Chapter 4

Co-quasi-variational Inequalities

In this chapter, we consider two different classes of generalized co-quasi-variational inequalities in the setting of Banach spaces. By using the sunny nonexpensive retractions, we construct the projection iterative methods for finding the approximate solutions of our problems. Some existence and convergence results are also derived. In the last section, we consider the multivalued co-quasi-variational inequality problem for fuzzy mappings. Following the technique of the first two sections, we give an iterative algorithm and prove the convergence results for the approximate solutions obtained by our proposed algorithm. The existence result for a solution of this problem is also investigated.

4.1 Introduction

The projection iterative method is one of the most important and useful methods for finding the approximate solutions of fixed point problems, and variational and quasi-variational inequality problems; See for example [25, 38, 40, 56, 57, 60, 70, 71, 87, 88] and references therein. Most of the papers appeared in the literature on this topic, the metric projection operators, like in Hilbert spaces, are used. But it is impossible to use metric projection operator in the setting of Banach spaces because these operators are not nonexpansive. Recently, Takahashi and Kim [93] used sunny nonexpansive retraction to set up an iterative scheme for finding a fixed point of a nonexpansive and nonself mappings in Banach spaces. Inspired by the work of Takahashi and Kim [93], Alber and Yao [3] used sunny nonexpansive retraction to construct the projection iterative method for finding the approximate solutions of a class of multivalued quasi-variational inequalities in Banach spaces. They gave the name *co-quasi-variational inequality* for quasi-variational inequality in Banach spaces and presented an iterative algorithm. They also proved several convergence results for approximate solutions obtained by their algorithm and in particular several existence results are obtained. Recently, Chang [19] also studied the existence of solutions and convergence of Mann and Ishikawa iterative processes for a class of variational inclusions with accretive type mappings in Banach spaces. The mathematical approach in [19] is quite different from the one used by Alber and Yao [3]. In this chapter, we mainly follow the approach of Alber and Yao [3].

4.2 Generalized Multivalued Mixed Co-quasi- variational Inequalities

In this section, we consider generalized multivalued mixed co-quasi-variational inequality problem in the setting of Banach spaces. By extending the terminology and method of Alber and Yao [3], we suggest and analyze an iterative algorithm to compute the approximate solution of our problem with noncompact valued mappings. We also prove convergence result for the approximate solutions obtained by our algorithm.

Let B be a real Banach space with its dual B^* and $\langle x, f\rangle$ be a pairing between $x \in B$ and $f \in B^*$. Given single valued mappings $f, g, p, G : B \to B$ and multivalued mappings $M, S, T, K : B \to 2^B$ such that $\forall\, x \in B$, $K(x)$ is nonempty, closed and convex subset of B, we consider the following *generalized multivalued mixed co-quasi-variational inequality problem*:

$$\text{(GMMCQVIP)} \quad \begin{cases} \text{Find } x \in B,\ u \in M(x),\ v \in S(x), \text{ and } w \in T(x) \\ \text{such that } G(x) \in K(x) \text{ and} \\ \langle p(u) - (f(v) - g(w)), J(z - G(x))\rangle \geq 0, \quad \forall\, z \in K(x), \end{cases}$$

where $J : B \to B^*$ is the normalized duality mapping. If B is a real Hilbert space, then (GMMCQVIP) reduces to the generalized multivalued mixed quasi-variational inequality problem which is more general and seems to be new one.

Special Cases:

(i) If $p \equiv 0$, g and M are identity mappings and T is a single valued mapping, then (GMMCQVIP) reduces to the problem of finding $x \in B$ and $v \in S(x)$ such that $G(x) \in K(x)$ and

$$\langle T(x) - f(v), J(z - G(x))\rangle \geq 0, \quad \forall\, z \in K(x). \tag{4.2.1}$$

A problem similar to (4.2.1) is considered and studied by Alber and Yao [3]. They employed the sunny nonexpansive retraction method to formulate characterization of solutions of such problem. An iterative algorithm for finding the approximate solutions is suggested by them. They also derived some existence and convergence results.

(ii) If B is a Hilbert space, f, g and M are identity mappings, $G(x) \equiv p(x)$ and $K(x) = K$, then (GMMCQVIP) reduces to the following *generalized variational inequality problem* considered and studied by Verma [98]:

$$\text{(GVIP)} \quad \begin{cases} \text{Find } x \in B,\ v \in S(x), \text{ and } w \in T(x) \text{ such that} \\ \langle p(x) - (v - w), z - p(x)\rangle \geq 0, \quad \forall\, z \in K. \end{cases}$$

It is clear that the generalized multivalued mixed co-quasi-variational inequality problem includes many kinds of quasi-variational inequalities, variational inequality and complementarity problems as special cases, such as in [40, 87, 98, 100] and references therein.

Now we first derive some characterizations for a solution of (GMMCQVIP).

Lemma 4.2.1. *Let B be a Banach space, $f, g, p, G : B \to B$ be single valued mappings and $M, S, T : B \to CB(B)$, and $K : B \to 2^B$ multivalued mappings such that $\forall\, x \in B$, $K(x)$ is a nonempty, closed and convex subset. Then the following statements are equivalent:*

(a) *The set of elements* (x, u, v, w) *such that* $x \in B$, $u \in M(x)$, $v \in S(x)$ *and* $w \in T(x)$, *is a solution of (GMMCQVIP).*

(b) $x \in B$, $u \in M(x)$, $v \in S(x)$, $w \in T(x)$ *and*

$$G(x) = Q_{K(x)}[G(x) - \tau(p(u) - (f(v) - g(w))] \quad \text{for any} \quad \tau > 0.$$

For the proof of above lemma, we refer to [3] and references therein.

Combining Proposition 1.2.10 and Lemma 4.2.1, we obtain the following result on the characterization of solutions for (GMMCQVIP).

Lemma 4.2.2. *Let* B *be a real Banach space,* X *a nonempty, closed and convex subset of* B. *Let* $f, g, p, m, G : B \to B$ *be single valued mappings, and* $M, S, T : B \to CB(B)$ *and* $K : B \to 2^B$ *be multivalued mappings such that* $\forall\ x \in B$, $K(x) = m(x) + X$. *Then the set of elements* (x, u, v, w) *such that* $x \in B$, $u \in M(x)$, $v \in S(x)$ *and* $w \in T(x)$ *is a solution of (GMMCQVIP) if and only if*

$$x = x - G(x) + m(x) + Q_X[G(x) - \tau(p(u) - (f(v) - g(w))) - m(x)], \quad \text{for any} \quad \tau > 0.$$

To compute the approximate solutions of (GMMCQVIP), we propose the following iterative algorithm.

Algorithm 4.2.1. *Let* $K(x) = m(x) + X$, *where* X *is a nonempty, closed and convex subset of* B *and* $\tau > 0$ *be fixed. Let* $f, g, p, G : B \to B$ *be single valued mappings and* $S, M, T : B \to CB(B)$ *be multivalued mappings. For given* $u_0 \in M(x_0)$, $v_0 \in S(x_0)$ *and* $w_0 \in T(x_0)$, *we let*

$$x_1 = x_0 - G(x_0) + m(x_0) + Q_X[G(x_0) - \tau(p(u_0) - (f(v_0) - g(w_0))) - m(x_0)].$$

Since $u_0 \in M(x_0) \in CB(B)$, $v_0 \in S(x_0) \in CB(B)$, $w_0 \in T(x_0) \in CB(B)$, *by Nadler [69], there exist* $u_1 \in M(x_1)$, $v_1 \in S(x_1)$, $w_1 \in T(x_1)$ *such that*

$$\|u_0 - u_1\| \leq (1 + 1)\mathcal{H}(M(x_0), M(x_1)),$$

$$\|v_0 - v_1\| \leq (1 + 1)\mathcal{H}(S(x_0), S(x_1)),$$

$$\|w_0 - w_1\| \leq (1+1)\mathcal{H}(T(x_0), T(x_1)).$$

Let

$$x_2 = x_1 - G(x_1) + m(x_1) + Q_X[G(x_1) - \tau(p(u_1) - (f(v_1) - g(w_1))) - m(x_1)].$$

By induction, we can obtain the sequences $\{x_n\}$, $\{u_n\}$, $\{v_n\}$ *and* $\{w_n\}$ *as*

$$x_{n+1} = x_n - G(x_n) + m(x_n) + Q_X[G(x_n) - \tau(p(u_n) - (f(v_n) - g(w_n))) - m(x_n)], \quad (4.2.2)$$

$$u_n \in M(x_n), \quad \|u_n - u_{n+1}\| \leq (1 + (n+1)^{-1})\ \mathcal{H}(M(x_n), M(x_{n+1})),$$

$$v_n \in S(x_n), \quad \|v_n - v_{n+1}\| \leq (1 + (n+1)^{-1})\ \mathcal{H}(S(x_n), S(x_{n+1})),$$

$$w_n \in T(x_n), \quad \|w_n - w_{n+1}\| \leq (1 + (n+1)^{-1})\ \mathcal{H}(T(x_n), T(x_{n+1})),$$

$n = 0, 1, 2, 3, \ldots.$

Now we prove that the approximate solutions of (GMMCQVIP) obtained by Algorithm (4.2.1) converge to the exact solution of (GMMCQVIP).

Theorem 4.2.3. *Let B be a real uniformly smooth Banach space with the module of smoothness $\tau_B(t) \leq Dt^2$ for some $D > 0$. Let X be a nonempty, closed and convex subset of $B, f, g, p, G : B \to B$ be single valued mappings, and $M, S, T : B \to CB(B)$ and $K : B \to 2^B$ be multivalued mappings such that $\forall\ x \in B,\ K(x) = m(x) + X$. Suppose that the following conditions are satisfied:*

(i) *f, g, and p are Lipschitz continuous with corresponding constants ξ, r and σ, respectively.*

(ii) *G is both strongly accretive with constant γ and Lipschitz continuous with constant δ.*

(iii) *M, S, T are $\mathcal{H}$-Lipschitz continuous with corresponding constants s, h and d, respectively.*

(iv) *m is Lipschitz continuous with constant θ.*

(v) $0 < 2(1 - 2\gamma + 64D\delta^2)^{\frac{1}{2}} + 2\theta + \tau\sigma s + [1 + \tau(\xi h - rd)] < 1.$

Then there exists a set of elements (x,u,v,w) such that $u \in M(x)$, $v \in S(x)$ and $w \in T(x)$ which is a solution of (GMMCQVIP) and $x_n \to x$, $u_n \to u$, $v_n \to v$, $w_n \to w$ as $n \to \infty$, where $\{x_n\}$, $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are the sequences obtained by Algorithm (4.2.1).

Proof. By the iterative scheme (4.2.2) and Proposition 1.2.10, we have

$$\begin{aligned}
||x_{n+1} - x_n|| &= ||x_n - G(x_n) + m(x_n) + Q_X[G(x_n) - \tau(p(u_n) \\
&\quad -(f(v_n) - g(w_n))) - m(x_n)] - (x_{n-1} - G(x_{n-1}) + m(x_{n-1}) \\
&\quad -Q_X[G(x_{n-1}) - \tau(p(u_{n-1}) - (f(v_{n-1}) - g(w_{n-1}))) - m(x_{n-1})]|| \\
&\leq ||x_n - x_{n-1} - (G(x_n) - G(x_{n-1}))|| + 2||m(x_n) - m(x_{n-1}]|| \\
&\quad +||x_n - x_{n-1} - (G(x_n) - G(x_{n-1}))|| + \tau||p(u_n) - p(u_{n-1})|| \\
&\quad +||x_n - x_{n-1} + \tau(f(v_n) - f(v_{n-1})) - \tau(g(w_n) - g(w_{n-1}))|| \\
&= 2||x_n - x_{n-1} - (G(x_n) - G(x_{n-1}))|| + 2||m(x_n) - m(x_{n-1})|| \\
&\quad +\tau||p(u_n) - p(u_{n-1})|| + ||x_n - x_{n-1} \\
&\quad +\tau(f(v_n) - f(v_{n-1})) - \tau(g(w_n) - g(w_{n-1}))||. \qquad (4.2.3)
\end{aligned}$$

By Proposition 1.2.8, we have (see, for example, the proof of Theorem 3 in [3]).

$$||x_n - x_{n-1} - (G(x_n) - G(x_{n-1}))||^2 \leq (1 - 2\gamma + 64D\delta^2)||x_n - x_{n-1}||^2. \qquad (4.2.4)$$

It is clear that

$$||m(x_n) - m(x_{n-1})|| \leq \theta||x_n - x_{n-1}||. \qquad (4.2.5)$$

Since M, S and T are $\mathcal{H}$-Lipschitz continuous, and f, g and p are Lipschitz continuous, we have

$$||p(u_n) - p(u_{n-1})|| \leq \sigma||u_n - u_{n-1}|| \leq \sigma s\left(1 + \frac{1}{n}\right)||x_n - x_{n-1}||, \qquad (4.2.6)$$

$$||f(v_n) - f(v_{n-1})|| \leq \xi||v_n - v_{n-1}|| \leq \xi h\left(1 + \frac{1}{n}\right)||x_n - x_{n-1}||, \qquad (4.2.7)$$

$$||g(w_n) - g(w_{n-1})|| \leq r||w_n - w_{n-1}|| \leq rd\left(1 + \frac{1}{n}\right)||x_n - x_{n-1}||. \qquad (4.2.8)$$

From (4.2.7) - (4.2.8), it follows that

$$\begin{aligned}
&\|x_n - x_{n-1} + \tau(f(v_n) - f(v_{n-1})) - \tau(g(w_n) - g(w_{n-1}))\| \\
\leq\ & \|x_n - x_{n-1}\| + \tau\|f(v_n) - f(v_{n-1})\| - 2\tau\|g(w_n) - g(w_{n-1})\| \\
\leq\ & \|x_n - x_{n-1}\| + \tau\xi h(1 + 1/n)\|x_n - x_{n-1}\| - \tau r d(1 + 1/n)\|x_n - x_{n-1})\| \\
=\ & [1 + \tau(1 + 1/n)(\xi h - rd)]\|x_n - x_{n-1}\|. \qquad (4.2.9)
\end{aligned}$$

Combining (4.2.3) - (4.2.6) and (4.2.9), it follows that

$$\|x_{n+1} - x_n\| \leq t_n\|x_n - x_{n-1}\|,$$

where $t_n = 2(1 - 2\gamma + 64D\delta^2)^{\frac{1}{2}} + 2\theta + \tau\sigma s\left(1 + \frac{1}{n}\right) + [1 + \tau(1 + 1/n)(\xi h - rd)]$.

Let $t = 2(1-2\gamma+64D\delta^2)^{\frac{1}{2}}+2\theta+\tau\sigma s+[1+\tau(\xi h-rd)]$. Then $t_n \to t$ as $n \to \infty$. It follows from (v) that $t < 1$. Hence $t_n < 1$ for n sufficiently large. Consequently $\{x_n\}$ is a Cauchy sequence in B. Let $x_n \to x \in B$. Now we prove that $u_n \to u \in M(x)$, $v_n \to v \in S(x)$ and $w_n \to w \in T(x)$. In fact, it follows from Algorithm 4.2.2 that

$$\|u_n - u_{n-1}\| \leq \left(1 + \frac{1}{n}\right) s\|x_n - x_{n-1}\|,$$

$$\|v_n - v_{n-1}\| \leq \left(1 + \frac{1}{n}\right) h\|x_n - x_{n-1}\|,$$

$$\|w_n - w_{n-1}\| \leq \left(1 + \frac{1}{n}\right) d\|x_n - x_{n-1}\|,$$

which implies that $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are also Cauchy sequences in B. Let $u_n \to u$, $v_n \to v$, $w_n \to w$ as $n \to \infty$. Since Q_X, G, f, g, p, M, S, T and m are continuous in B, we have

$$x = x - G(x) + m(x) + Q_X[G(x) - \tau(p(u) - (f(v) - g(w))) - m(x)].$$

Further, we have

$$\begin{aligned}
d(v, S(x)) &= \inf\{\|v - y\| : y \in S(x)\} \\
&\leq \|v - v_n\| + d(v_n, S(x)) \\
&\leq \|v - v_n\| + \mathcal{H}(S(x_n), S(x)) \\
&\leq \|v - v_n\| + h\|x_n - x\| \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Hence $v \in S(x)$. Similarly we can prove that $u \in M(x)$, $w \in T(x)$. The result then follows from Theorem 4.2.2.

□

4.3 Completely Generalized Multivalued Co-quasi-variational Inequalities

In this section, we consider the completely generalized multivalued co-quasi-variational inequality problem in the setting of Banach spaces which is also considered and studied by Noor et al [77]. By using the sunny nonexpansive retraction operator, we give the characterization of solutions of our problem. This characterization is used to suggest and analyze an iterative algorithm for finding the approximate solutions of our problem. The existence and convergence results are also studied. Several special cases of our problem are also mentioned. Our approach is different from the one used by Noor et al [77].

Let B be a real Banach space with its dual B^* and $\langle x, f\rangle$ be a pairing between $x \in B$ and $f \in B^*$. Let $N(.,.) : B \times B \to B$ and $G : B \to B$ be nonlinear mappings and let $T, A : B \to C(B)$ and $K : B \to 2^B$ be multivalued mappings such that $\forall\ x \in B$, $K(x)$ is a nonempty, closed and convex set. We consider the following *completely generalized multivalued co-quasi-variational inequality problem* :

$$\text{(CGMCQVIP)} \quad \begin{cases} \text{Find } x \in B,\ u \in T(x), \text{ and } v \in A(x) \\ \text{such that } G(x) \in K(x) \text{ and} \\ \langle N(u,v), J(z - G(x))\rangle \geq 0, \quad \forall\ z \in K(x), \end{cases}$$

where $J : B \to B^*$ is the normalized duality operator. This problem is also considered by Noor et al [77].

SPECIAL CASES:

(i) If T is a single valued nonlinear operator and $N(u, v) = T(x) + A(v)$, then (CGMCQVIP) is equivalent to the following problem considered and studied by Alber and Yao [3]:

$$\text{(GMCVIP)} \quad \begin{cases} \text{Find } x \in B \text{ and } v \in A(x) \text{ such that} \\ G(x) \in K(x) \text{ and} \\ \langle T(x) + A(v), J(z - G(x))\rangle \geq 0, \quad \forall\ z \in K(x). \end{cases}$$

It is called *generalized multi-valued co-variational inequality problem.* Alber and Yao [3] suggested an iterative algorithm for finding the approximate solutions of this problem. They also derived several convergence results for their algorithm and in particular several existence results for a solution of this problem.

(ii) When B is a Hilbert space, J reduces to the identity mapping. Consequently, (GMCVIP) reduces to the following problem:

$$\text{(GMVVIP)} \quad \begin{cases} \text{Find } x \in B \text{ and } v \in A(x) \text{ such that} \\ G(x) \in K(x) \text{ and} \\ \langle T(x) + A(v), z - G(x) \rangle \geq 0, \quad \forall\, z \in K(x). \end{cases}$$

It is called *generalized multi-valued variational inequality problem*, and it is introduced and studied by Jou and Yao [59].

It is clear that (CGMCQVIP) is more general and unifying one which includes many known problems considered and studied in the literature.

Definition 4.3.1. Let $T, A : B \to C(B)$ be two multivalued mappings and $N(.,.) : B \times B \to B$ be a nonlinear mapping. The mapping $u \mapsto N(u, v)$ is said to be *strongly accretive with respect to* T if for any $x_1, x_2 \in B$, there exists a constant $t > 0$ such that $\forall\, u_1 \in T(x_1),\ u_2 \in T(x_2)$ and $\forall\, v \in B$,

$$\langle N(u_1, v) - N(u_2, v), J(x_1 - x_2) \rangle \geq t\|x_1 - x_2\|^2.$$

Definition 4.3.2. The mapping $N(.,.) : B \times B \to B$ is said to be *Lipschitz continuous with respect to first argument* if there exists a constant $\beta > 0$ such that

$$\|N(u_1, .) - N(u_2, .)\| \leq \beta\|u_1 - u_2\|, \quad \forall\, u_1, u_2 \in B.$$

Now we first give some characterizations of solutions of (CGMCQVIP) by using sunny nonexpansive retraction operator. Then by using these characterizations, we suggest iterative algorithm for computing the approximate solutions of (CGMC-QVIP).

Lemma 4.3.1. *Let B be a real Banach space and $N(.,.) : B \times B \to B$ be a nonlinear mapping. Let $T, A : B \to C(B)$ and $K : B \to 2^B$ be multivalued maps such that*

$\forall\ x \in B$, $K(x)$ is nonempty, closed and convex subset of B. Then the following statements are equivalent:

(a) The triplet (x,u,v), where $x \in B$, $u \in T(x)$, and $v \in A(x)$, is a solution of (CGMCQVIP).

(b) $x \in B$, $u \in T(x)$, $v \in A(x)$ and $G(x) = Q_{K(x)}[G(x) - \tau(N(u,v))]$ for any $\tau > 0$.

Proof. It is similar to the proof of Theorem 1 in [3] (see also the proof of Theorem 3.1 in [40] and Theorem 8.1 in [2]). □

By combining Proposition 1.2.10 and Lemma 4.3.1, we have the following result.

Lemma 4.3.2. *Let X be a nonempty, closed and convex subset of a real Banach space B. Let $N(.,.) : B \times B \to B$ and $g : B \to B$ be nonlinear mappings and $T, A : B \to C(B)$ and $K : B \to 2^B$ be multivalued maps such that $\forall\ x \in B$, $K(x) = m(x) + X$. Then the triplet (x,u,v), where $x \in B$, $u \in T(x)$, and $v \in A(x)$, is a solution of (CGMCQVIP) if and only if*

$$x = x - G(x) + m(x) + Q_X[G(x) - \tau N(u,v) - m(x)], \quad \text{for any } \tau > 0.$$

We now construct the iterative algorithm for finding the approximate solutions of (CGMCQVIP).

Algorithm 4.3.1. *Let $K(x) = m(x) + X$, where X is a nonempty, closed and convex subset of a real Banach space B and $\tau > 0$ be fixed. Given $x_0 \in B$, take any $u_0 \in T(x_0)$, $v_0 \in A(x_0)$ and let*

$$x_1 = x_0 - G(x_0) + m(x_0) + Q_X[G(x_0) - \tau N(u_0,v_0) - m(x_0)].$$

Since $T(x_0)$ and $A(x_0)$ are nonempty and compact sets, there exist $u_1 \in T(x_1)$ and $v_1 \in A(x_1)$ such that

$$\|u_0 - u_1\| \leq \mathcal{H}(T(x_0), T(x_1))$$

and

$$\|v_0 - v_1\| \leq \mathcal{H}(A(x_0), A(x_1)).$$

Let

$$x_2 = x_1 - G(x_1) + m(x_1) + Q_X[G(x_1) - \tau N(u_1, v_1) - m(x_1)].$$

By induction, we obtain sequences $\{x_n\}$, $\{u_n\}$ *and* $\{v_n\}$ *such that*

$$u_n \in T(x_n), \ \|u_n - u_{n+1}\| \leq \mathcal{H}(T(x_n), T(x_{n+1}))$$

$$v_n \in A(x_n), \ \|v_n - v_{n+1}\| \leq \mathcal{H}(A(x_n), A(x_{n+1})),$$

and

$$x_{n+1} = x_n - G(x_n) + m(x_n) + Q_X[G(x_n) - \tau(N(u_n, v_n)) - m(x_n)], \tag{4.3.1}$$

$n = 0, 1, 2 \ldots.$

Next we prove that the approximate solutions of (CGMCQVIP) obtained by Algorithm 4.3.1 converge to the exact solution of (CGMCQVIP), and in particular, we prove the existence of a solution of (CGMCQVIP).

Theorem 4.3.3. *Let* B *be a real uniformly smooth Banach space with the module of smoothness* $\tau_B(t) \leq Ct^2$ *for some* $C > 0$*. Let* X *be a nonempty, closed and convex subset of* B, $N(.,.) : B \times B \to B$ *and* $G, m : B \to B$ *be nonlinear mappings and* $T, A : B \to C(B)$ *and* $K : B \to 2^B$ *be multivalued maps such that* $\forall\ x \in B,\ K(x) = m(x) + X$*. Suppose that the following conditions are satisfied:*

- *(i)* $N(.,.)$ *is strongly accretive with respect to* T *and* A *with corresponding constants* $t > 0$ *and* $s > 0$, *respectively.*
- *(ii)* $N(.,.)$ *is Lipschitz continuous in both the arguments with corresponding constants* $\beta > 0$ *and* $\alpha > 0$, *respectively.*
- *(iii)* G *is both strongly accretive with constant* $\gamma > 0$ *and Lipschitz continuous with constant* $\delta > 0$.
- *(iv)* m *is Lipschitz continuous with constant* $\theta > 0$.
- *(v)* T *and* A *are* $\mathcal{H}$*-Lipschitz continuous with constant* $\xi > 0$ *and* $\eta > 0$, *respectively.*

(vi) $0 < 2(1-2\gamma+64C\delta^2)^{\frac{1}{2}} + 2\theta + (1-2\tau(t+s)+64C\tau^3(\alpha^2\eta^2+\beta^2\xi^2))^{\frac{1}{2}} < 1.$

Then there exist $x \in B$, $u \in T(x)$, and $v \in A(x)$ such that the triplet (x,u,v) is a solution of (CGMCQVIP) and the sequences $\{x_n\}$, $\{u_n\}$ and $\{v_n\}$ generated by Algorithm 4.3.1 converge strongly to x, u and v, respectively, that is, $x_n \to x$, $u_n \to u$ and $v_n \to v$ as $n \to \infty$.

Proof. By the iterative scheme (4.3.1) and Proposition 1.2.10, we have

$$\begin{aligned}
\|x_{n+1}-x_n\| &= \|x_n - G(x_n) + m(x_n) + Q_X[G(x_n) - \tau N(u_n,v_n) - m(x_n)] \\
&\quad -x_{n-1} - G(x_{n-1}) + m(x_{n-1}) \\
&\quad -Q_X[G(x_{n-1}) - \tau N(u_{n-1},v_{n-1}) - m(x_{n-1}))]\| \\
&\leq \|x_n - x_{n-1} - (G(x_n) - G(x_{n-1}))\| + 2\|m(x_n) - m(x_{n-1})\| \\
&\quad +\|x_n - x_{n-1} - (G(x_n) - G(x_{n-1}))\| \\
&\quad +\|x_n - x_{n-1} - \tau(N(u_n,v_n) - N(u_{n-1},v_{n-1}))\| \\
&= 2\|x_n - x_{n-1} - (G(x_n) - G(x_{n-1}))\| \\
&\quad +2\|m(x_n) - m(x_{n-1})\| \\
&\quad +\|x_n - x_{n-1} - \tau(N(u_n,v_n) - N(u_{n-1},v_{n-1}))\|. \qquad (4.3.2)
\end{aligned}$$

By Proposition 1.2.8, we have (see, for example, the proof of the Theorem 3 in [3])

$$\|x_n - x_{n-1} - (G(x_n) - G(x_{n-1}))\|^2 \leq (1-2\gamma+64C\delta^2)\|x_n - x_{n-1}\|^2. \qquad (4.3.3)$$

Since $N(.,.)$ is strongly accerative with respect to the mappings T and A, $N(.,.)$ is Lipschitz continuous in both the arguments, by using Proposition 1.2.8 and Algorithm 4.3.1, we have

$$\begin{aligned}
&\|x_n - x_{n-1} - \tau(N(u_n,v_n) - N(u_{n-1},v_{n-1}))\|^2 \\
&\leq \|x_n - x_{n-1}\|^2 - 2\tau\langle N(u_n,v_n) - N(u_{n-1},v_{n-1}), \\
&\qquad J(x_n - x_{n-1}) - \tau(N(u_n,v_n) - N(u_{n-1},v_{n-1}))\rangle \\
&= \|x_n - x_{n-1}\|^2 - 2\tau\langle N(u_n,v_n) - N(u_{n-1},v_{n-1}), \\
&\qquad J(x_n - x_{n-1})\rangle - 2\tau\langle N(u_n,v_n) - N(u_{n-1},v_{n-1}), J(x_n - x_{n-1}) \\
&\qquad -\tau(N(u_n,v_n) - N(u_{n-1},v_{n-1}))) - J(x_n - x_{n-1})\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \|x_n - x_{n-1}\|^2 - 2\tau\langle N(u_n, v_n) - N(u_{n-1}, v_n) \\
&\quad + N(u_{n-1}, v_n) - N(u_{n-1}, v_{n-1}), J(x_n - x_{n-1})\rangle \\
&\quad - 2\tau\langle N(u_n, v_n) - N(u_{n-1}, v_{n-1}), J(x_n - x_{n-1}) \\
&\quad - \tau(N(u_n, v_n) - N(u_{n-1}, v_{n-1}))) - J(x_n - x_{n-1})\rangle \\
&= \|x_n - x_{n-1}\|^2 - 2\tau\langle N(u_n, v_n) - N(u_{n-1}, v_n), \\
&\quad J(x_n - x_{n-1})\rangle - 2\tau\langle N(u_{n-1}, v_n) - N(u_{n-1}, v_{n-1}), \\
&\quad J(x_n - x_{n-1})\rangle - 2\tau\langle (N(u_n, v_n) - N(u_{n-1}, v_{n-1}), J(x_n - x_{n-1} \\
&\quad - \tau(N(u_n, v_n) - N(u_{n-1}, v_{n-1}))) - J(x_n - x_{n-1})\rangle \\
&\leq \|x_n - x_{n-1}\|^2 - 2\tau t\|x_n - x_{n-1}\|^2 - 2\tau s\|x_n - x_{n-1}\|^2 \\
&\quad + 4d^2\rho_B\tau\left(\frac{4\tau\|N(u_n, v_n) - N(u_{n-1}, v_{n-1})\|}{d}\right) \\
&= (1 - 2\tau(t + s))\|x_n - x_{n-1}\|^2 + 4d^2\rho_B\tau(\frac{4\tau}{d}\|N(u_n, v_n) \\
&\quad - N(u_n, v_{n-1}) + N(u_n, v_{n-1}) - N(u_{n-1}, v_{n-1})\|) \\
&\leq (1 - 2\tau(t + s))\|x_n - x_{n-1}\|^2 \\
&\quad + 4d^2\rho_B\tau(\frac{4\tau}{d}\|N(u_n, v_n) - N(u_n, v_{n-1})\| \\
&\quad + \|N(u_n, v_{n-1}) - N(u_{n-1}, v_{n-1})\|) \\
&\leq (1 - 2\tau(t + s))\|x_n - x_{n-1}\|^2 + 64C\tau^3(\|N(u_n, v_n) \\
&\quad - N(u_n, v_{n-1})\|^2 + \|N(u_n, v_{n-1}) - N(u_{n-1}, v_{n-1})\|^2) \\
&\leq (1 - 2\tau(t + s))\|x_n - x_{n-1}\|^2 \\
&\quad + 64C\tau^3(\alpha^2\|v_n - v_{n-1}\|^2 + \beta^2\|u_n - u_{n-1}\|^2) \\
&\leq (1 - 2\tau(t + s))\|x_n - x_{n-1}\|^2 \\
&\quad + 64C\tau^3(\alpha^2\mathcal{H}^2(A(x_n), A(x_{n-1})) + \beta^2\mathcal{H}^2(T(x_n), T(x_{n-1}))) \\
&\leq (1 - 2\tau(t + s))\|x_n - x_{n-1}\|^2 \\
&\quad + 64C\tau^3(\alpha^2\eta^2\|x_n - x_{n-1}\|^2 + \beta^2\xi^2\|x_n - x_{n-1}\|^2) \\
&= (1 - 2\tau(t + s) + 64C\tau^3(\alpha^2\eta^2 + \beta^2\xi^2))\|x_n - x_{n-1}\|^2. \qquad (4.3.4)
\end{aligned}
$$

It is clear from the Lipschitz continuity of m that

$$\|m(x_n) - m(x_{n-1})\| \leq \theta\|x_n - x_{n-1}\|. \qquad (4.3.5)$$

From (4.3.3) - (4.3.5), we have the following inequality

$$||x_{n+1} - x_n|| \leq k||x_n - x_{n-1}||,$$

where

$$k = 2(1 - 2\gamma + 64C\delta^2)^{\frac{1}{2}} + 2\theta + (1 - 2\tau(t+s) + 64C\tau^3(\alpha^2\eta^2 + \beta^2\xi^2))^{\frac{1}{2}}$$

and $0 < k < 1$ by (vi). Consequently, $\{x_n\}$ is a Cauchy sequence and thus converges to some $x \in B$. Now we prove that $u_n \to u \in T(x)$ and $v_n \to v \in A(x)$. From Algorithm 4.3.1, we have

$$||u_{n+1} - u_n|| \ \leq \mathcal{H}(T(x_{n+1}), T(x_n)) \leq \xi||x_{n+1} - x_n||$$

and

$$||v_{n+1} - v_n|| \ \leq \mathcal{H}(A(x_{n+1}), A(x_n)) \leq \eta||x_{n+1} - x_n||$$

which imply that the sequences $\{u_n\}$ and $\{v_n\}$ are Cauchy in B. Let $u_n \to u$ and $v_n \to v$. Since Q_X, G, T, A, $N(.,.)$ and m are continuous, we have

$$x = x - G(x) + m(x) + Q_X[G(x) - \tau N(u,v) - m(x)].$$

It remains to show that $u \in T(x)$ and $v \in A(x)$. In fact,

$$\begin{aligned} d(u, T(x)) &= \inf\{||u - w|| : w \in T(x)\} \\ &\leq ||u - u_n|| + d(u_n, T(x)) \\ &\leq ||u - u_n|| + \mathcal{H}(T(x_n), T(x)) \\ &\leq ||u - u_n|| + \xi||x_n - x|| \to 0 \text{ as } n \to \infty. \end{aligned}$$

Hence $d(u, T(x)) = 0$ and therefore $u \in T(x)$. Similarly, we can prove that $v \in A(x)$. The result then follows from Lemma 4.3.2. □

4.4 Generalized Multivalued Co-quasi-variational Inequalities with Fuzzy Mappings

In an unpublished work, Ansari [4] introduced the concept of variational inequalities for fuzzy mappings, called *fuzzy variational inequalities*, in his Ph.D. thesis. Separately, Chang and Zhu [23] also studied a class of variational inequalities for fuzzy mappings. Several kinds of variational and quasi-variational inequalities for fuzzy mappings are considered and studied by Chang [18], Chang and Huang [22], Noor [73] and Lee et al [62]. Motivated and inspired by the work going in this field, in this section, we consider the multivalued co-quasi-variational inequality problem for fuzzy mappings in the setting of Banach spaces. Following the technique of previous sections, we give an iterative algorithm for computing the approximate solutions of our problem. The existence and convergence results are also studied.

Let B be a real Banach space and B^* be its topological dual space. Let $\langle .,. \rangle$ be the dual pairing between B and B^*. Let $\mathcal{F}(B)$ be a collection of all fuzzy sets over B. A mapping $P : B \to \mathcal{F}(B)$ is said to be a *fuzzy mapping.* For each $x \in B, P(x)$ (denoted by P_x in the sequel) is a fuzzy set on B and $P_x(y)$ is the membership function of y in P_x.

A fuzzy mapping $P : B \to \mathcal{F}(B)$ is said to be *closed* if for each $x \in B$, the function $y \mapsto P_x(y)$ is upper semicontinuous, that is, for any given net $\{y_\alpha\} \subset B$ satisfying $y_\alpha \to y_0 \in B$, $\lim sup_\alpha \; P_x(y_\alpha) \leq P_x(y_0)$. For $C \in \mathcal{F}(B)$ and $\lambda \in [0,1]$, the set $(C)_\lambda = \{x \in B : C(x) \geq \lambda\}$ is called a *λ-cut set* of C.

A closed fuzzy mapping $A : B \to \mathcal{F}(B)$ is said to satisfy condition $(*)$: if there exists a function $a : B \to [0,1]$ such that for each $x \in B, (A_x)_{a(x)}$ is a nonempty and bounded subset of B.

It is clear that if A is a closed fuzzy mapping satisfying condition $(*)$, then for each $x \in B$, the set $(A_x)_{a(x)} \in CB(B)$.

In fact, let $\{y_\alpha\}_{\alpha\in\Gamma} \subset (A_x)_{a(x)}$ be a net and $y_\alpha \to y_0 \in B$. Then $(A_x)(y_\alpha) \geq a(x)$ for each $\alpha \in \Gamma$. Since A is closed, we have

$$A_x(y_0) \geq \lim sup_{\alpha\in\Gamma} A_x(y_\alpha) \geq a(x).$$

This implies that $y_0 \in (A_x)_{a(x)}$ and so $(A_x)_{a(x)} \in CB(B)$.

Let $P, Q : B \to \mathcal{F}(B)$ be two closed fuzzy mappings satisfying condition $(*)$. Then there exist two functions $a, b : B \to [0,1]$ corresponding to P and Q, respectively, such that for each $x \in B$, we have $(P_x)_{a(x)}, (Q_x)_{b(x)} \in CB(B)$. Therefore, we can define two multivalued mappings $\tilde{P}, \tilde{Q} : B \to CB(B)$ by

$$\tilde{P}(x) = (P_x)_{a(x)}, \; \tilde{Q}(x) = (Q_x)_{b(x)}, \quad \forall\, x \in B.$$

In the sequel, $\tilde{P}$ and $\tilde{Q}$ are called the *multivalued mappings induced by the fuzzy mappings* P and Q respectively.

Let $T, A, G : B \to B$ be single valued mappings, $P, Q : B \to \mathcal{F}(B)$ be two fuzzy mappings. Let $a, b : B \to [0,1]$ be given functions. Let $K : B \to 2^B$ such that $\forall\, x \in B$, $K(x)$ is a nonempty, closed and convex. We consider the following *multivalued fuzzy co-quasi-variational inequality problem*:

$$\text{(MFCVQIP)} \quad \begin{cases} \text{Find } x, u, \text{ and } v \in K \text{ such that} \\ P_x(u) \geq a(x), Q_x(v) \geq b(x), \; G(x) \in K(x) \text{ and} \\ \langle T(u) + A(v), J(z - G(x)) \rangle \geq 0 \qquad \forall\, z \in K(x), \end{cases}$$

where $J : B \to B^*$ is the normalized duality mapping.

Lemma 4.4.1. *Let B be a real Banach space, $T, A, G : B \to B$ be single valued mappings, $\tilde{P}, \tilde{Q} : B \to CB(B)$ and $K : B \to 2^B$ be multivalued mappings such that $\forall\, x \in B$, $K(x)$, is nonempty, closed and convex. Then the following statements are equivalent:*

- *(a) The set of elements $x \in B, u \in \tilde{P}(x)$, and $v \in \tilde{Q}(x)$ is a solution of (MFC-QVIP).*
- *(b) $x \in B, u \in \tilde{P}(x), v \in \tilde{Q}(x)$ and $G(x) = Q_{K(x)}[G(x) - \tau(T(u) + A(v))]$ for any $\tau > 0$.*

Proof. It is similar to the proof of Theorem 1 in [3]. □

As in previous sections, we give the following characterization of solutions of (MFCVQIP).

Lemma 4.4.2. *Let B be a real Banach space and X be a nonempty, closed and convex subset of B. Let $T, A, G, m : B \to B$ be single valued mappings, $\tilde{P}, \tilde{Q} : B \to CB(B)$ and $K : B \to 2^B$ be multivalued mappings such that $\forall\ x \in B,\ K(x) = m(x) + X$. Then the set of elements $x \in B, u \in \tilde{P}(x)$, and $v \in \tilde{Q}(x)$ is a solution of (MFCQVIP) if and only if*

$$x = x - G(x) + m(x) + Q_X[G(x) - \tau(T(u) + A(v)) - m(x)],\ \ for\ any\ \tau > 0.$$

Based on the above mentioned observations, we suggest the following iterative algorithm for finding the approximate solutions of (MFCQVIP).

Algorithm 4.4.1. *Let $K(x) = m(x) + X$, where X is a nonempty, closed and convex subset of B and $\tau > 0$ be fixed. Let $P, Q : B \to \mathcal{F}(B)$ be two closed fuzzy mappings satisfying condition $(*)$ and $\tilde{P}, \tilde{Q} : B \to CB(B)$ be multivalued mappings induced by the fuzzy mappings P, Q, respectively. For given $x_0 \in B, u_0 \in \tilde{P}(x_0), v_0 \in \tilde{Q}(x_0)$, we let*

$$x_1 = x_0 - G(x_0) + m(x_0) + Q_X[G(x_0) - \tau(T(u_0) + A(v_0)) - m(x_0)].$$

By Nadler [69] , there exist $u_1 \in \tilde{P}(x_1), v_1 \in \tilde{Q}(x_1)$ such that

$$\|u_0 - u_1\| \leq \tilde{\mathcal{H}}(\tilde{P}(x_0), \tilde{P}(x_1))$$

$$\|v_0 - v_1\| \leq \tilde{\mathcal{H}}(\tilde{Q}(x_0), \tilde{Q}(x_1)).$$

Let

$$x_2 = x_1 - G(x_1) + m(x_1) + Q_X[G(x_1) - \tau(T(u_1) + A(v_1)) - m(x_1)].$$

By induction, we can obtain sequences $\{x_n\}, \{u_n\}$ and $\{v_n\}$ satisfying

$$\begin{aligned} u_n \in \tilde{P}(x_n), \|u_n - u_{n+1}\| &\leq \tilde{\mathcal{H}}(\tilde{P}(x_n), \tilde{P}(x_{n+1})) \\ v_n \in \tilde{Q}(x_n), \|v_n - v_{n+1}\| &\leq \tilde{\mathcal{H}}(\tilde{Q}(x_n), \tilde{Q}(x_{n+1})) \end{aligned}$$

and

$$x_{n+1} = x_n - G(x_n) + m(x_n) + Q_X[G(x_n) - \tau(T(u_n) + A(v_n)) - m(x_n)]. \quad (4.4.1)$$

We have the following existence and convergence result.

Theorem 4.4.3. *Let B be a real uniformly smooth Banach space with the module of smoothness $\tau_B(t) \leq Dt^2$ for some $D > 0$. Let X be a nonempty, closed and convex subset of B. Let $P, Q : B \to \mathcal{F}(B)$ be two closed fuzzy mappings satisfying condition $(*)$ and $\tilde{P}, \tilde{Q} : B \to CB(B)$ be multivalued mappings induced by the fuzzy mappings P, Q, respectively. Let $\tilde{P}$ and $\tilde{Q}$ be $\tilde{\mathcal{H}}$-Lipschitz continuous mappings with constants β and η, respectively. Let $T, A, m : B \to B$ be Lipschitz continuous with constant α, λ and θ, respectively, $G : B \to B$ be both strongly accretive with constant $\gamma > 0$ and Lipschitz continuous with constant $\delta > 0$. If*

$$0 < (1 - 2\gamma + 64D\delta^2) + 2\theta + \delta + \tau(\beta\alpha + \lambda\eta) < 1. \tag{4.4.2}$$

Then there exist $x \in B, u \in \tilde{P}(x), v \in \tilde{Q}(x)$ such that (x, u, v) is a solution of (MFCQVIP) and the sequences $\{x_n\}, \{u_n\}$ and $\{v_n\}$ generated by the Algorithm 4.4.1 converge strongly to x, u and v, respectively, that is, $x_n \to x, u_n \to u$ and $v_n \to v$ as $n \to \infty$.

Proof. By (4.4.1), we have

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|x_n - G(x_n) + m(x_n) + Q_X[G(x_n) - \tau(T(u_n) + A(v_n)) \\ &\quad -m(x_n)] - (x_{n-1} - G(x_{n-1}) + m(x_{n-1})) \\ &\quad -Q_X[G(x_{n-1}) - \tau(T(u_{n-1}) + A(v_{n-1})) - m(x_{n-1})]\| \\ &\leq \|x_n - x_{n-1} - (G(x_n) - G(x_{n-1})\| \\ &\quad +2\|m(x_n) - m(x_{n-1})\| + \|G(x_n) - G(x_{n-1})\| \\ &\quad +\tau\|T(u_n) - T(u_{n-1})\| + \tau\|A(v_n) - A(v_{n-1})\|. \end{aligned} \tag{4.4.3}$$

From the proof of Theorem 3 in [3], we have

$$\|x_n - x_{n-1} - (G(x_n) - G(x_{n-1}))\|^2 \leq (1 - 2\gamma + 64D\delta^2)\|x_n - x_{n-1}\|^2. \tag{4.4.4}$$

It follows from the Lipschitz property of the corresponding functions that

$$\|m(x_n) - m(x_{n-1})\| \leq \theta\|x_n - x_{n-1}\| \tag{4.4.5}$$

$$\|T(u_n) - T(u_{n-1})\| \leq \beta\alpha\|x_n - x_{n-1}\| \tag{4.4.6}$$

$$\|A(v_n) - A(v_{n-1})\| \leq \lambda\eta\|x_n - x_{n-1}\| \tag{4.4.7}$$

$$\|G(x_n) - G(x_{n-1})\| \leq \delta\|x_n - x_{n-1}\|. \tag{4.4.8}$$

From (4.4.3) - (4.4.8), we have

$$\|x_{n+1} - x_n\| \leq t\|x_n - x_{n-1}\|,$$

where $t = (1-2\gamma+64D\delta^2)+2\theta+\delta+\tau(\beta\alpha+\lambda\eta)$ and $0 < t < 1$ by (4.4.2). Consequently $\{x_n\}$ is a Cauchy sequence, and thus it converges to some $x \in B$. By (4.4.1) ,we have

$$\|u_n - u_{n-1}\| \leq \tilde{\mathcal{H}}(\tilde{P}(x_n), \tilde{P}(x_{n-1})) \leq \beta\|x_n - x_{n-1}\|,$$

$$\|v_n - v_{n-1}\| \leq \tilde{\mathcal{H}}(\tilde{Q}(x_n), \tilde{Q}(x_{n-1})) \leq \eta\|x_n - x_{n-1}\|,$$

and hence $\{u_n\}$ and $\{v_n\}$ are also Cauchy sequences in B. Let $\{u_n\}$ and $\{v_n\}$ converge to some $u \in B$ and $v \in B$, respectively. Since $Q_X, G, \tilde{P}, \tilde{Q}, T, A$ and m are all continuous, we have

$$x = x - G(x) + m(x) + Q_X[G(x) - \tau(T(u) + A(v)) - m(x)].$$

Further, we have

$$\begin{aligned} d(u, \tilde{P}(x)) &= \inf\{\|u - z\| : z \in \tilde{P}(x)\} \\ &\leq \|u - u_n\| + d(u_n, \tilde{P}(x)) \\ &\leq \|u - u_n\| + \tilde{\mathcal{H}}(\tilde{P}(x_n), \tilde{P}(x)) \\ &\leq \|u - u_n\| + \beta\|x_n - x\| \to 0 \ as \ n \to \infty \end{aligned}$$

and hence $u \in \tilde{G}(x)$. Similarly we can show that $v \in \tilde{D}(x)$. The result then follows from Lemma 4.4.2. □