CHAPTER VII

FUZZY α-CUTS ON FUZZY GROUPS AND L-FUZZY NUMBER

We start with the following definitions:

7.2 Section II: Preliminaries

7.2.1 Definition: With a fuzzy set A, a collection of crisp sets is associated known as α-cuts (Alpha cuts). The α-cut fuzzy set A_α of $A = \{ x \in X / \mu_A (x) \geq \alpha \}$.

7.2.2 Remark: The strong α-cuts is denoted α^{+A} (where A is crisp set) and it is defined as $\alpha^{+A} = \{x / A(x) > \alpha)$

7.2.3 Definition: The set of levels $\alpha \in [0,1]$ that represent distinct α–cut of a given fuzzy set A is called a level set of 'A' and it is defined as $\wedge (A) = \{a / A (x) = \alpha\}$ for some $x \in X$ where α is the parameter $0 < \alpha \leq 1$.

7.2.4 Note: $\hat{G}_\alpha = \{(x, \mu_A(x)) / x \in \hat{G}_\alpha\}$.

The following are the results based on alpha cut fuzzy groups

7.2.1 Proposition: Let $\hat{G}$ be a fuzzy group and $\hat{G}_\alpha$ be its α -cut. Then $\hat{G}_\alpha$ forms a fuzzy group.

Proof: Let $\hat{G}$ be a fuzzy group and $\hat{G}_\alpha = \{x \in G / \mu_{\hat{G}\alpha} (x) \geq \alpha.\}$ be α -cut of fuzzy subset of $\hat{G}$.

(FGI) Let $x, y \in \hat{G}_\alpha$. $\mu_G (x) \geq \alpha$, $\mu_G (y) \geq \alpha$

$$
\begin{aligned}
\mu_G (xy) &= \mu_A (xy) \\
&\geq \min \{\mu_A(x), \mu_A(y)\} \\
&\geq \min \{\hat{G}_\alpha (x), \hat{G}_\alpha (y)\} \\
&= \min \{\alpha, \alpha\} \\
&= \alpha.
\end{aligned}
$$

It implies $xy \in \hat{G}_\alpha$

(FG2) Let $x \in \hat{G}_\alpha$ implies $\mu_\alpha (x) \geq \alpha$

It follows that $\mu_{\hat{G}}(x) \geq \alpha$ so that $\mu_{\hat{G}}(x^{-1}) \geq \alpha$. Therefore $x^{-1} \in \hat{G}$.

(FG3) Let $x \in \hat{G}_\alpha$, It gives that $\mu_{\hat{G}}(e) = 1$

$\mu_{\hat{G}}(e) > \alpha$ implies $e \in \hat{G}_\alpha$. Also $\mu_G (e) = 1$. So $\hat{G}_\alpha$ is a fuzzy group.

7.2.2 Proposition: If $\hat{G}_\alpha^i$ and $\hat{G}_\alpha^j$ are α-cut fuzzy groups, then $\hat{G}_\alpha^i \cup \hat{G}_\alpha^j$ is also a α-cut fuzzy group if $i < j$.

Proof: **S**ince $i < j$, then $\mu_{\hat{G}\alpha}^i > \mu_{\hat{G}\alpha}^j$.

(FGI) $\mu_{\hat{G}\alpha i \cup \hat{G}\alpha j}(xy)$ $= \max\{\mu_{\hat{G}\alpha}^i(xy), \mu_{\hat{G}\alpha}^j(xy)\}$

$= \max\{\mu_{\hat{G}\alpha}(xy)^i, \mu_{\hat{G}\alpha}(xy)^j\}$

$= (\mu_{\hat{G}\alpha}(xy))^i$

$\geq \min\{\mu_{\hat{G}\alpha}^i(x), \mu_{\hat{G}\alpha}^i(y)\}$

$\geq \min\{\max\{\mu_{\hat{G}\alpha}^i(x), \mu_{\hat{G}\alpha}^i(y)\}, \max\{\mu_{\hat{G}\alpha}^i(x), \mu_{\hat{G}\alpha}^j(y)\}\}$

$= \min\{\max\{\mu_{\hat{G}\alpha}^i(x), \mu_{\hat{G}\alpha}^j(x)\}, \max\{\mu_{\hat{G}\alpha}^i(y), \mu_{\hat{G}\alpha}^i(y)\}\}$

$= \min\{\mu_{\hat{G}\alpha i \cup \hat{G}\alpha j}(x), \mu_{\hat{G}\alpha i \cup \hat{G}\alpha j}(y)\}$.

FGI is satisfied.

(FG2) $\mu_{\hat{G}\alpha i \cup \hat{G}\alpha j}(x)$ $= \max\{\mu_{\hat{G}\alpha}^i(x), \mu_{\hat{G}\alpha}^i(x)\}$

$= \max\{\mu_{\hat{G}\alpha}(x)^i, \mu_{\hat{G}\alpha}(x)^j\}$

$= \max\{\mu_{\hat{G}\alpha}(x^{-1})^i, \mu_{\hat{G}\alpha}(x^{-1})^j\}$

$= \max\{\mu_{\hat{G}\alpha}^i(x^{-1}), \mu_{\hat{G}\alpha}^j(x^{-1})\}$

$= \mu_{\hat{G}\alpha i \cup \mu\hat{G}\alpha j}(x^{-1})$.

FG2 is satisfied.

(FG3) $\mu_{\hat{G}\alpha i} \cup \mu_{\hat{G}\alpha j}(e)$ $= \max\{\mu_{\hat{G}\alpha}^i(e)\ \mu_{\hat{G}}\alpha^j(e)\}$

$= \max\{1,1\}$ ($\hat{G}_\alpha$ is a fuzzy group)

$= 1$.

$\hat{G}_\alpha^i \cup \hat{G}_\alpha^j$ forms a fuzzy group.

7.2.3 Proposition: $\hat{G}_\alpha^i$ and $\hat{G}_\alpha^j$ are fuzzy groups. Then $\hat{G}_\alpha^i \cap \hat{G}_\alpha^j$ also a fuzzy group if $i < j$

Proof: Let $i < j$.

(FGI) $\mu_{\hat{G}\alpha i \cap \hat{G}\alpha j}(xy)$ $= \min\{\mu_{\hat{G}\alpha}^i(xy), \mu_{\hat{G}\alpha}^j(xy)\}$

$= \min\{\mu_{\hat{G}\alpha}(xy)^i, \mu_{\hat{G}\alpha}(xy)^j\}$

$= (\mu_{\hat{G}\alpha}(xy))^i$

$\geq \min\{\mu_{\hat{G}\alpha}^i(x), \mu_{\hat{G}\alpha}^i(y)\}$

$= \quad \min \{\mu_{\hat{G}\alpha}(x)^j, \mu_{\hat{G}\alpha}(y)^j\}$

$= \quad \min \{\min \{\mu_{\hat{G}\alpha}(x)^i, \mu_{\hat{G}\alpha}(x)^j\}, \min \{\mu_{\hat{G}\alpha}(y)^j, \mu_{\hat{G}\alpha}(y)^j\}$

$= \quad \min \{\min\{\mu_{\hat{G}\alpha}{}^i(x), \mu_{\hat{G}\alpha}{}^j(x)\}, \min \{\mu_{\hat{G}\alpha}{}^i(x), \mu_{\hat{G}\alpha}{}^j(y)\}\}$

$= \quad \min \{(\mu_{\hat{G}\alpha i \cap \hat{G}\alpha j}) (x), (\mu_{\hat{G}\alpha i \cap \hat{G}\alpha}{}^j) (y)\}.$

FGI is satisfied.

(FG2) $(\mu_{\hat{G}\alpha I \cap \hat{G}\alpha j}) (x) = \quad \min \{\mu_{\hat{G}\alpha}{}^i(x), \mu_{\hat{G}\alpha}{}^j(x)\}$

$= \quad \min \{\mu_{\hat{G}\alpha}{}^i(x^{-1}) , \mu_{\hat{G}\alpha}{}^i(x^{-1})$

(FG3) $\mu_{\hat{G}\alpha i \cap \hat{G}\alpha j} (e) = \quad \min \{\mu_{\hat{G}\alpha}{}^i(e) , \mu_{\hat{G}\alpha}{}^j(e)\}$

$= \quad \min \{1,1\} = 1$

FG3 is satisfied.

7.3 Section III: L-Fuzzy number

A fuzzy number is a fuzzy set on the real axis with convex, normal and continuous. For a mappingλ: $R \rightarrow [0, 1] = L$ associating with each real number t , its grade of membership λ (t). A fuzzy number λ is called convex if $\lambda (t) > \min \{\lambda(s), \lambda(\eta)\}$, $s < t < \eta$, $s, t, \eta \in R$. If λ is normal, then there exists $t_0 \in R$ such that $\lambda (t_o) = 1$.

A fuzzy number λ will be called upper semi continuous provided for all $t \in R$ with $\lambda (t) \leq \alpha$, there is $\delta > 0$ such that $|s-t| < \delta \rightarrow \lambda (s) < \alpha$. The fuzzy real line R_L consists of L-fuzzy numbers which fulfill regularity condition. If R_L denote the set of all regular L-fuzzy numbers, then R_L will be called the L-fuzzy real line. A fuzzy number λ is called non-negative if $\lambda(t) = 0$. The equality of fuzzy number λ and μ is defined by $\lambda = \mu$ if and only if $\lambda(t) = \mu (t)$ for all $t \in R$. The arithmetic operation +,-,. and /on $R_L \times R_L$ are defined by

(i) $(\lambda + \mu) (t) = V_{S\varepsilon R} \{\min (\lambda(s) , \mu(t-s)\} \; t \in R.$

(ii) $(\lambda - \mu) (t) = V_{S\varepsilon R} \{\min (\lambda(s) , \mu(s-t)\} \; t \in R.$

(iii) $(\lambda\mu) (t) = V_{S\varepsilon R} \{\min (\lambda s) , \mu(t/s)\} \; t \in R.$

Additive and multiplicative identities in R_L are $0(t) = \begin{cases} 1 & \text{if } t \neq 0 \\ 0 & \text{if } t = 0 \end{cases}$

$$\tilde{I} = \begin{cases} 1 & \text{if } t = 1 \\ 0 & \text{if } t \neq 1 \end{cases}$$

The operations + and • are associative and commutative with the identities 0 and 1 respectively.

The level set of sum, different and product fulfill the following conditions

$[\lambda + \mu] = [\lambda]_\alpha + [\mu]_\alpha$

In general, it is not oblivious that for f: $X \times Y \to [0, 1]$ and λ, μ fuzzy subsets of X and Y respectively. We would always have $f\{(\lambda, \mu)\} = f\{[\lambda]_\alpha, [\mu]_\alpha\}$ for all $\alpha \in L$. However it may be shown that if L = [0, 1] and f: $R \times R \to R$ be continuous. Then f $[(\lambda, \mu)]_\alpha = f[[\lambda]_\alpha, [\mu]_\alpha$ for all $\alpha \in L$ and all upper semi continuous L-fuzzy sets λ and μon R. A fuzzy number λ is regular if

(i) λ is convex

(ii) λ is normal

(iii) λ is upper semi continuous.

(iv) Each level of λ is bounded.

7.3.1 Proposition: λ is fuzzy number with convex function. Then λ is constant function.

Proof: $\lambda: R \to [0,1]$ is a map. Let a, b $\in$ R and if a < b. Let $\beta \in R$ such that $\beta < a$ and so α is convex function. $a < \alpha < b$ is got implies that $\lambda(\alpha) > \lambda(a)$ --- (i) and $\lambda(\alpha) > \lambda(b)$ --- (ii).

$a < b < b + \alpha$ implies that $\lambda(b) > \lambda(a)$ --- (iii) and $\beta < a < b$ so that $\lambda(a) > \lambda(b)$ --- (iv).

$a < b < b + \alpha$. Thus $\lambda(b) > \lambda(\alpha)$ gives $\lambda(\alpha) = \lambda(b)$ by (ii) and (iv).

It follows that $\lambda(x) = \lambda(b)$ for all $x \in (a, b) = \lambda(a)$. a > b is following in a similar way.

Let $\beta \in R$ such that $\beta < a < b$ by $\beta < a$ and $\lambda(\beta) = \lambda(a)$. Let $\gamma \in R$ such that $a < b < \gamma$, by $b < \gamma$ implies $\lambda(b) = \lambda(\gamma)$ so that $\lambda(\gamma)$ we conclude that $\lambda(a) = \lambda(b)$ for all a, b $\in$ R. λ is a constant function.

7.3.2 Proposition: Let a fuzzy number λ be a convex function. Then (R, +, 0) is a fuzzy group.

Proof: λ is a fuzzy number and convex function by (7.3.1). So λ is a constant function.

(FG1) Let x, y $\in$ R. It gives that $\lambda(x) = C, \lambda(y) = C$

$$\lambda (xy) = C = \min \{\lambda(x), \lambda(y)$$

(FGI) is satisfied in R. (FG2) $\lambda(x) =$ C for all x $\in$ R. $\lambda(x) = \lambda(-x)$. FG2 is satisfied in R.

7.4 section IV: Some Structure Properties of Intuitionistic Fuzzy Group and Fuzzy S-semi groups

The intuitionistic fuzzification of the concept of groups is considered and some properties of such groups are investigated. In this chapter, intuitionistic fuzzy groups (IFG) intuitionistic normal groups (ING) is defined and some characterizations of them are taken. The concept of Intuitionistic fuzzy set was introduced by Atanassov [1986] as a generalization of the notion of fuzzy sets. In this section, we present now some preliminaries on the theory of intuitionistic fuzzy sets.

7.4.1 Definition: An Intuitionistic fuzzy set A in the Universe of discourse

U is characterized by two membership functions given by

1. A truth Membership function $t_A : U \rightarrow [0,1]$
2. A false membership function.$f_A : U \rightarrow [0.1]$

where t_A (x) is a Lower bound of the grade of membership of x derived from the evidence for x and f_A (x) is a Lower bound on the negation of x derived from the evidence against x and $t_A (x)+ f_A (x) \leq 1$. The Intuitionistic fuzzy set A is written as $\tilde{A} = \{(x, (t_A (x) , f_A (x))\} / x \in U \}$ where the interval $[t_A (x) ,1- f_A (x)]$ is called intuitionistic values of x in A and denoted by I_A (x). In an intuitionistic fuzzy sets are independently proposed by the decision maker but they are mathematically not independent. This makes a Major difference in the judgement about the grade of membership.

7.4.2 Definition: An intuitionistic fuzzy set A of a Set U with $t_A(x) = 0$ and $f_A(x) = 1$ for all $x \in U$ is called the Zero intuitionistic fuzzy set of U. An intuitionistic Fuzzy set A of a Set U with $t_A(x) = 1$ and $f_A(x) = 0$ for all $x \in U$ is called unit intuitionistic Fuzzy set of U.

7.4.3 Definition; An intuitionistic fuzzy set A of a Set U with $t_A(x) = \alpha$ and $f_A(x) = 1 - \alpha$ for all $x \in U$ is called for α–Intuitionistic fuzzy set of U where $\alpha \in (0, 1)$.

7.4.4 Definition: Let (X, *) be a group. An intuitionistic fuzzy set A of X is called Intuitionistic fuzzy group of X if the following conditions are true.

$I_A(xy) \geq \text{Min}\{I_A(x), I_A(y)\}$ for all $x, y \in X$.

1. $t_A(xy) \geq \text{Min}\{t_A(x), t_A(y)\}$ implies that $t_A(x^{-1}) \geq t_A(x)$
2. $1\text{-}f_A(xy) \geq \text{Min}\{1\text{-}f_A(x), 1\text{-}f_A(y)\}$ implies that $1\text{-}f_A(x^{-1}) \geq 1\text{-}f_A(x)$

(Here the element xy stands for x*y).

The following propositions are proved on intuitionistic fuzzy groups

7.4.1 Proposition: If A is an intuitionistic Fuzzy group of X, then $I_A(x^{-1}) = I_A(x)$ for all $x \in X$.

Proof: So $t_A(x^{-1}) = t_A(x)$ and $1 - f_A(x^{-1}) = 1 - f_A(x)$ for all $x \in X$.

7.4.2 Proposition: Unit Intuitionistic fuzzy set, Zero Intuitionistic fuzzy set and α -Intuitionistic fuzzy set of a group 'X' are intuitionistic fuzzy group of X.

Proof: It is trivial.

7.4.3 Proposition: A set of necessary and sufficient conditions for al intuitionistic fuzzy set A to be an intuitionistic fuzzy group of X is that $I_A(xy^{-1}) \geq \min\{I_A(x), I(y)\}$.

Proof: Let A be an Intuitionistic fuzzy group of X. Then $t_A(xy^{-1}) \geq \min\{t_A(x), t_A(y^{-1})\} \geq \min\{t_A(x), t_A(y)\}$.

Similarly $1 - f_A(xy^{-1}) > \min\{1 - f_A(x), 1 - f_A(y)\}$.

For the Reverse part, suppose that A be an intuitionistic fuzzy set of the group X of which e is the identity element.

Now $t_A(yy^{-1}) \geq \min\{t_A(y), t_A(y)$ or $t_A(e) \geq t_A(y)$ - (i)

It follows that $t_A(ey^{-1}) \geq \min\{t_A(e)\, t_A(y)\}$ or $t_A(y^{-1}) \geq t_A(y)$ - (ii)

From (i) and (ii), $t_A(y^{-1}) = t_A(y)$

Also $t_A(xy) \geq \min\{t_A(x), t_A(y^{-1})\} \geq \min\{t_A(x), t_A(y)\}$.

Similarly it can be proved that

$$1- f_A(x^{-1}) \geq 1 - f_A(x) \text{ and } f_A(xy) \geq \min\{1 - f_A(x), 1 - f_A(y)\}.$$

7.4.4 Proposition: If A and B an two intuitionistic fuzzy groups of a group X, then $A \cap B$ is also intuitionistic Fuzzy Group of 'X'.

Proof: Since A and B are two IFG's of X.

$$t_{A \cap B}(xy^{-1}) = \min\{t_A(xy^{-1}), t_B(xy^{-1})\}$$
$$\geq \min\{\min\{t_A(x), t_A(y)\}, \text{Min}\{t_B(x), t_B(y)\}$$
$$\geq \min\{t_{A\cap B}(x), t_{A\cap B}(y)\}$$
$$1- f_{A \cap B}(xy^{-1}) = \min\{1- f_A(xy^{-1}), 1- f_B(xy^{-1})\}$$
$$\geq \min\{\text{Min}\{1- f_A(x), 1- f_A(y)\}, \text{Min}\{1- f_B(x), 1- f_B(y)\}\}$$
$$1- f_{A \cap B}(xy^{-1}) \geq \min\{1- f_{A\cap B}(x), 1- f_{A\cap B}(y)\}.$$

7.4.5 Proposition: If A and B are two intuitionistic Fuzzy Group of a Group X, then $A \cup B$ is also intuitionistic Fuzzy group of X

Proof: Let A and B be two IFG's of X.

$$t_{A \cup B}(xy^{-1}) = \max\{t_A(xy^{-1}), t_B(xy^{-1})\}$$
$$\geq \max\{\min\{t_A(x), t_A(y)\}, \text{Min}\{t_B(x), t_B(y)\}\}$$
$$\geq \min\{\max\{t_A(x), t_B(x)\}, \text{Max}\{t_A(y), t_B(y)\}\}$$
$$\geq \min\{t_{A\cup B}(x), t_{A\cup B}(y)\}$$
$$1- f_{A \cup B}(xy^{--1}) = \max\{\min\{1- f_A(x), 1- f_B(x)\}, \text{Min}\{1- f_A(y), 1- f_B(y)\}\}.$$
$$\geq \min\{\max\{1- f_A(x), 1- f_B(x)\}, \text{Max}\{1- f_A(y), 1- f_B(y)\}\}$$
$$\geq \min\{1- f_{A\cup B}(x), 1- f_{A\cup B}(y)\}.$$

This completes the proof.

7.4.6 Proposition: If $A = < x, t_A, f_A >$ is an Intuitionistic fuzzy group of a group X, then

i. t_A is a fuzzy group of X (ii) $1- f_A$ is a fuzzy group of X

Proof: The proof is straight forward

7.4.5: Definition: For α, $\beta \in [0, 1]$, (α, β) – cut and α - cut of a Intuitionistic Fuzzy Sets are defined. Let A be an intuitionistic fuzzy set of a Universe X with the true membership function t_A and the false membership $f_{A.}$ Then (α,β) – cut is defined as $(\alpha, \beta) = \{x \in X, I_A(x) \geq (\alpha, \beta)$ and α-cut is defined as $A_\alpha = \{x \in X, t_A (x) \geq \alpha \}$.

7.4.7: Proposition: Let A be an intuitionisitc fuzzy group of a group X. Then (α, β) – cut forms a subgroup of X for all α, β.

Proof: For all x, y $\in (\alpha, \beta)$, it follows that $t_A (x) \geq \alpha$, $1 - f_A(x) \geq \beta$ and $t_A (y) \geq 1 - f_A(y) \geq \beta$.

Now (i) $t_A (xy^{-1}) \geq \min \{t_A(x), t_A(y)\} \geq \min\{\alpha, \alpha\} \geq \alpha$.

(ii)$1- f_A (xy^{-1}) \geq \min \{1 - f_A(x), 1 - f_A(y)\} \geq \min \{\beta, \beta\} \geq \beta$

$xy^{-1} \in (\alpha, \beta)$ – cut. This completes the proof.

7.4.6 Definition: The sub groups like (α, β) cut are also called intuitionistic – cut sub group of a group X.

7.4.7 Definition: Let X be a group. A fuzzy set S is called a fuzzy semi group if $S (xy) \geq \min \{S(x), S(y)\}$ for all x, y $\in$ X.

7.4.8: Definition: An intuitionistic fuzzy set $A = (t_A, f_A)$ in S called an intuitionistic fuzzy sub semigroup of S if $t_A (xy) \geq \min \{t_A(x), t_A(y)\}$ and $f_A(xy) \leq \max \{ f_A(x) , f_A(y)\}$ for all x, y $\in$ S.

Example: Let S= {a,b,c} be the semi group with the following Multiplication table;

	a	b	C
a	a	a	B
b	c	b	A
c	b	a	C

Let $\mu : S\rightarrow [0,1]$ defined by $\mu = 0.5/a +0.6/b + 0.6/c$ but it is not a fuzzy sub semi group.

7.4.8: Proposition: An intuitionistic fuzzy Sub semi group $A = (t_A, f_A)$ of S is called an intuitionistic fuzzy group of 'S' if $t_A(x^{-1}) \geq t_A(x)$; $f_A(x^{-1}) \leq f_A(x)$

7.4.9 Proposition: Every intuitionistic fuzzy sub semi group of S is constant.

Proof: Let $A = (t_A, f_A)$ be an intuitionistic fuzzy group of a group S and let $x\in S$ then.

$t_A (x) = t_A (ee) = t_A(xx^{-1}x^{-1}x) = t_A(x(x^{-1}x^{-1})x) \geq \text{Min } \{t_A(x), t_A(y)\} = t_A(x)$

and $f_A = f_A (ee) = f_A (\{xx^{-1}x^{-1}x) \leq \text{Max } \{f_A (x), f_A (x)\}$ $f_A (x)$ where e is the identity of S.

It follows that $t_A(x) = t_A(e)$ and $f_A(x) = f_A(e)$ which means that $A = (t_A, f_A)$ is constant.

7.4.9: Definition: Let A be an intuitionistic fuzzy group of a group X. Then A is called intuitionistic normal group if $I_A(xy) = I_A(yx)$ for all $x, y \in X$.
Alternatively $I_A(x) = I_A(yxy^{-1})$ for all $x, y \in X$.

7.4.10 Note: The notion [x, y] is used for the expression $x^{-1} y^{-1} xy$.

7.4.10 Proposition: Let A be a intuitionistic normal group of a group X. Then $I_A(x, y) = I_A(e)$ for all $x, y \in X$.

Proof: Since A is a Intuitionistic normal group of X, it gives that $I_A(x) = I_A(yxy^{-1})$ for all $x, y \in X$.
Replacing $x = y^{-1}$ and $y = x^{-1}$, it gets that $I_A(y^{-1}) = I_A(x^{-1}y^{-1}xy)$ or $I_A(x^{-1}y^{-1}xyy^{-1}) = I_A(y^{-1})$ or $I_A[(x, y)y^{-1}] = I_A(y^{-1})$ or $I_A[x, y] = I_A(e)$. Hence it completes the proof.

Conclusion: Group theory has many applications in Physics, Chemistry and Computer science problem. we have defined intuitionistic fuzzy groups and studied some properties of intuitionistic Fuzzy Groups. The concept is analogous notion of fuzzy groups introduced by Rosenfeld [1971], of the unknown Part $[1- t_A(x) - f_A(x)]$ is zero for all x (of the group X) ,