CHAPTER V
STRUCTURE PROPERTIES OF M - FUZZY GROUPS

Introduction: Fuzzy group was extended by Roventa [2001] who have introduced the concept of fuzzy groups operating on fuzzy sets. Wu [1981] studied the fuzzy normal subgroups. Gu [1994] put forward the notion of fuzzy groups with operators. In this chapter, we introduce the concept of M- fuzzy groups with operators and obtain some related results. For the sake of convenience, we set out the former concepts.

5.2 Section-II: Preliminaries

5.2.1 Definition: A group with operators is an algebraic system consisting of a group G, a set M and a function defined in the product set M×G and having the values in G such that if mx denotes the element in G determined by the element x of G and the element m of M, and m $\in$ M, then G is called M- group with operators.

5.2.2 Definition: A subgroup A of an M- group G is said to be the fuzzy subgroup if mx $\in$ A for all m $\in$ M and x$\in$ A.

5.2.3 Definition: Let A be a fuzzy set in U and $\bullet$: G × G → G be a composition law, such that (G, $\bullet$) forms M- group. Let two conditions be (FG1) $A(m(xy)) \geq \min \{ A(mx) , A(my)\}$ and (FG2) $A(mx^{-1}) = A(mx)$ for all x, y in A. If the supplementary conditions $A(me_G) = 1$ is also satisfied, then the M-fuzzy group is called a standardized M- fuzzy group, where e_G is an identity of M- group (G, $\bullet$).

Example: Let A be an M- subgroup of M- group G and let A be a fuzzy set in G defined by

$\mu_A(x)$ = 0.7 if x έ A; 0.2 otherwise for all x έ X. Then it is easy to verify that A is M- fuzzy group of G.

Example: Let G be an M- group while A is a non empty subset of G. If χ_A is the characteristic function of A, then A is an M –subgroup of G if and only if A is M – fuzzy subgroup of G.

In this chapter, the theory of fuzzy sets has developed in many directions and is finding application in a wide variety of fields. Rosenfeld [1971] used this concept to develop the theory of fuzzy groups, We have given independent proof of several theorems on M- fuzzy groups. We discuss about M- fuzzy groups and investigate some of their structures on the concept of M-fuzzy group family.

The following are the propositions on M- fuzzy groups.

5.2.1.Proposition: Let A be M- fuzzy group and S be a fuzzy subset of A then S is a M- fuzzy subgroup of A if and only if $A_S(m(x\ y)) \geq \min (\{ A_S(mx), A_S(my)\}$ for all x, y in S.

Proof: Case-(1). Let S be a M- fuzzy subgroup of A. S itself is a M- fuzzy group. Thus (FG1) and (FG2) are satisfied in S. Therefore it gives

$A_S(m(xy^{-1})) \geq \min \{ A_S(mx), A_S(my^{-1})\} = \min \{ A_S(mx), A_S(my)\}$.

Case (2): Let $A_S(m(xy^{-1})) \geq \min\{A_S(mx), A_S(my)\}$ for all $x,y \in S$ ----(1). If y = x in (1) , $A_S(m(xy^{-1})) = A_S(me) \geq \min \{A_S(mx), A_S(my)\} = A_S(mx)$. If x = e in (1). $A_S(my^{-1}) = A_S(m(ey^{-1})) \geq \min \{ A_S(me), A_S(my^{-1})\} = A_S(my)$, for all $x,y \in S$.

Implies that $A_S(my) \geq A_S(my^{-1})$ for all x, y in S.since substituting $y=y^{-1}$ in the above equation Therefore $A_S(my) = A_S(my^{-1})$. FG2 is satisfied in S. Further $A_S(m(x(y^{-1})) \geq \min (A_S(mx), A_S(m(y^{-1})\} = \min \{ A_S(mx), A_s(my)\}$, FG1 is satisfied in S.

5.2.2 Lemma: For all $a, b \in [0, 1]$, $m \in M$ and p is any positive integer, verify that (i) If $ma \leq mb$, then $(ma)^p \leq (mb)^p$ and (ii) $\min \{ma, mb\}^p = \min \{(ma)^p, (mb)^p\}$.

Proof: It is obvious.

5.2.3 Proposition: If A is a M- fuzzy group, then $A^p = ((mx,(A(mx)^p)/mx \in A\}$ is M-fuzzy group.

Proof: Let A is M- fuzzy group. Where (A, •) is M- group. Thus (A^p, •) is M- fuzzy group for all the positive integer p. Let $x,y \in A^p$ where p is any positive integer.

(FG1) $A^p(m(xy)) = (A(m(xy)^p)) \geq \min \{A(mx) , A(my)\}^p = \min \{ A(mx)^p , A(my)^p\}$. FG1is satisfied in A^p. (FG2) $A^p(mx) = (A(mx)^p = (A(mx^{-1})^p = A^p(mx^{-1})$.($A^p$, •) is M- fuzzy group by (5.2.2). $A^p(me) = (A(me)^p = (1)^p$ and so (A^p, •) is standardized M- fuzzy group.

5.2.4 Corollary: The M- fuzzy group A^q is a M- fuzzy subgroup A^p ,if $q \leq p$.

Proof: Clearly A^q and A^p are M- fuzzy groups by (5.2.3). For all $x \in [0, 1]$, $x^q \geq x^p$ implies that $A^q \subset A^p$ (since $A^q(x) \leq A^p(x)$ for all $x \in A$.

5.2.5 Proposition: If A^i and A^j are M-fuzzy groups, then $A^i \cup A^j$ is also M- fuzzy group for all natural numbers i and j.

Proof: Let $i < j$.

(FG1) $A^i \cup A^j (m(xy)) = \max \{ A^i(m(xy)) , A^j(m(xy)\}$

$$= \max \{ (A(m(xy))^i , (A(m(xy))^j\}$$

$$= (A(m(xy))^i$$

$$= A^i(m(xy))$$

$$\geq \min \{ A^i(mx) , A^j(my)\} \text{ by (5.2.3)}$$

$$= \min \{\max \{A^i(mx) , A^j(mx)\} , \max \{A^i(my) , A^j (my)\}\}.$$

$$\geq \min \{ A^i \cup A^j(mx) , A^i \cup A^j (my)\}.$$

Therefore $A^i \cup A^j$ is M-fuzzy group by (5.2.4).

(FG2) $A^i \cup A^j (mx) = \max \{ A^i(mx) , A^j(mx\}$

$$= \max \{ (A(mx))^i , (A(mx))^j \}$$

$$= \max \{ (A(mx^{-1})^i , A (mx^{-1})^j \}$$

$$= A^i \cup A^j (mx^{-1}).$$

5.2.6 Proposition: If A^i and A^j are M-fuzzy groups, then $A^i \cap A^j$ is also M-fuzzy groups, where i and j are natural numbers.

Proof: It is obvious.

5.2.7 Lemma: Prove that $A^p \subset A$ for all p..

Proof: Let $x \in A$. Therefore $A^p(mx) \leq A(mx)$ for all natural number p since $(A(mx)^P \leq A(mx)$.

5.2.5 Definition: Let $f: G \rightarrow G^1$ be a homomorphism's of M- groups. For any fuzzy set $A \in G^1$, a new fuzzy set $A^f \in G$ is defined by $A^f(mx) = A(f(mx))$ for all $x \in G$.

5.2.8 Proposition : Let G and G^1 be M – groups and f an M- homomorphism from G onto G^1. (i) if A is M – fuzzy group of G^1, then A^f is M- fuzzy group of G. (ii) if A^f is M- fuzzy group of G, then A is M- fuzzy group of G^1.

Proof: (i) Let $x,y \in G$ and $m \in M$. It gives that

(FG1) $A^f(m(xy)) = A(f(mx), f(my)\}$

$\geq \min\{Af(mx), Af(my)\}$

$= \min\{A^f(mx), A^f(my)\}$

(FG2) $A^f(mx) = A(f(mx)) = A(f(mx)^{-1})$

$= Af(mx^{-1}) = A^f(mx^{-1})$ .Therefore A^f is M- fuzzy group of G.

(ii) For any $x,y \in G$ and $m \in M$, there exists $a, b \in G$ such that $f(ma) = x$ and $f(mb) = y$.
(FG1) $A(m(xy)) = A\{f(ma), f(mb)\}$

$= A^f(m(ab))$

$\geq \min\{A^f(ma), A^f(mb)\}$

$= \min\{Af(ma), Af(mb)\}$

$= \min\{A(mx), A(my)\}$

(FG2) $A(mx^{-1}) = A(f(ma))^{-1} = Af(ma^{-1}) = A^f(ma^{-1}) = A^f(ma) = A(mx)$.

5.2.9 Proposition: Let < A > = { A , A^1 , A^2, …, A^P,…, E}. Then $\cup A^P = A$ and $\cap A^P = E$ where p varies 1 to ∞.

Proof: Let $x \in A$; $m \in M$. Clearly $A \subset \cup A^P$ …(i) (since $A(mx) \leq \cup A^p(mx)$) where p varies 1 to ∞. $\cup A^P(mx) = \max\{A(mx), A^2(mx), \ldots, \}$

$$= A(mx) \quad (\text{since } A(mx) \leq (A(mx))^p \text{ for all natural number p.}$$

It gives that $\cup A^p \subset A$ --- (ii). From (i) and (ii), $\cup A^p = A$.

Claim: $E = \cap A^p$. Let $me \in A^P$ implies that $(me,1) \in A^p$ for all p.

implies that $(me,1) \in \cap A^p$ and $E \subset \cap A^p$ … (iii) where p varies 1 to ∞.

Let $mx \in \cap A^p$ implies that $mx \in A^p$ for all p varing1 to ∞.

$\cap A^P(mx) = \min\{A(mx), A^2(mx)\} = 0$ if $mx \neq me$; $= 1$ if $mx = me$.

$x \in \cap A^p$ implies that $mx = me$ thus $\cap A^p \subset E$.-- (iv). From (iii) and (iv), $E = \cap A^P$.

5.2.10 Proposition: Let G and G^1 be M- groups and f be homomorphism from G onto G^1. If A^f is M- fuzzy group of G , then A is M- fuzzy group of G^1.

Proof: By proposition (5.2.8) , A is M- fuzzy group of G^1. For any $y \in G^1$ and $m \in M$.

(FG1) $A^f(my) = \sup_{z \in f^{-1}(my)} A(z) \geq \sup_{mf(x) = my} A(mx) \geq \sup_{f(x)=y} A(x) = A^f(y)$

(FG2) $A^f(mx^{-1}) = \sup_{z \in f^{-1}(mx^{-1})} A(z) = \sup_{mx \in f^{-1}(mx^{-1})} A(mx)$ where $x \in f^{-1}(y)$

$$= \sup_{f(mx) = mx^{-1}} A(mx) = \sup_{mf(x) = mx^{-1}} A(mx)$$

$$\geq \sup_{f(x) = x^{-1}} A(x^{-1})$$

$$= A^f(x^{-1}).$$

So A is M- fuzzy group of G^1.

5.2.6 Definition: Let A be a M- fuzzy group.. Then the following set of M- fuzzy groups $\{A, A^1, A^2,..A^p ,\ldots, E\}$ is called M- fuzzy group family generated by A. It is denoted by < A >.

5.2.11 Proposition: Let A be a M- fuzzy group. Then $A \supset A^2 \supset A^3 \ldots. \supset A^p ,..., E$..

Proof: It is known that $A(ma) \in [0, 1]$ implies that $A(ma) \geq A(ma)^2$, $A(ma^2) \geq (A(ma^2))^2 ,\ldots,$ $A(ma^n) \geq (A(ma^n))^{2.}$ by using the definition of fuzzy subsets. This gives that $A \supset A^2$. By generalizing it for any natural numbers i and j with $i \leq j$, $(A_i(ma))^i \geq (A_i (ma))^j$, $(A_i(ma^2))^i \geq (A(ma^2)^j$, …, $(A_i (ma^n))^i \geq A_j(ma^n))^{j.}$. So $A^i \supset A^j$ for any natural numbers i and j with $i \leq j$ which means that $A \supset A^2 \supset A^3 \ldots. \supset (A^{p)n} \ldots$. Finally $E = \cap A^p$ which is immediate from proposition (5.2.9). Since $A (ma)^p = 1$ as $n \to \infty$ if ma = me. = 0. If ma $\neq$ me, the required relations are obtained.

Conclusion: W.M.Wu [1981] and A. Rosenfeld [1971] introduced the concept of fuzzy normal subgroups and fuzzy groups. we investigate the concept of M- fuzzy groups and obtain some Results.

5.3 Section III: Characteristics of weaker fuzzy groups in terms of Rosenfeld's fuzzy groups

Introduction: Ray [1999] discussed that in the short communication, some properties of the product of two fuzzy subsets and fuzzy subgroups. Rovento [2001] proved that the crisp environment the notions of normal subgroup and group operating on a set are well known due to many applications. we study extensions of these classical notions to the larger universe of fuzzy sets. We obtain a characterization of operations of fuzzy group on a fuzzy set in terms of homomorphisms of crisp groups. Ray (Fuzzy sets and systems 105 (1999) 181-183 studied some results of the product of fuzzy sets and fuzzy subgroups. Ray's results will be generalized.

Further more, we define p-level subset and p-level subgroups and then we study of their properties.

Kuroki [1992] showed that the lattice FNS(G) of all fuzzy normal subgroups of a group G is isomorphic to the lattice fuzzy congruence of G of all fuzzy congruence's on G. Also they explained FNS (G) forms a modular lattice for every $\alpha \in [0,1]$ and let G and G^1 be groups and $f : G \rightarrow G^1$ be homomorphism's. If [Ã] is a fuzzy (normal) subgroup of G, then f[Ã] is a fuzzy (normal) subgroup of G^1. Finally they described let G, G^1 and 'f'be as above. If [Ã] is a fuzzy normal subgroup of G^1 then G/f^{-1} [Ã] $\equiv$ f(G) / [Ã] . Hence the author proved that in the theory of groups, there exists a close relationship between normal subgroups and congruence's. It is a natural question to extend the relationship of these to the case of fuzzy group theory.

We have define fuzzy congruence's on groups and fuzzy quotient groups by fuzzy congruence's and investigate their properties. Rosenfeld defined fuzzy subgroupoids and proved that a homomorphic image of a fuzzy subgroupoid with the sup property was a fuzzy groupoid and hence that a homomorphic image of a fuzzy subgroup with sup property was a fuzzy subgroup. This theorem needs the sup property, but we can show the theorem without sup property. Moreover Mukherjee and Bhattacharyya showed that if [Ã] is a fuzzy subgroup of a finite group G such that all the level subgroups of G are normal subgroups, then [Ã] is a fuzzy normal subgroup. They can also prove the theorem without finites using the transfer principle which is a fundamental tool developed here.

Wanging [1981] proved that if A is fuzzy subgroup of G, then gAg^{-1} is also fuzzy subgroup of G for all g in G and $\cap\ gAg^{-1}$ is a normal subgroup of G (under a t- norm as A). They showed that let A and B be two fuzzy subgroups of G under the t- norms T_1 and T_2 respectively, then $A\cap B$ is a fuzzy subgroups under any t- norm T such that $T_1,T_2 \geq T$. The intersection of any two normal fuzzy subgroups of G is also a normal fuzzy subgroup of G under any t- norm weaker than the t- norms of the two fuzzy subgroups. They also explained let $f : G \rightarrow H$ be a group homomorphism.

If A is a normal fuzzy subgroups of H, then $f^{-1}(A)$ is a normal fuzzy subgroup of G and if f is an epimorphism then f(A) is a normal fuzzy subgroup of H. we derived that $G^t = \{ x \in G /$

$A(x) \geq t$ } is a subgroup of G. The normaliser of a fuzzy subgroup of G is a subgroup of G. The concept of a normal fuzzy subgroup and proved some properties of this new concept.

H. Sherwood [1983] introduced the concept of product of fuzzy groups (Fuzzy sets & fuzzy systems) and its properties. A new class of fuzzy groups that is introduced is weaker than the standard fuzzy groups defined by Rosenfeld [1971] and characterize some properties of weaker fuzzy groups.

. Anthony and Sherwood [1983] redefined fuzzy groups in terms of t- norm which is replaced the min operations of Rosenfeld's definition. Some properties of these redefined fuzzy groups, which we call t- fuzzy groups, have been developed by Sherwood [1983], Sessa [1984], sidky and misherf [1991]. However the definition of t- fuzzy groups seems to be too general. We define a new class of fuzzy group which is weaker than the fuzzy groups defined by Rosenfeld's [1971] and characterize some properties of weaker fuzzy groups.

5.3.1 Definition: A function A from a set X to the closed unit interval [0, 1] in U is called a fuzzy set in X, for every $x \in A$, A(x) is called membership grade of x in A. The set $\{ x \in A / A(x) > 0 \}$ is called the support of A and it is denoted by supp(A). For fuzzy sets λ and μ in a set X, then λ o μ has been defined in most articles by

$$(\lambda o \mu)(x) = \begin{cases} \sup\limits_{ab=x} \min\{\lambda(a), \mu(b)\}, & \text{if } ab=x \\ 0 & \text{if } ab \neq x \end{cases}$$

We weaker this definition as follows.

5.3.2 Definition: Let X be a set and let λ ,μ be two fuzzy sets in X. So λ o μ is defined by

$$(\lambda o \mu)(x) = \begin{cases} \sup\{\lambda(a), \mu(b)\} & \text{if } ab = x \\ 0 & \text{if } ab \neq x. \end{cases}$$

5.3.3 Definition: Let X be a group. We define λ^{-1} by $\lambda^{-1}(x) = \lambda(x^{-1})$ for $x \in X$. The standard definition of a fuzzy group is that a fuzzy set 'A' in a group X is a fuzzy group if and only if $A(xy) \geq \min\{A(x), A(y)\}$ and $A(x^{-1}) = A(x)$ for all $x,y \in X$. This definition is weaken as follows.

5.3.4 Definition: Let S be a groupoid. A function $A: S \to [0, 1]$ is a weaker groupoid in S if and only if for every x, y in S , (WF1): $A(xy) \geq A(x)\, A(y)$, a weaker fuzzy groupoid is denoted by a w- fuzzy groupoid. If X is a group, a weaker fuzzy groupoid 'A. in X if and only if for $x \in X$, (WF2) : $A(x^{-1}) = A(x)$, a weaker fuzzy group X is denoted by a w- fuzzy group. Since $\min(a, b) \geq ab$, our definition of a w-fuzzy group is weaker than the by Rosenfeld [1971]. It is easy to see that if G is fuzzy group in a group X and e is the identity of X , $G(e) \geq G(x)$ for all $x \in X$. If G is a w-fuzzy group in a group X, $G(e) = G(xx^{-1}) \geq G(x)\, G(x^{-1}) = [G(x)]^2$ for all $x \in X$

The following are expressed on the properties of w- fuzzy groups

5.3.1 Proposition: Let A be a fuzzy subset in a group X such that $A(e) = 1$, where e is the identity of X. Then A is a w- fuzzy group if and only if $A(xy^{-1}) \geq A(x) . A(y)$ for all $x, y \in X$.

Proof: Suppose A is w-fuzzy group. Then $A(xy^{-1}) \geq A(x)\, A(x^{-1}) = A(x)\, A(y)$ .Then $A(xy^{-1}) \geq A(x)\, A(y)$. Then $A(x^{-1}) = A(ex^{-1}) \geq A(e)\, A(x^{-1}) \geq A(x) = A(ex) \geq A(e)\, A(x) = A(x^{-1})$. $A(x) = A(x^{-1})$ and $A(xy^{-1}) = A(x(y^{-1})) \geq A(x)\, A(y^{-1}) = A(x)\, A(y)$.

5.3.5 Definition: Let A be a w- fuzzy groupoid in a group X such that $A(a) = A(a^{-1})$. Let $e_\lambda: X \to X$ be identity defined by $e_\lambda(x) = x\lambda$ for all $x \in A$. Similar we define the left identity of A.

5.3.2 Proposition: If w-fuzzy groupoid A on X has left identity e_λ and a right identity e_μ , then $e_\lambda = e_\mu$.

Proof: $e_\lambda(A)$ (x) = sup A(z) = $A(x^{-1}a)$. So $z \in e_\lambda^{-1}(x)$ ≥ A(x) $A(a^{-1})$ = A(x) A(a) = A(x) = $A(xa^{-1}a) \geq A(xa^{-1})$ A(a) = A (xa^{-1}) = $e_\lambda(A)(x)$.

Thus $e_\lambda(A)$ (x) ≥ A(x) ≥ $e_\lambda(A)(x)$. That is $e_\lambda(A)$ = A. similarly we may show $e_\mu(A)$ = A.

5.3.6 Definition: Let f: G→ G^1 be a homomorphism of fuzzy groups. For any fuzzy set A ∈ G^1, we define a new fuzzy set A^f in G by $A^f(x)$ = Af(x) for all x ∈ G, and $f(x^{-1})$ =f(x)= $(f(x))^{-1}$.

5.3.7 Definition: Let A and B be two fuzzy subsets of X. Then the direct product A × B is defined by (A × B) (x, y) = min {A(x) , B(y)} and (x, y) • (z, p) = (xz, yp) for all x, y, z, p in X.

5.3.3 Proposition: Let G and G^1 be groups and f a homomorphism from G onto G^1. (i) if A is w-fuzzy group of G^1,then A^f is w- fuzzy group of G. (ii) if A^f is w- fuzzy group of G, then A is w-fuzzy group of G^1.

Proof: (i) Let x, y ∈ G.

It follows that (WF1) $A^f(xy)$ = Af(xy) = A (f(x) f(y)) ≥ Af(x) Af(y) = $A^f(x)$ $A^f(y)$.

(WF2) $A^f(x^{-1})$ = $Af(x^{-1})$ = Af(x) = $A^f(x)$. So A^f is w- fuzzy group of G^1.

(ii) for any x,y ∈ G^1, There exists a ,b ∈ G such that f(a) = x and f(b) = y.

(WF1) A(xy) = A(f(a) f(b)) = Af(ab) = $A^f(ab) \geq A^f(a)$ $A^f(b)$ = Af(a) Af(b) = A(x) A(y)

(WF2) $A(x^{-1})$ = $A(f(a^{-1}))$ = $Af(a^{-1})$ = $A^f(a^{-1})$ = $A^f(a)$ = Af(a) = A(x). A is w-fuzzy group of G^1.

5.3.3 Proposition : If A and B be w- fuzzy groups of G_1 and G_2 respectively, then A × B is w- fuzzy group of $G_1 \times G_2$

Proof: Let (a_1,b_1) , (a_2,b_2) $\in$ $G_1 \times G_2$

(WF1) $A \times B((a_1,b_1)(a_2,b_2)) = (A \times B)(a_1a_2, b_1b_2)$

$= \min\{A(a_1a_2), B(b_1b_2)\}$

$\geq \min\{A(a_1)A(a_2), B(b_1)B(b_2)\}$

$\geq \min\{A(a_1)B(b_1), A(a_2)B(b_2)\}$

$\geq \min\{A(a_1)B(b_1)\} \min\{A(a_2)B(b_2)\}$

$\geq A \times B(a_1,b_1)\ A \times B(a_2,b_2)$

(WF2) $A \times B(a_1, b_1)^{-1} = A \times B(a_1^{-1},b_1^{-1})$

$= \min\{A(a_1^{-1}), B(b_1^{-1})\}$

$= \min\{A(a_1), B(b_1)\}$

$= A{\times}B(a_1,b_1)$

5.3.4 Corollary: If A_1, A_2 ,…, A_n are w- fuzzy groups of G_1, G_2…, G_n respectively, then $A_1 \times A_2 \times$,…, $\times A_n$ is w- fuzzy groups of $G_1 \times G_2 \times \ldots, \times G_n$.

Proof: This result can easily show by induction method.

5.3.5 Proposition: Let A and B be fuzzy subsets of G_1 and G_2 respectively such that $A \times B$ is a w- fuzzy group of $G_1 \times G_2$ then A and B is w- fuzzy group of G_1 and G_2 respectively.

Proof: $(A \times B)(e_1, e_2) = \min(A(e_1), A(e_2)\} \geq (A \times B)(x,y)$ for all $(x, y) \in G_1 \times G_2$. Then $A(x) \leq A(e_1)$ or $B(y) \leq B(e_2)$. If $A(x) \leq A(e_1)$ then $A(x) \leq B(e_2)$ or $B(y) \leq B(e_2)$.

Let $A(x) \leq B(e_2)$. Then for all $x, y \in G$ $(A \times B)(x,e_2) = A(x)$

(WF1) $A(xy) = (A \times B)(xy, e_2)$

$= (A \times B)((x, e_2)(y, e_2))$

$\geq (A \times B)(x, e_1)\ (A \times B)(y, e_2)$

$= \min \{ A(a_1 a_2), B(b_1 b_2)\}$

$\geq \min \{ A(a_1) A(a_2), B(b_1) B(b_2)\}$

$\geq \min \{ A(a_1) B(b_1), A(a_2) B(b_2)\}$

$\geq \min \{ A(a_1) B(b_1)\} \min \{A(a_2) B(b_2)\}$ $\geq$
$A \times B (a_1,b_1) \; A\times B(a_2,b_2)$

(WF2) $A\times B(a_1,b_1)^{-1} = A \times B (a_1^{-1}, b_1^{-1})$

$= \min \{A(a_1^{-1}), B(b_1^{-1})\}$

$= \min (A(a_1), B(b_1)\}$

$= A \times B (a_1,b_1)$

5.3.6 Proposition: Let A and B be fuzzy sets of G_1 and G_2 respectively such that $A \times B$ is w- fuzzy group of $G_1\times G_2$. Then A and B are w-fuzzy groups of G_1 and G_2 respectively.

Proof: $(A \times B)(e_1,e_2) = \min \{(A(e_1), A(e_2)\} \geq (A \times B)(x, y)$ for all $(x, y) \in G_1 \times G_2$. Then $A(x) \leq A(e_1)$ or $B(y) \leq B(e_2)$. If $A(x) \leq A(e_1)$ then $A(x) \leq B(e_2)$ or $B(y) \leq B(e_2)$.

Let $A(x) \leq B(e_2)$. Then for all $x, y \in G$ $(A\times B)(x,e_2) = A(x)$

(WF1) $A(xy) = (A\times B)(xy,e_2) = (A\times B)((x,e_2)(y,e_2))$

$\geq (A\times B)(x,e_1) \; (A\times B)(y,e_2)$

$\geq A(x) A(y)$

(WF2) $A(x^{-1}) = (A \times B)(x^{-1},e_2)$

$= (A \times B)(x^{-1},e_2^{-1})$

$= (A \times B)(x,e_2)^{-1}$

$= (A \times B)(x,e_2)$

$= A(x)$ therefore A is w- fuzzy group of G.

Now suppose that $A(x) \leq B(e_2)$ is not true for all $x \in G_1$. If $A(x) \geq B(e_2)$, there exists $x \in G_1$, Then $B(y) \leq B(e_2)$ for all $y \in G_2$. Therefore

$(A \times B)(e_1, y) = B(y)$ for all $y \in G_2$. Similarly for all $x, y \in G_2$,

$$\begin{aligned} B(xy) &= (A \times B)(e_1, xy) \\ &= (A \times B)((e_1, x)(e_2, y)) \\ &\geq (A \times B)(e_1, x)(A \times B)(e_1, y) \\ &= B(x)B(y) \end{aligned}$$

And

$$\begin{aligned} B(x^{-1}) &= (A \times B)(e^1, x^{-1}) \\ &= (A \times B)(e_1^{-1}, x^{-1}) \\ &= (A \times B)(e_1, x)^{-1} \\ &= (A \times B)(e_1, x) = B(x). \end{aligned}$$

Hence B is w- fuzzy group of G_2 consequently either A or B is w- fuzzy group of G_1 or G_2 respectively.

5.3.8 Definition: Let $f: G \to G^1$ be a group homomorphism's and A be w- fuzzy group of G^1.

Then $Af(x) = (Aof)(x) = f^{-1}(A)(x)$.

5.3.7 Proposition:: Let $f: G \to G^1$ be a group homomorphism and A be a w- fuzzy group of G^1.

Then $f^{-1}(A)$ is w- fuzzy group of G.

Proof: Let $x, y \in G$. It gives that

(WF1) $$\begin{aligned} f^{-1}(A)(xy) &= (A \circ f)(xy) = A f(xy) \\ &= A(f(x).f(y)) \\ &\geq Af(x).Af(y) \\ &\geq (A \circ f)(x)\,(A \circ f)(y) \\ &\geq f^{-1}(A)(x)\,.f^{-1}(A)(y) \end{aligned}$$

(WF2) $f^{-1}(A)(x^{-1}) = (A \circ f)(x^{-1}) = Af(x^{-1}) = A(f(x)) = f^{-1}(A)(x)$

5.3.8 Proposition: Let A be a w- fuzzy group of group G and A* be a fuzzy set in G defined by $A^*(x) = A(x) + 1 - A(e)$ for all $x \in G$. Then A* is w- fuzzy group of G containing A.

Proof: For $x, y \in G$, it gives that

(WF1) $A^*(xy) = A(xy) + 1 - A(e)$

$\geq (A(x)\ A(y)) + 1 - A(e)$

$\geq (A(x)+1-A(e))\ (A(y)+1-A(e))$

$\geq A^*(x)\ A^*(y)$

(WF2) $A^*(x^{-1}) = A(x^{-1}) + 1 - A(e)$

$= A(x) + 1 - A(e)$

$= A^*(x)$

A* is w- fuzzy group of G containing A.