgroup of R, then $\mu \times \chi$ is a anti fuzzy right R- subgroup of R.

43: Let μ and χ be sensible S- anti fuzzy right R- subgroups of a near- ring R. Then $\mu \times \chi$ is a sensible S- anti fuzzy right R- subgroup of R×R.

44: If $\mu \times \chi$ is a sensible S- anti fuzzy right R- subgroup of R×R , Then $\mu \times \chi$ need not be sensible S- anti fuzzy right R- subgroup of R.

Section 4: Some structure properties of Q-Cyclic Fuzzy group family and BCK-Algebra

Introduction: The original concept of fuzzy sets was firstly introduced in the pioneering work [1965] of Zadeh as an extension of crisp (usual) sets, by enlarging the truth value set of grade of membership from the two sets {0, 1} to the unit interval [0, 1] of real numbers. There has been tremendous interest in the fuzzy set theory due to its many applications ranging from engineering and computer science to social behavior studies. More details and historical background of fuzzy set theory.

There is a quite substantial literature on fuzzy group theory. The study of fuzzy groups was started firstly by Rosenfeld [1971]. He used the *min* operating to define his fuzzy groups and showed how some basic notions of fuzzy group theory should be extended in an elementary manner to develop the theory of fuzzy groups. It was extended by Anthony and Sherwood [1979]. They used the *t- norm* operating instead of the *min* to define the t- fuzzy groups. Roventa and Spircu [2001] introduced the fuzzy group operating on fuzzy sets. Sidkey and Misherf [1991] defined t- cyclic fuzzy groups by using t- level sets in the crisp environment. Ray [1993] defined a cyclic fuzzy group of a given fuzzy group family. In this chapter, we give a sufficient condition for a Q- fuzzy subset to be a Q- cyclic fuzzy group. By using this Q- cyclic fuzzy group, we then define a Q- cyclic fuzzy group family and investigate its structure properties with applications.

Ray [1993] established subgroups and normal subgroups of a fuzzy group and their criteria's. The fuzzy order of an element of a group also defined and its relationship with the order of the element is examined. The construction of the smallest fuzzy group containing a given arbitrary fuzzy set is systematized. A cyclic fuzzy group that is a restriction of a fuzzy group that is a restriction of a fuzzy group or that is generated by a fuzzy point is brought to discussion. Finally he explained the usual interactions between a cyclic and abelian fuzzy group. He also pointed out the possibility of an abelian fuzzy group might be isomorphic to a direct product of some of its cyclic subgroups is touched upon.

Kim [2006] considered the intuitionistic Q- fuzzification of the concept of sub algebras in BCK/BCI algebra and he explained (i) Let A be an intuitionistic Q- fuzzy sub algebra of X, Then $X_A^{(\alpha,\beta)}$ is a sub algebra of X with $\alpha + \beta \leq 1$. (ii) Any sub algebra of X can be realized as both a μ- level sub algebra and a γ- level algebra of some intuitionistic Q- fuzzy sub algebra of X. (iii) Let $f : X \rightarrow Y$ be a homomorphism from a BCK/BCI algebra X onto a BCK/BCI algebra. If A is an intuitionistic Q- fuzzy sub algebra of X, then the image f(A) is intuitionistic Q- fuzzy sub algebra of X.

Kim [2006] introduced the new class of algebra's related to BCK algebra's and semi groups called KS-semi group and define an ideal of a KS- semi groups and a strong KS-semi groups. He also defined a congruence relation on KS-semi groups and quotient KS-semi group and proved (i) every p-ideal of a KS-semi group X in an ideal but the converse is not true. (ii) Let X be a strong KS-Semi group with a unity 1 and A any non- empty subset of X. If $g \in A$ and $x \leq y$ imply $x \in A$ then A is an ideal of X. (iii) Let X be a KS-semi group. Then an equivalence relation ρ on X is congruence if and only if is both left and right compatible. (iv) Let $f : X \rightarrow Y$ be a homomorphism of KS-semi groups. Then kerlf is an ideal of X.

Abu osman [1987] explained the closure operator on the set of fuzzy relation on S and to show that (i) The closed hull of a fuzzy relation (s, μ) is given by $\acute{\Gamma}(s, \mu) = (s\sigma, \mu)$. (ii) The composition of two closed fuzzy relations need not be closed fuzzy relations.

Kim [2000] introduced the notion of sensible fuzzy R- subgroups in near rings, and he showed that (i) Every sensible fuzzy R- subgroup of R with respect to S-norm s s an anti fuzzy R- subgroup of R. (ii) An onto homomorphic image of a fuzzy right R- subgroup with respect to s is a fuzzy right R- subgroup. (iii) If μ is a fuzzy right R- subgroup of R with respect to s and θ is an endomorphism's of R, then μ[θ] is a fuzzy right R-subgroup of R with respect to s.

1.4.1 Definition: Let A and B be fuzzy sets. Then A is a subset of B if $\mu_A(x) \leq \mu_B(x)$ for every $x \in U$ and it is denoted by $A \subseteq B$ or $B \supseteq A$.

1.4.2 Definition: Two fuzzy sets A and B are called equal if $\mu_A(x) = \mu_B(x)$ for every $x \in U$ and it is denoted by A= B

1.4.3 Definition: Let A and B be fuzzy sets. Then the algebraic product of two fuzzy sets A and B is defined by $A \bullet B = \{ (x, \mu_{\tilde{A}}(x)) / x \in U, \mu_{A\bullet B} = \mu_A \bullet \mu_B \}$

1.4.4 Definition: Let A and B be fuzzy sets. Then the Union $A \cup B$ and Intersection $A \cap B$ are respectively defined by the equations.

$A \cup B = \{(x, \mu_{A \cup B}(x)) / x \in U, \mu_{A \cup B}(x) = \max\{(\mu_A(x), \mu_B(x)\}\}$ and

$A \cap B = \{(x, \mu_{A \cap B}(x)) / x \in U, \mu_{A \cap B}(x) = \min\{(\mu_A(x), \mu_B(x)\}\}$.

1.4.5 Remark: These definitions can be generated for countable number of fuzzy sets. If $\tilde{A}_1, \tilde{A}_2, ...,$are fuzzy sets with membership functions $\mu_{\tilde{A}1}(x), \mu_{\tilde{A}2}(x), ...,$then the membership functions of $X = U \tilde{A}_i$ and $Y = \cap \tilde{A}_i$ are defined as

$\mu_X(x) = \max\{\mu_{\tilde{A}1(x)}, \mu_{\tilde{A}2(x)...}\}, x \in U$ and $\mu_Y(x) = \min\{\mu_{\tilde{A}1}(x), \mu_{\tilde{A}2}(x), ...\}, x \in U$ respectively.

1.4.6 Definition: Let f be a non- fuzzy function from X to Y. The image $f(\tilde{A})$ of a fuzzy set $\tilde{A}$ on X is defined by means of the extension principle as

$$f(\tilde{A}) = \{ (y, q) , \mu_{f(\tilde{A})}(y, q) \ / \ y = f(x) , x \in X \} ,$$

where
$$\mu_{f(\tilde{A})}(y, q) = \begin{cases} \sup_{(x,q)\varepsilon f^{-1}(y,q)} \mu_{\tilde{A}}(x, q) & \text{if } f^{-1}(y, q) \neq \Phi \\ 0 & \text{otherwise} \end{cases}$$

1.4.7 Definition: Let G = < a > be a cyclic group. If $\tilde{A} = \{ (a^n, \mu(a^n)) / n \in z \}$ is a fuzzy group, then $\tilde{A}$ is called a cyclic fuzzy group generated by $(a, \mu(a))$ and denoted by < a, $\mu(a)n$ >.

1.4.8 Definition: (Q- Cyclic fuzzy group**)** Let A = < a, q> be a Q- cyclic group of a group G.

If $\tilde{A} = \{ ((a^n, q) , \mu(a^n, q)) \ / n \in z \}$ is a Q- fuzzy group, then $\tilde{A}$ is called a Q- cyclic fuzzy group generated by < (a, q) , μ (a, q) >.

1.4.9 Definition: Let e be the identity element of the group A. we define the identity Q- fuzzy group E by E = { {e, q) , $\mu_{\tilde{A}}$(e, q) / $\mu_{\tilde{A}}$(e , q) = 1}.

1.4.10 Remark: Since a Q- cyclic fuzzy group is an abelian group, it is clear that μ(xy, q) = μ(yx, q) for x,y ∈ A, q ∈Q. Therefore, the Q- cyclic fuzzy groups $\tilde{A}^m$, $\tilde{A}_i \cup \tilde{A}_j$ and $\tilde{A}_i \cap \tilde{A}_j$ are also normal Q- fuzzy groups.

1.4.11 Definition: Let Ã be a Q- cyclic fuzzy group, then the following set of Q- cyclic fuzzy groups { Ã , $\tilde{A}^2$, $\tilde{A}^3$, …, $\tilde{A}^m$,…. E } is called the Q- cyclic fuzzy group family generated by Ã. It will be denoted by < Ã >.

1.4.12 Definition : An algebra (X,*,0) is called BCK-algebra if it satisfies the following conditions

1. ((x*y) * (x*z) * (z*y) = 0
2. (x*(x*y))*y) = 0
3. x*x = 0
4. x*y=0, y*x = 0 → x=y
5. 0*x = 0 for all x,y,z in X

1.4.13 Definition: A fuzzy set 'A' in d-algebra X is called a fuzzy sub algebra of X if it satisfies 4A(x*y) > min {A(x),A(y)} for all x, y in X and 'B' be two fuzzy sets in X. The Cartesian product (A×B) (x,y) = min {A(x), A(y)} for all in x,y in X.

1.4.14 Definition: An intuitionistic fuzzy set A = < t_A, f_A > in X is called an intuitionistic fuzzy sub algebra of 'X' if it satisfies the following conditions

t_A(x*y) > min { t_A(x) , t_A(y)}

f_A(x*y) < mix { f_A(x) , f_A(y)} for all x, y in X.

1.4.15 Definition: A S-semi group is a non-empty set X with two binary operations * and a constant 0 satisfies the following axioms

(s1) (X,*,0) is a d-algebra.

(s2) (X,˙) is a semi group.

(s3) x ˙ (y*z) = (x˙ y) * (x˙ z) and (x*y) ˙z = (x˙ z) * (y˙ z) for all x, y, z in X

1.4.16 Remarks: X denotes the S-semi group unless or otherwise specified. For the sake of convenience, we shall write the implication x.y by xy.

1.4.17 Definition: Let X be a non-empty set. A fuzzy subset of X is a function $\mu : X \rightarrow [0,1]$. Let μ be a fuzzy subset of X. for a fixed $0 < t < 1$, the set $\mu_t = \{x \in X / \mu(x) \geq t\}$ is called level set of μ.

1.4.18 Definition: A non-empty subset A of X with binary operations * and is called sub S-semi group of X if it satisfies the following conditions $x*y \in A$ and $xy \in A$ for all x, y $\in X$.

1.4.19 Definition: Let X be a sub S-semi group if it satisfies the following conditions.(i) $A(x*y) \geq \min\{A(x), A(y)\}$ and (ii)$A(xy) \geq \min\{A(x), A(y)\}$ for all x, y in X.

The following results are proved in chapter IV.

45. Let A be a Q- fuzzy group of G .Then (i)$A(x, q) \leq A(e, q)$ for all $x \in G$ and $q \in Q$. (ii) The subset $G_A = \{ x \in G / A(x, q) = A(e, q)\}$ is a Q- fuzzy group of G.

46: Let G be a finite group and A be a Q- fuzzy group of G. Consider the subset H of G given by $H = \{x \in G / A(x, q) = A(e, q)\}$ .Then H is a crisp subgroup of G.

47: If A is Q- fuzzy group of G, then the set U(A; t) is also Q-fuzzy group for all $q \in Q$, $t \in Im(A)$.

48: If A is Q- fuzzy set in G such that all non- empty level subset U(A; t) is Q- fuzzy group of G then A is Q- fuzzy group of G.

49*:* A set of necessary and sufficient conditions for a Q- fuzzy set of a group G to be a Q- fuzzy group of G is that $A(xy^{-1}, q) \geq \min (A(x, q) , A(y, q)\}$ for all x,y in G and q in Q.

50: If G is a group, then $\tilde{A}^m = \{ (a^n, q) , \mu_{\tilde{A}}(a^n, q)^m / n \varepsilon z \}$ is also a Q –cyclic fuzzy group.

51: Let A be a Q- cyclic group with 12 elements and generated by (a, q). Let Ã be a Q- fuzzy set of the group A defined as follows. $\mu_{\tilde{A}}(a^0,q) = 1$, $\mu_{\tilde{A}}(a^4,q) = \mu_{\tilde{A}}(a^8,q) = t_1$, $\mu_{\tilde{A}}(a^2,q) = \mu_{\tilde{A}}(a^6,q) = \mu_{\tilde{A}}(a^{16},q) = t_2$ and $\mu_{\tilde{A}}(x) = t_3$ for all other elements x in A, where $t_1,t_2,t_3 \varepsilon [0,1]$ with $t_1 > t_2 > t_3$. It is clear that Ã is a Q- fuzzy group of A, thus $\tilde{A} = \{ (a^k, q) , \mu_{\tilde{A}}(a^k,q) / k \in z \}$ is a Q- cyclic fuzzy group generated by ((a, q) , $\mu_{\tilde{A}}(a, q)$).

52: The Q- fuzzy group $\tilde{A}^n$ is a Q- fuzzy subgroup of $\tilde{A}^m$, if $m \leq n$.

53: If $\tilde{A}^i$ and $\tilde{A}^j$ are Q- cyclic fuzzy groups, then $\tilde{A}^i \cup \tilde{A}^j$ is also Q- cyclic fuzzy group if $i < j$

54: If $\tilde{A}_i$ and $\tilde{A}_j$ are Q- cyclic fuzzy groups, then $\tilde{A}_i \cap \tilde{A}_j$ is also a Q- cyclic fuzzy group.

55: Since a Q- cyclic fuzzy group is an abelian group, it is clear that $\mu(xy, q) = \mu(yx, q)$ for $x,y \in A$, $q \in Q$. Therefore, the Q- cyclic fuzzy groups $\tilde{A}^m$, $\tilde{A}_i \cup \tilde{A}_j$ and $\tilde{A}_i \cap \tilde{A}_j$ are also normal Q- fuzzy groups.

56: Let $< A > = \{ A , A^1 , A^2$,----, A^m, ---, E. Then $\cup\ A^P = A$ and $\cap\ A^P = E$ where m varies 1 to ∞.

57: Let $\tilde{A}$ be a Q- cyclic fuzzy group. Then $A \supset A^2 \supset A^3 \ldots . \supset A^m \ldots \supset E$.

58: Let $< \tilde{A} > = \{ \tilde{A} , \tilde{A}^2 , \tilde{A}^3 \ldots \tilde{A}^m \ldots E \}$. Then $\tilde{A} \supset \tilde{A}^2 \supset \ldots . \supset \tilde{A}^m \ldots . \supset E$.

59: Let f be a group homomorphism's of a Q- cyclic fuzzy group $\tilde{A}$. Then the image of $\tilde{A}$ under f is a Q- cyclic fuzzy group.

60: Let $\{\tilde{A}^m, \tilde{A}^{m-1}, \ldots, \tilde{A}\}$ be a finite Q-cyclic fuzzy group family. So $\tilde{A}^m \times \tilde{A}^{m-1} \times \ldots . \times \tilde{A} = \tilde{A}^m$.

Section 5: Some Algebraic properties of BCK-algebra's and Fuzzy S-algebra

Y. Imai and K. Iseki [1966] introduced two classes of abstract algebras; BCK - algebras and BCI–algebras. It is known that the notion of BCI-algebras is a generalization of BCK–algebras. J. Neggers and H.S Kim [1999] introduced the class of d-algebras which is another generalization of BCK-algebras and investigated relations between d-algebras and BCK-algebras. A. Rosenfeld [1971] introduced the notion of fuzzy group. Following the idea of fuzzy groups, O.G Xi [1991] introduced the notion of fuzzy BCK-algebras. After that, Y.B Jun

[1992] studied fuzzy BCK-algebras. Recently, the new class of algebraic structure introduced by Kim [2006], called S-semi group which is the combination of BCK-algebras and semi groups. In this section, we fuzzify the new class of algebraic structure introduced by Kim [2006]. We have proved some interesting results which are very closer to the results in BCK – algebras.

1.5.1 Definition: A group with operators is an algebraic system consisting of a group G, a set M and a function defined in the product set M×G and having the values in G such that if mx denotes the element in G determined by the element x of G and the element m of M, and $m \in M$, then G is called M- group with operators.

1.5.2 Definition: A subgroup A of an M- group G is said to be the fuzzy subgroup if $mx \in A$ for all $m \in M$ and $x \in A$.

1.5.3 Definition: Let A be a fuzzy set in U and $\bullet : G \times G \to G$ be a composition law, such that $(G, \bullet)$ forms M- group. Let two conditions be (FG1) $A(m(xy)) \geq \min \{ A(mx), A(my)\}$ and (FG2) $A(mx^{-1}) = A(mx)$ for all x, y in A. If the supplementary conditions $A(me_G) = 1$ is also satisfied, then the M-fuzzy group is called a standardized M- fuzzy group, where e_G is an identity of M- group $(G, \bullet)$.

1.5.4 Definition: Let $f: G \to G^1$ be a homomorphism's of M- groups. For any fuzzy set $A \in G^1$ we define a new fuzzy set $A^f \in G$ by $A^f(mx) = Af(mx)$ for all $x \in G$.

1.5.5 Definition: Let A be a M- fuzzy group, then the following set of M- fuzzy groups $\{A, A^1, A^2,..A^p….E \}$ is called M- fuzzy group family generated by A. It is denoted by $< A >$.

1.5.6 Definition: A function A from a set X to the closed unit interval [0,1] in U is called a fuzzy set in X, for every $x \in A$, A(x) is called membership grade of x in A. The set $\{x \in A/A(x) > 0\}$

is called the support of A and it is denoted by supp(A). For fuzzy sets λ and μ in a set X, then $\lambda o \mu$ has been defined in most articles by

$(\lambda o \mu)(x) = \sup \min \{ \lambda(a), \mu(b) \}$, if $ab=x$; $= 0$; if $ab \neq x$

We weaker this definition as follows.

1.5.7 Definition: Let X be a set and let λ ,μ be two fuzzy sets in X, $\lambda o \mu$ is defined by

$(\lambda o \mu)(x) = \sup \{ \lambda(a), \mu(b) \}$ if $ab = x$; $= 0$, if $ab \neq x$.

1.5.8 Definition: Let X be a group, we define λ^{-1} by $\lambda^{-1}(x) = \lambda(x^{-1})$ for $x \in X$. The standard definition of a fuzzy group by Rosenfeld[1971] is that a fuzzy set 'A' in a group X is a fuzzy group if and only if $A(xy) \geq \min \{A(x), A(y)\}$ and $A(x^{-1}) = A(x)$ for all $x, y \in X$ we weaken this definition as follows.

1.5.9 Definition: Let S be a groupoid. A function $A : S \rightarrow [0,1]$ is a weaker groupoid in S if and only if for every x, y in S , (WF1) $A(xy) \geq A(x) A(y)$, we denote a weaker fuzzy groupoid by a w- fuzzy groupoid. If X is a group, a weaker fuzzy groupoid 'A. in X if and only if for $x \in X$, (WF2)$A(x^{-1}) = A(x)$, we denote a weaker fuzzy group X by a w- fuzzy group .Since $\min(a,b) \geq ab$, our definition of a w- fuzzy group is weaker than the standard definition by Rosenfeld[1971]. It is easy to see that if G is fuzzy group in a group X and 'e' is the identity of X , then $G(e) \geq G(x)$ for all $x \in X$. If G is a w-fuzzy group in a group X, $G(e) = G(xx^{-1}) \geq G(x) G(x^{-1}) = [G(x)]^2$ for all $x \in X$

1.5.10 Definition: Let A be a w-fuzzy groupoid in a group X such that $A(a) = A(a^{-1})$. Let $e_\lambda: X \rightarrow X$ be identity defined by $e_\lambda(x)=x\lambda$ for all $x \in A$. Similar we define the left identity.

1.5.11 Definition: Let f: G→ G^1 be a homomorphism of fuzzy groups. For any fuzzy set A ∈ G^1 we define a new fuzzy set A^f in G by $A^f(x) = Af(x)$ for all x ∈ G, and $f(x^{-1}) = f(x) = (f(x))^{-1}$.

1.5.12 Definition: Let A and B be two fuzzy subsets of X then the direct product A × B is defined by (A × B) (x, y) = min {A(x) , B(y)} and (x, y) • (z, p) = (xz, yp) for all x, y, z, p in X.

1.5.13 Definition: Let f: G→ G^1 be a group homomorphism's and 'A' be w- fuzzy group of G^1 then $A^f(x) = (Aof)(x) = f^{-1}(A)(x)$.

The following results are showed on S-semi groups in chapter V.

61: Intersection of two BCK-algebras is BCK-algebra w.r.t* as well as Δ

62: Union of two BCK-algebras is BCK-algebra w.r.t.* if one is contained in other.

63: A subset of BCK-algebra is BCK-algebra.

64: Homomorphic image of BCK-algebra is BCK-algebra.

65: Union of fuzzy sub algebra is a fuzzy sub algebra.

66: Intersection of two fuzzy sub algebra is a fuzzy sub algebra.

67 : α - cut of fuzzy sub algebra is a fuzzy sub algebra.

68: Product of two fuzzy sub algebra is fuzzy sub algebra.

69: Union of two intuitionistic fuzzy sub algebra is an intuitionstic fuzzy sub algebra.

70: The intersection of two intuitionistic fuzzy sub algebra's is an intuitionistic fuzzy sub algebra.

71 : α - cut of an intuitionistic fuzzy sub algebra is an intuitionistic fuzzy sub algebra.

72: Product of intuitionistic fuzzy sub algebra is also an intuitionistic fuzzy sub algebra.

73: A fuzzy set μ of X is a fuzzy sub S-semi group if and only if the upper level set μ_t is either empty or a sub S-semi group of X, for every $0 < t < 1$.

Structure properties of M-Fuzzy groups

Several researches were conducted on the generalizations of the notion of fuzzy sets. The study of fuzzy group was started by Rosenfeld [1971] and it was extended by Roventa [2001] who have introduced the concept of fuzzy groups operating on fuzzy sets. Wu [1981] studied the fuzzy normal subgroups. Gu [1994] put forward the notion of fuzzy groups with operators. In this paper, we introduce the concept of M- fuzzy groups with operators and obtain some related results. For the sake of convenience, we set out the former concepts.

The following results are found in chapter V.

74: Let A be M- fuzzy group and S be a fuzzy subset of A. Then S be M- fuzzy subgroup of A if and only if $A_S(m(xy)) \geq \min (\{ A_S(mx) , A_S(my)\}$ for all x, y in S.

75: For all a, b $\in [0,1]$, m $\in$ M and p is any positive integer , verify that (i) If $ma \leq mb$ then $(ma)^p \leq (mb)^p$ and (ii) $\min \{ma,mb\}^p = \min \{(ma)^p , (mb)^p\}$

76: If A is a M- fuzzy group, then $A^p= ((mx, (A(mx)^p) / mx \varepsilon A \}$ is M- fuzzy group.

77: The M- fuzzy group A^q is a M- fuzzy subgroup A^p , if $q \leq p$.

78: If A^i and A^j are M-fuzzy groups, $A^i \cup A^j$ is also M-fuzzy group for positive integers i & j.

79: If A^i and A^j are M- fuzzy groups, then $A^i \cap A^j$ is also M- fuzzy groups, where i and j are natural numbers.

80: Prove that $A^p \subset A$ for all p..

81: Let G and G^1 be M – groups and f an M- homomorphism from G onto G^1,. (i) if A is M – fuzzy group of G^1 then A^f is M- fuzzy group of G. (ii) if A^f is M- fuzzy group of G then A is M- fuzzy group of G^1.

82: Let < A > = { A , A^1 , A^2,----, A^P,---, E}. Then $\cup A^P = A$ and $\cap A^P = E$ where p varies 1 to ∞.

83: Let G and G^1 be M- groups and f is homomorphism from G onto G^1. If A^f is M- fuzzy group of G , then A is M- fuzzy group of G^1.

84: Let A be a M- fuzzy group, then $A \supset A^2 \supset A^3 \ldots\ldots \supset A^p \ldots E$.

The following results are found for w-fuzzy groups in chapter V.

85: Let A be a fuzzy subset in a group X such that A(e) =1, where e is the identity of X then A is a w- fuzzy group if and only if $A(xy^{-1}) \geq A(x)\, A(y)$ for all $x, y \in X$.

86: If w- fuzzy groupoid A on X has left identity e_λ and a right identity e_μ then $e_\lambda = e_\mu$.

87: Let G and G^1 be groups and f a homomorphism from G onto G^1, (i) if A is w- fuzzy group of G^1 then A^f is w-fuzzy group of G.(ii) if A^f is w-fuzzy group of G then A is w-fuzzy group of G^1.

88: If A and B be wfuzzy groups of G_1 & G_2 respectively then $A \times B$ is w-fuzzy group of $G_1 \times G_2$.

89: If $A_1, A_2, \ldots, A_n$ are w- fuzzy groups of $G_1, G_2, \ldots, G_n$ respectively then $A_1 \times A_2 \times \ldots \times A_n$ is w- fuzzy groups of $G_1 \times G_2 \times \ldots \times G_n$.

90: Let A and B be fuzzy subsets of G_1 and G_2 respectively such that A×B is a w- fuzzy group of $G_1 \times G_2$ then A and B is w- fuzzy group of G_1 and G_2 respectively.

91: Let $f: G \rightarrow G^1$ be a group homomorphism and let 'A' be a w- fuzzy group of G^1 then $f^{-1}(A)$ is w- fuzzy group of G.

92: Let A be a w- fuzzy group of group G and A* be a fuzzy set in G defined by $A^*(x) = A(x) + 1 - A(e)$ for all $x \in G$. Then A* is w- fuzzy group of G containing A.

Section 6: Generalized product of fuzzy groups and P-level subgroups

Since its inception, there has been a tremendous interest in the fuzzy set theory due to its many applications ranging from engineering and computer sciences to social behavior studies. There is quite substantial literature on fuzzy group theory. The study of fuzzy groups was started firstly by Rosenfeld [1971]. He used the min operating to define his fuzzy groups and showed how some basic notions of group theory should be extended in an elementary manner to develop the theory of fuzzy groups. It was extended by Antony and Sherwood [1982]. They used the-t-norm operating instead of the min to define the t-fuzzy groups. Roventa and Spircu [2001] introduced the fuzzy group operating on fuzzy sets.

We first generalized the results of the product of fuzzy groups which were done by Ray [1999]. We also define p-level subset and P-level subgroups, and then we study some of their properties.

1.6.1 Definition: Let $\mu_{\tilde{A}} : U \to [0,1]$ be any function and A be a crisp set in the universe U then the ordered pairs $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) / x \in U\}$ is called a fuzzy set and $\mu_{\tilde{A}}$ is called a membership function.

1.6.2 Definition: Let $\hat{G}_i$ be a fuzzy group under a minimum operation in a group X_i (i = 1,2,..,n) , then the membership function of the product $G = \hat{G}_1 \times \hat{G}_2 \ldots \times \hat{G}_n$ in $X = X_1 \times X_2 \times \ldots \times X_n$ is defined by
$(\hat{G}_1 \times \hat{G}_2 \times \hat{G}_3 \ldots \times \hat{G}_n)(x_1, x_2 \ldots x_n) = \min \{\hat{G}_1(x_1), \hat{G}_2(x_2), \ldots. \hat{G}_n(x_n)\}$.

1.6.3 Definition: A fuzzy group $\hat{G}$ of a group X is said to be conjugate to a fuzzy subgroup H of x there exists x in G such that for all $g \in X$, $\hat{G}(g) = H(x^{-1} g x)$

1.6.4 Definition: Let X be a groupoid and T a t-norm. A function $A : X \to [0,1]$ is $\hat{G}$ called t-fuzzy group of X if and only if (i) $A(xy) \geq T\{A(x), A(y)\}$(ii) $A(x^{-1}) \geq A(x)$ for every x, y in X

1.6.5 Definition: Let $\hat{G}_i$ be a t-fuzzy group of X_i for each i=1,2,3 …. n and T be t-norm. The T product of $\hat{G}_i$ (i = 1,2,3 …… n) is the function $(\hat{G}_1 \times \hat{G}_2 \times \ldots \times \hat{G}_n): (X_1 \times X_2 \times \ldots \times X_n) \to [0,1]$ defined by $(\hat{G}_1 \times \hat{G}_2 \times \ldots. \times \hat{G}_n)(x_1, x_2 \ldots., x_n) = T(\hat{G}_1(x_1), \hat{G}_1(x_2), \ldots., \hat{G}_n(x_n))$.

1.6.6 Definition: Given a fuzzy set $A: X \to [0,1]$ and let $P \in [0,1]$. The set $A_p = \{x \in X / A(x) \geq p\}$ is called a P-cut of a fuzzy set A

1.6.7 Definition: Let A be a fuzzy subset of a set X , T a t-norm and $c \in [0,1]$. Then we define a P-level subset of a fuzzy set A as $A_c T = \{x \in X : T(A(x), c) \geq c\}$.

1.6.8 Definition: Let X be a set. A mapping $A: X \to [0,1]$ is called interval valued fuzzy set (briefly i-v fuzzy set) of X and $A(x) = [A^-(x), A^+(x)]$, for all $x \in X$ where A^- and A^+ are fuzzy sets in X.

1.6.9 Definition: A mapping $\max^i : [0,1] \times [0,1] \to [0,1]$ given by

$\max^i(a, b) = [\max(a^-, b^-), \max(a^+, b^+)]$ for all a, b in [0,1] is called interval max- norm.

1.6.10 Definition: Let $\max^{i}$ be a norm. For arbitrary $a \in [0, 1]$ if it satisfies $\max^{i}(a,a) = a$, then max-norm is called idempotent norm.

1.6.11 Definition: An interval number on [0, 1] say a is a closed subinterval of [0, 1] So $a = [a^-,a^+]$ Where $0 < a^- < a^+ < 1$ for any interval numbers $a = [a^-, a^+]$ and $b = [b^-, b^+]$ on [0, 1] we define $a < b$ if and only if $a^- < b^-$ and $a^+ < b^+$ $a = b$ if and only if $a^- = b^-$ and $a^+ = b^+$, $a + b = [a^- + b^- , a^+ + b^+]$, whenever $a^- + b^- < 1$ and $a^+ + b^+ < 1$.

1.6.12 Definition: Let X be a fuzzy set. A mapping $A: X \to [0,1]$ is called a fuzzy set in X. Let $\alpha \in [0,1]$ define $L(A ; \alpha) = \{ x \in X / A(x) \leq \alpha \}$, then $L(A; \alpha)$ is called lower level set of A.

1.6.13 Definition: A semi ring $(R , + \bullet)$ is called a hemi ring if + is commutative and there exists an element $0 \in R$ such that $0 \bullet a = a \bullet 0 = 0$ for all $a \in R$.

1.6.14 Definition : A fuzzy set A of a semi ring is said to be fuzzy left ideal of R if $A(x + y) \geq \min^{i}\{A(x) , A(y) \}$ and $A(xy) \geq A(y)$ for all x, y in R. Note that if A is a fuzzy left h- ideal of a hemi ring R then $A (0) \geq A (x)$.

1.6.15 Definition : A fuzzy subset A of a hemi ring R is said to be $\max^{i}$ - anti fuzzy left h- ideal of 'R' (i) $A(x+y) \leq \max^{i}\{ A(x), A(y) \}$ (ii) $A (xy) \leq A(y)$ (iii) $x + a + z = b + z \to A(x) \leq \max^{i}\{ A(a) , A(b) \}$ for all x, y a, b in R.

1.6.16 Remark: $(A\cup B) (x) = \{ \max^{i}\{A^-(x) , B^-(x) \} , \max^{i}\{ A^+(x) , B^+(x) \}\}$

$(A\cap B) (x) = \{ \min^{i}\{ A^-(x), B^-(x) \} , \min^{i}\{ A^+(x) , B^+(x) \}\}$

1.6.17 Definition : Let $R_1$and R_2 be two hemi rings and f be a function of R_1 into R_2.If A is a fuzzy subset in R_2, then pre image of A under 'f' is the fuzzy set in R_1 defined by $f^{-1} (A) (x) = A f(x)$, for all x in R.

1.6.18 Definition: Let R_1 and R_2 be any two sets and let $f: R_1 \to R_2$ be any function. A fuzzy subset A of R_1 is called f – invariant if $f(x) = f(y) \to A(x) = A(y)$.

1.6.19 Definition: max ^{i} - anti fuzzy left h- ideal of A of hemi ring R is said to be max ^{i} - anti fuzzy characteristics if $A^{f}(x) = A(x)$ for $x \in R$ and $f \in Aut(R)$.

1.6.20 Definition: If a fuzzy set A is normal interval valued anti fuzzy left h- ideal of R, then A(0) = 1.

The following results are found in Product of fuzzy groups & t-fuzzy groups in chapter VI

93: If $\hat{G}_1$, $\hat{G}_2$, ….,$\hat{G}_n$ be a fuzzy groups of the groups X_1, X_2,....,X_n respectively, then $\hat{G}_1 \times \hat{G}_2 \times .... \times \hat{G}_n$ is fuzzy groups of $X_1 \times X_2 \times \ldots \times X_n$.

94: Let $\hat{G}_1 \times \hat{G}_2 \times \ldots \times \hat{G}_n$ be fuzzy groups of the groups $X_1 \times X_2 \times \ldots \times X_n$, respectively then $\hat{G}_1 \times \hat{G}_2 \times \ldots \times \hat{G}_n$ is a fuzzy normal subgroup of $X_1 \times X_2 \times \ldots \times . X_n$.

95: Let the fuzzy groups $\hat{G}_1 \times \hat{G}_2 \times \ldots \times \hat{G}_n$ of $X_1 \times X_2 \times \ldots \times X_n$ conjugate to fuzzy subgroups H_1,H_2, ,…, H_n. Then $\hat{G}_1 \times \hat{G}_2 \times \ldots \times \hat{G}_n$of the group $X_1 \times X_2 \times \ldots \times X_n$ is conjugate to the H_1, H_2,…., H_n of $X_1 \times X_2 \times \ldots \times X_n$

The following results are found in p-level subgroups in chapter VI

96: Let X a group and A a t-fuzzy group of X, then the P-level subset A_cT for $c \in [0,1]$ and $c < T(A(e), r)$, is a subgroup of X, where of X, where e is the identity of ‘X’.

97: Let A and B be P-level subsets of the sets X and Y respectively and let $c \in [0, 1]$, then A×B is also a P-level subset of X×Y.

98: Let X and Y be two groups , A and B a t-fuzzy group of X and Y respectively. Then the P-level subset $(A\times B)_c t$, for $c \in [0,1]$ is a fuzzy group of X×Y, where e_x and e_y are identies of X and Y respectively.

99: Let X be a group and A_cT a P-level subgroup of X. If A is a normal t-fuzzy subgroup of X, then A_cT is a normal subgroup of X.

100: Let A and B be fuzzy subsets of X and Y respectively, T be a t-norm and c $\in$ [0, 1]. Then $A_cT \times B_cT = (AB)_cT$.

Characterization of interval valued anti fuzzy left h-ideals over hemi ring in interval valued anti fuzzy characteristic function

Introduction: Fuzzy set theory has been developed in many directions by many researchers and has evoked great interest among mathematicians working in different fields of mathematics, such as topological spaces , functional analysis , loop , group , ring , near ring, vector spaces , automation. There have been wide - ranging applications if the theory of fuzzy sets, from the design of robots and computer simulations to engineering and water resource planning. Since then many researchers have been involved in extending the concepts and results of abstract algebra to the broader frame work of the fuzzy setting. The notion of fuzzy left h- ideals in hemi ring was introduced in Y.V.Jun et.al.[2004] . The notion of (i-v) fuzzy set , a kind of well-known generalization of ordinary fuzzy set, was introduced by Zadeh [1965], Biswas [1994] investigate (i-v) fuzzy subgroup, Wang and Li [2000] investigated TH- interval valued fuzzy subgroup and SH – interval valued fuzzy subgroup Zeng [2006] proposed concepts of cut set of (i-v) fuzzy set and investigated decompositions of theorems and representation theorems of (i-v) fuzzy set and so on. These works show the importance of (i-v) fuzzy set. In this paper, we apply the notion of (i-v) fuzzy sets to anti fuzzy left h- ideals of hemi ring. We introduce the notion of (i-v) anti fuzzy left h- ideals of R with respect to max norm and investigate some of their properties. Using lower level set, we give a characterization of $\max^{i}$ – anti fuzzy left h- ideal. Finally we establish the theorems of the homomorphic image and the inverse image.

Jun. Y.B [2004] showed that the fuzzy setting of a left h- ideal in a hemi ring is constructed and basic properties are investigated. Using a collection of left h- ideals of a hemi ring S are established. He also explained the notion of a finite valued fuzzy left h- ideals and its characterization. fuzzy relations on a hemi ring S are also discussed. X.P.Li, [2000] explained the idempotent interval co norm SH induced by a T- co norm on the space on the interval valued fuzzy sets on fuzzy groups and SH interval valued fuzzy groups. In the mean time some of its basic properties and structural characterizations are discussed. Also he showed that the theorems of the homomorphic image and the inverse image are given. D. M. Olson [1978] showed that the fundamental homomophisms theorems for rings is not generally applicable in hemi ring theory. He explained also the class of N- homomorphisms of hemi rings the fundamental theorem is valid. In addition, the concept of N- homomorphism is used to prove that every hereditarily semi subtractive hemi ring is of type (k). W. J. Liu [1987] proved that some basic concepts of fuzzy algebra as a fuzzy invariant subgroups, fuzzy ideals and some fundamental properties. He also showed that characteristic of a field by fuzzy ideals.

The following theorems are showed in chapter VI

101: Let R be a hemi ring and A be a fuzzy set in R then A is max ^{i} - anti fuzzy h- ideal in R if and only if A ^{c} is a fuzzy left h- ideal in R.

102: Let A and B are max ^{i} - anti fuzzy left h- ideal of R , then A $\cup$ B also max ^{i} - anti fuzzy left h- ideal in R.

103: Let f : $R_1 \rightarrow R_2$ be an onto homomorphism of hemi ring. If A is max ^{i} - anti fuzzy left h- ideal of R_2, then $f^{-1}(A)$ is a max ^{i} - anti fuzzy left h- ideal of R_1.

104: Let f: $R_1 \rightarrow R_2$ be an epimorphisms of hemi ring. Let A be an f- invariant max ^{i} - anti fuzzy left h- ideal of R_1, then max ^{i} - anti fuzzy left h- ideal of R_2..

105: Let A be max^i – anti fuzzy left h – ideal in a hemi ring R such that $L(A; \alpha)$ is a left h – ideal of R , for each $\alpha \in$ Im(A), $\alpha \in [0,1]$ then A is max^i- anti fuzzy left h – ideal of R.

106: Let A be max^i – anti fuzzy left h – ideal in a hemi ring R. Let A^+ be a fuzzy lower cut set in R defined by $A^+(x) = A(x) +1 - A(0)$ for all x in R then A^+ is lower cut of max^i – anti fuzzy left h – ideal in R which contains A.

Section 7: Fuzzy α-cuts on fuzzy groups and L-fuzzy number

We give some properties on α-cuts fuzzy groups. The concept of fuzzy subset of a non-empty set first was introduced by Zadeh [1965] . In Rosenfeld formulated the concept of fuzzy subgroup of a group. This work was the first fuzzification of any algebraic structures and thus opened a new direction, new exploration, new path of thinking to mathematicians, engineers, computer scientists any many others in various ways of various tests. The fuzzy concept is taking the vital role in all engineering fields especially for the design of analysis part.

1.7.1 Definition: With any fuzzy set A we can associate a collection of crisp sets known as α-cuts (Alpha cuts). The α-cut fuzzy set A denoted by A_α it is defined as $A_\alpha = \{x \in X / \mu_A(x) \geqq \alpha\}$.

1.7.2 Remark: The strong α-cuts is denoted α^{+A} (where A is crisp set) and it is defined as
$\alpha^{+A} = \{x/A(x) > \alpha)$.

1.7.3 Definition: The set of levels $\alpha \in [0,1]$ that represent distinct α–cut of a given fuzzy set.
A is called a level set of A and it is defined as^ (A) = $\{a / A(x) = \alpha\}$ for some $x \in X$ where α is the parameter $0 < \alpha \leq 1$

1.7.4 Note: $\hat{G}_\alpha = \{(x, \mu_A(x)\} / x \in \hat{G}_\alpha\}$.

1.7.5 Definition: An Intuitionistic fuzzy set A in the Universe of discourse
U is characterized by two membership functions given by

1. A truth Membership function $t_A : U \rightarrow [0,1]$
2. A false membership function.$f_A : U \rightarrow [0.1]$

where t_A (x) is a Lower bound of the grade of membership of x derived from the evidence for x and f_A (x) is a Lower bound on the negation of x derived from the evidence against x and $t_A(x)+f_A(x) \leq 1$. The Intuitionistic fuzzy set A is written as $\tilde{A} = \{(x, (t_A(x), f_A(x))\} / x \in U\}$ where the interval $[t_A(x), 1- f_A(x)]$ is called intuitionistic values of x in A and denoted by $I_A(x)$. In an intuitionistic fuzzy sets are independently proposed by the decision maker but they are mathematically not independent. This makes a Major difference in the judgement about the grade of membership.

1.7.6 Definition: An intuitionistic fuzzy set Aof a Set U with $t_A(x) = 0$ and $f_A(x) = 1$ for all $x \in U$ is called the Zero intuitionistic fuzzy set of U. An intuitionistic Fuzzy set A of a Set U with $t_A(x) = 1$ and $f_A(x) = 0$ for all $x \in U$ is called unit intuitionistic Fuzzy set of U.

1.7.7 Definition; An intuitionistic fuzzy set A of a Set U with $t_A(x) = \alpha$ and $f_A(x) = 1- \alpha$ for all $x \in U$ is called for α–Intuitionistic fuzzy set of U where $\alpha \in (0,1)$

1.7.8 Definition: Let (X, *) be a group. An intuitionistic fuzzy set A of X is called Intuitionistic fuzzy group of X if the following conditions are true.

$I_A(xy) \geq \min\{I_A(x), I_A(y)\}$ for all $x,y \in X$.

1. $t_A(xy) \geq \min\{t_A(x), t_A(y)\}$

   $t_A(x^{-1}) \geq t_A(x)$

2. $1-f_A(xy) \geq \min\{1-f_A(x), 1-f_A(y)\}$

   $1-f_A(x^{-1}) \geq 1- f_A(x)$ (Here the element xy stands for x*y).

1.7.9 Definition: For $\alpha, \beta \in [0,1]$ we now define (α, β) – cut and α - Cut of a intuitionistic Fuzzy Sets Let A be an intuitionistic fuzzy set of a Universe X with the true membership function t_A and the false membership f_A them (α,β) – cut is defined as $(\alpha,\beta) = \{x \in X, I_A(x) \geq (\alpha,\beta)$ and α - cut is defined as $A_\alpha = \{x \in X, t_A(x) \geq \alpha\}$

1.7.10 Remark: The sub groups like (α,β) cut are also called intuitionistic – cut sub group of X.

1.7.11 Definition: Let X be a group. A fuzzy set S is called a fuzzy semi group if S (xy) $\geq$ min {S(x), S(y)}, for all x, y$\in$X.

1.7.12 Definition: An intuitionistic fuzzy set A = (t_A, f_A) in S called an intuitionistic fuzzy sub semigroup of S if t_A (xy) $\geq$ Min {t_A(x), t_A(y)} and f_A(xy) $\leq$ Max { f_A(x) , f_A(y)} for all x, y $\in$ S

1.7.13 Definition: Let A be an intuitionistic fuzzy group of a group X. Then A is called intuitionistic normal group if I_A (xy) = I_A(yx) for all x, y $\in$ X. Alternatively I_A (x) = $I_A(yxy^{-1})$ for all x, y $\in$ X.

1.7.14 Note: The notion [x, y] is used for the expression $x^{-1} y^{-1} xy$.

The following theorems are showed in chapter VII

107: Let Ĝ be a fuzzy group and $\hat{G}_\alpha$ be its α -cut. Then $\hat{G}_\alpha$ forms a fuzzy group.

108: If $\hat{G}_\alpha{}^i$ and $\hat{G}_\alpha{}^j$ are α-cut fuzzy groups, then $\hat{G}_\alpha{}^i \cup \hat{G}_\alpha{}^j$ is also a α-cut fuzzy group if i < j

109: $\hat{G}_\alpha{}^i$ and $\hat{G}_\alpha{}^j$ are fuzzy groups then $\hat{G}_\alpha{}^i \cap \hat{G}_\alpha{}^j$ also a fuzzy group if i < j.

110: λ is fuzzy number with convex function. Then λ is constant function.

111: Let a fuzzy number λ be a convex function. Then (R, +, 0) is a fuzzy group.

Some Structure Properties of Intuitionistic Fuzzy Group

We consider the intuitionistic fuzzification of the concept of groups and investigate some properties of such groups. In this paper defined intuitionistic fuzzy groups (IFG) intuitionistic normal groups (ING) and takes some characterizations of them.

The concept of Intuitionistic fuzzy set was introduced by Atanassov [1986] as a generalization of the notion of fuzzy sets. In this section, we present now some preliminaries on the theory of intuitionistic fuzzy sets. Since then it has been applied in wide varieties of field like computer science, management science, Medical sciences, Engineering problems etc., to list a few only. Consequently, there is a genuine necessity of a different kind of fuzzy sets. We consider the intuitionistic fuzzification of the concept of groups and investigate some properties of such groups.

The following theorems are showed in Intuitionistic fuzzy group in chapter VII

112: If A is an intuitionistic Fuzzy group of X Then $I_A(x^{-1}) = I_A(x)$ for all $x \in X$.

113: Unit Intuitionistic fuzzy set, Zero Intuitionistic fuzzy set and α -Intuitionistic fuzzy set of a group X are intuitionistic fuzzy group of X.

114: A set of necessary and sufficient conditions for al intuitionistic fuzzy set A to be an intuitionistic fuzzy group of X is that $I_A(xy^{-1}) \geq \min\{I_A(x), I(y)\}$

115: If A and B an two intuitionistic fuzzy groups of a group X then $A \cap B$ is also intuitionistic Fuzzy Group of X.

116: If A and B are two intuitionistic Fuzzy Group of X then $A \cup B$ is also intuitionistic Fuzzy group of X

117: If $A = <x, t_A, f_A>$ is an Intuitionistic fuzzy group of a group X, then t_A is a fuzzy group of X and $1- f_A$ is a fuzzy group of X

118: Let A be an intuitionisitc fuzzy group of a group X. Then (α,β) – cut forms a subgroup of X for all α, β.

The following theorems are showed in Intuitionistic Fuzzy Semi group and sub semi group in chapter VII

119: An intuitionistic fuzzy Sub semi group $A = (t_A, f_A)$ of S is called an intuitionistic fuzzy group of S if $t_A(x^{-1}) \geq t_A(x)$; $f_A(x^{-1}) \leq f_A(x)$, and every intuitionistic fuzzy sub semi group of S is constant.

120: Let A be a intuitionistic normal group of a group X then $I_A(x, y) = I_A(e)$ for all $x, y \in X$.

Section 8: Pseudo Fuzzy cossets of Fuzzy normal groups

In this section, the theory of fuzzy sets has developed in many directions and is finding applications in a wide variety of fields. Rosenfeld in 1971 used this concept to develop the theory of fuzzy groups. we have given independent proof of several theorems on fuzzy groups. We discuss about Fuzzy groups and investigate some of their structures on the concept of Fuzzy Group family.

1.8.1 Definition : Let $\mu_{\tilde{A}} : U \rightarrow \{ 0,1\}$ be any function and A be a crisp set in the universal set U, then $\tilde{A} = \{(x, \mu_{\tilde{A}} (x) / x \in U\}$ is called a fuzzy set and the function $\mu_{\tilde{A}}$ is called a membership function of Ã

Note that the membership function $\mu_{\tilde{A}} (x)$ specifies the grade or degree to which any

Element x in U belongs to the fuzzy set Ã . Larger valued of $\mu_{\tilde{A}}(x)$ indicate higher degrees of membership.

The classical sets can be considered as a special case of fuzzy sets with all membership grades equal to 1. We will identity any fuzzy set with its membership function and use these two concepts as interchangeable.

(FG_3) $\mu (e_G) = 1$ are satisfied, then the fuzzy group is called a standardized fuzzy group where ρ_G is an identity of the groups (G, •)

1.8.2 Definition: Let μ be a fuzzy sub group of a group G. for any a ∈ G. are defined by $(a\ \mu)\ (x) = \mu\ (a^{-1} x)$ for every x ∈ G is called the fuzzy cosset of the group G determined by a and μ

1.8.3 Definition: Let μ be the fuzzy sub group of a group G. then for any a, b ∈ G a fuzzy middle cosset a μ b of the group G is defined by $(a\ \mu\ b)\ (x) = \mu\ (a^{-1}\ x\ b^{-1})$ for every x ∈ G.

1.8.4 Definition: Let μ be a fuzzy sub group of G and a ∈ G. Then the pseudo fuzzy cosset $(a\mu)^{p}$ is defined by $(a\ \mu)p\ (x) = p\ (a)\ \mu\ (x)$ for every x ∈ G and for some p∈P.

Example: Let G = { 1, w,w^2} be a group with respect to multiplication where w denotes the cube root of unity . Define a map μ : G → [0,1] by μ (x) =0.7if x = 1; = 0.3 if x = w, w^2.

pseudo fuzzy cosset $(a\ \mu)^p$ for p (x) = 0.4 for every x ∈ G to be equal to 0.28 if x =1 and 0.12 if x = w, w^2.

1.8.5 Definition: Let μ and λ be any two fuzzy subsets of a set X and p ∈ P. the pseudo fuzzy double cosset to $(\mu \times \lambda)^p$ is defined as $((\mu \times \lambda)^p = (x\ \mu)^p \cap (x\ \mu)^p$ for x ∈ X.

1.8.6 Definition: Let λ and μ be two fuzzy subgroups of a group G then λ and μ are said to be Conjugate fuzzy subgroups of G if for some g ∈ Gλ (x) = μ $(g^{-1} x g)$ for every x ∈ G.

1.8.7 Definition: Let μ be a fuzzy sub group of a group G and x ∈ G. then μ (x y) = μ (y) for every y ∈ G if and only if μ (x) = μ (e)

1.8.8 Definition: A fuzzy subgroups μ of a group G is said to be positive fuzzy subgroup of G if μ is positive fuzzy subset of the group G.

1.8.9 Definition: The strong Fuzzy α-cut is defined as $A^+_\alpha = \{x/A(x) > \alpha\}$ where A is any Fuzzy set .

1.8.10 Definition: Let A be a Fuzzy set in a set S. The strongest Fuzzy relation S on A is defined as $\mu_A(x, y) = \min \{(A(x), A(y)\}$.

1.8.11 Definition: Cartesian Product: Let λ and μ be any two Fuzzy sets in X then the Cartesian Product of λ and μ is λ× μ: X × X→ [0, 1] defined by (λ×μ) (x, y) = Min {λ(x) , μ(y)} for all x, y ∈ X.

1.8.12 Definition: i) $\min (a,b)^i = \min \{a^i,b^i)$ for all Positive integer i

$$\text{ii) } \mu_A i(x,y) = (\mu_A(x,y))^i$$

$$= \min \{A(x), A(y)\}^i$$

$$= \min \{A^i(x), A^i(y)\}.$$

The following results are obtained in chapter VIII

121: If G is a group, then prove that $G^m = \{ (x , \mu_G(x)^m / x \in G \}$ is a fuzzy group, and the fuzzy group G^n is a fuzzy subgroup of G^m , if $m \leq n$.

122: If $\hat{G}^i$ and $\hat{G}^j$ are fuzzy groups, then $\hat{G}^i \cup \hat{G}^j$ is also a fuzzy group if $i < j$

123: $\hat{G}^i$ and $\hat{G}^j$ are fuzzy groups then $\hat{G}^i \cap \hat{G}^j$ also a fuzzy group if $i < j$. Thus if $< A > = \{ A , A^1 , A^2 ,----, A^P ,----, E\}$ then $\cup A^P = A$ & $\cap A^P = E$ where p varies 1 to ∞.

We have given Independent proof of several theorems on pseudo fuzzy cossets of fuzzy normal subgroups we introduce the notion of pseudo fuzzy double cossets, pseudo fuzzy middle cossets of a group and consider its fundamental properties

We are formulated the fuzzification of any algebraic structures and thus opened a new directions new exploration, new path of thinking to mathematicians engineers, computer scientists and many others in various ways of various tests. The fuzzy concept is taking the vital role in all Engineering fields especially for the design and Analysis port. Mukherjee and Bhattacharya [1986] introduced the fuzzy right cossets and fuzzy left cossets of a group. Here we introduce the notion of pseudo fuzzy cossets of a group and how they are related with fuzzy normal subgroups.

The following results are obtained in Pseudo fuzzy cossets In chapter VIII

124: A fuzzy subset μ of a group G is a fuzzy subgroup of $\hat{G}$ if and only if $\mu (x y^{-1}) \geq \min \{ \mu (x), \mu (y)$ for every x, y in G

125: Let μ be a fuzzy subgroup of a group G then the pseudo fuzzy cosset $(a \mu)^p$ is a fuzzy sub group of G for every $a \in G$.

126: Every pseudo fuzzy double cosset is a fuzzy subgroup of a group G.

127: Every fuzzy middle cosset of a group G is a fuzzy subgroup of G.

128: Every pseudo fuzzy cosset is a fuzzy normal subgroup of a group G.

129: The intersection of two pseudo fuzzy cosset normal subgroup is also fuzzy normal subgroup of a group.

130: Pseudo fuzzy double cosset is a fuzzy normal subgroup of a group G.

131: Fuzzy middle cosset forms a fuzzy normal subgroup of G.

132: Let μ_A be a strongest Fuzzy relation on S and A^{+}_{α} be the strong α-cut the μ_A forms a strong α- cut Fuzzy group on S.

133: Let λ and μ be strong Fuzzy α- cuts on S then $\lambda \times \mu$ is a strong Fuzzy group α- cut.

134: Let $\mu_A{}^i$ and $\mu_A{}^j$ be two strong Fuzzy relations and $A_\alpha{}^+$ be strong Fuzzy α- cut then $\mu_A{}^i{}_{UA}{}^j$ forms a strong Fuzzy α-cut on S.

135: Let μ_A be the strongest Fuzzy Relation on S then $A_\alpha{}^+$ is a strong α-cut then μ_A forms a Fuzzy compatible.

136: Let μ_A be a fuzzy compatible then it forms a strong fuzzy α-cut.

Section 9: Multi-stage decision making approach to optimize the product mix in assignment level under fuzzy group parameters

Aryanezhad, M.B [2004] showed that one of the most important decisions made in production systems is determining the product mix in such a way that maximum throughput would be obtained. Several algorithms to determine the product mix under theory of constraints (TOC) have been developed. He also explained the inefficiency of the traditional TOC algorithm in handling the multiple bottleneck problems is discussed through an example, then the latest algorithm and its advantages will be discussed and an improved algorithm, which is much more efficient and can reach the optimum solution with considerable speed, will be presented. Finally showed that the improved algorithm and the integer linear programme (ILP) method will be compared with each other through the examples.

Chen. J [1992] explained a method of fuzzy multi attribute decision making based on rough sets, the uniqueness, as well as the crucial part is that we define rough sets under priority relationship. He also establish priority class using priority relationship R as the categorization criteria, then we study the mathematical properties of the concepts, define the priority degree between two objects and define the summary priority degree as well, by using inclusion degree priority degree measurement. Using summarization method, we can get the evaluation value for a given object and thus study the multi-attribute decision making. Finally he showed the decision making example given of rough sets, which helps give as insight into the method of fuzzy multi decision making based as rough sets.

Inventory models in which the demand rate on the inventory level is based on the common real life observation that greater product availability tends to stimulate more sales. Theory of constraints (TOC) is a production planning philosophy that tries to improve the throughput of the system management of inventory levels. Due to the existing of inventory levels in a production system the demands of all products can not be fully met. So one of the most important decisions made in production systems is product mix problem. Although many algorithms have been developed in the fields using the concept of theory of constraints. This paper benefits from a variety of advantages. In order to consider the importance of all inventory levels, group decision making approach is applied and the optimal product mix is reached. In the algorithm presented in this paper, each inventory level is considered as a decision maker. The new algorithm benefits from the concept of fuzzy group decision making and optimizes the product mix problem in inventory environment where all parameters are fuzzy values.

Theory of constraints (TOC) is a production planning philosophy that aims to improve the system through put by efficient use of inventory levels. In this paper product mix optimization is considered as a decision making problem. Two important criteria are throughput and the later delivery cost. Later delivery cost is the most of mission one unit of each product. Assuming each inventory level as a decision maker, product mix optimization is a group decision making problem. In all previous researchers all parameters (such as processing time, demand etc) are assumed as crisp values.

A new algorithm is developed to optimize the product mix problem with all inputs are fuzzy values and Borda methods is used in group decision making process as ordinal techniques are preferred to cardinal ones.

Contributions for application of fuzzy group in decision making in chapter IX

137: The improved algorithm benefits from the advantage of reaching optimal solution. In the previous researchers all inputs were considered as crisp values. The assumption is not in real cases. This paper considers product mix problem as a group decision making problem in which all inputs are fuzzy. In this chapter, a new algorithm for optimizing product mix under fuzzy parameters is developed. For this method, ordering methods are used in order to make decision in a fuzzy group decision making environment.