Chapter 3

Generalized Nonlinear Quasi-variational Inclusions

This chapter deals with the iterative methods for computing the approximate solutions of generalized nonlinear quasi-variational inclusion problems. The existence and convergence of solutions obtained by proposed algorithms are also studied. We have also mentioned several special cases.

3.1 Introduction

There are many numerical methods to compute the approximate solutions of variational inequalities and their generalizes, for example, projection method and its variant forms, auxiliary principle method, Newton and descart framework etc. The applicability of projection method is limited to the variational inequalities without involving the nonlinear term ϕ on the right hand side of the inequality. Such method cannot be used to suggest iterative algorithms for variational inequality problems of type (GSMVIP) and its generalized forms due to the presence of the nonlinear term ϕ. In 1994, Hassouni and Moudafi [41] first used the resolvent operator instead of projection operator to suggest iterative algorithm for finding the approximate solutions of variational inequalities involving the nonlinear term ϕ on the right hand side of the inequality. Adly [1] extended such approach for the more general variational inequality problems. The resolvent operators technique is extensively used by Noor, Rassias, Huang, et al; See, for example [46, 75, 76, 79] and references therein. The resolvent operators technique is quite general and flexible.

Recently, Ding [31] considered a class of generalized quasi-variational inclusions. By using the properties of the resolvent operator, he established an existence result for solutions and suggested a new iterative algorithm and a perturbed proximal point algorithm for finding the approximate solutions. He also proved the convergence results for solutions obtained by the proposed algorithms.

The Mann [66] and Ishikawa [58] type perturbed iterative process for nonlinear equations is well decomented in the literature. Huang [47] proposed Mann and Ishikawa types iterative algorithms for finding the approximate solutions of a more general problem known as generalized nonlinear implicit quasi-variational inclusions.

In this chapter, we consider generalized nonlinear quasi-variational inclusion problem. By using the resolvent operator technique, in the second section, we first convert our problem into a fixed point problem. Then we use this fixed point formulation to suggest an iterative algorithm for computing the approximate solutions of our problem. In the third section, we propose Ishikawa type perturbed iterative algorithm for finding the approximate solutions of our problem. The existence of solutions of our problem and the convergence results for approximate solutions obtained by the proposed algorithms are also studied. Several special cases of our problem are also mentioned.

Let H be a real Hilbert space whose norm and inner product are denoted by $\|.\|$ and $\langle .,. \rangle$, respectively. Given multivalued mappings $A, S, T : H \to 2^H$ and single valued mappings $g, m : H \to H$ and $N : H \times H \to H$, we consider the following problem which is called *generalized nonlinear quasi-variational inclusion problem:*

$$
\text{(GNQVIP)} \quad \begin{cases} \text{Find } x \in H, u \in S(x), v \in T(x), \text{ and } w \in A(x) \text{ such that} \\ g(x) - m(w) \in \text{dom } \partial\phi \text{ and} \\ \langle N(u,v), y - g(x) \rangle \geq \phi(g(x) - m(w), x) - \phi(y, x), \quad \forall\, y \in H, \end{cases}
$$

where $\phi : H \times H \to \mathbb{R} \cup \{+\infty\}$ is a functional such that for each fixed $y \in H$, $\phi(., y) : H \to R \cup \{+\infty\}$ is a proper, convex, lower semicontinuous functional on H and $\partial\phi$ denotes the subdifferential of ϕ.

SPECIAL CASES:

(i) If $\phi(x,y) = \phi(x),\ \forall\ y \in H$, then (GNQVIP) reduces to the following *generalized set-valued nonlinear quasi-variational inclusion problem* which is a special case of the problem considered by Shim et al [85] and appears to be a new one.

$$\text{(GSVNQVIP)}\quad \begin{cases} \text{Find } x \in H, u \in S(x), v \in T(x), \text{ and } w \in A(x) \text{ such that} \\ g(x) - m(w) \in \text{dom } \partial\phi \text{ and} \\ \langle N(u,v), y - g(x)\rangle \geq \phi(g(x) - m(w)) - \phi(y), \quad \forall\ y \in H. \end{cases}$$

(ii) If $m \equiv 0$ and A is identity mapping, then (GSVNQVIP) reduces to the following *generalized set-valued mixed variational inequality problem* considered and studied by Noor et al [78]:

$$\text{(GSMVIP)}\quad \begin{cases} \text{Find } x \in H, u \in S(x), \text{ and } v \in T(x) \text{ such that} \\ g(x) \in \text{dom } \partial\phi \text{ and} \\ \langle N(u,v), y - g(x)\rangle \geq \phi(g(x)) - \phi(y), \quad \forall\ y \in H. \end{cases}$$

(iii) If $N(u,v) = v - u,\ m \equiv 0$ and A is identity mapping, then (GNQVIP) reduces to the following *generalized quasi-variational inclusion problem* considered and studied by Ding [31]:

$$\text{(GQVIP)}\quad \begin{cases} \text{Find } x \in H, u \in S(x), \text{ and } v \in T(x) \text{ such that} \\ g(x) \in \text{dom } \partial\phi \text{ and} \\ \langle u - v, y - g(x)\rangle \geq \phi(g(x), x) - \phi(y, x), \quad \forall\ y \in H. \end{cases}$$

For a suitable choice of the mappings A, S, T, N, and ϕ, a number of other known variational inequalities and their generalizations can be obtained as special cases of our (GNQVIP).

3.2 Iterative Algorithms and Convergence Results

In this section, we first convert (GNQVIP) into a fixed point problem. Then by using this fixed point formulation, we suggest an iterative algorithm for computing the approximate solutions of (GNQVIP). We also study the existence of solutions of (GNQVIP) and the convergence of approximate solutions obtained by suggested algorithm.

In order to prove our main result, we need the following concepts and results.

Definition 3.2.1. [98] Let $S : H \to 2^H$ be a multivalued mapping. A mapping $N(.,.) : H \times H \to H$ is said to be *relaxed Lipschitz with respect to* S in the first argument if there exists a constant $k \geq 0$ such that

$$\langle N(u_1,.) - N(u_2,.), x - y\rangle \;\leq\; k\|x-y\|^2, \quad \forall\, u_1 \in S(x_1),\; u_2 \in S(x_2) \text{ and } x, y \in H.$$

Definition 3.2.2. [98] Let $T : H \to 2^H$ be a multivalued mapping. A mapping $N(.,.) : H \times H \to H$ is said to be *relaxed monotone with respect to* T in the second argument if there exists a constant $c > 0$ such that

$$\langle N(.,v_1) - N(.,v_2), x - y\rangle \geq c\|x-y\|^2, \quad \forall\, v_1 \in T(x_1),\; v_2 \in T(x_2) \text{ and } x, y \in H.$$

The following lemma ensures that (GNQVIP) is equivalent to a fixed point problem.

Lemma 3.2.1. *The set of elements* (x, u, v, w) *is a solution of (GNQVIP) if and only if* (x, u, v, w) *satisfies the relation:*

$$g(x) = m(w) + J_\eta^{\partial\phi(.,x)}[g(x) - \eta N(u,v) - m(w)], \tag{3.2.1}$$

where $\eta > 0$ *is a constant and* $J_\eta^{\partial\phi(.,x)} = (I + \eta\partial\phi(.,x))^{-1}$ *is the resolvent operator of* $\partial\phi(.,x)$ *and* I *stands for the identity mapping on* H.

The equation (3.2.1) can be written as

$$x = (1-\lambda)x + \lambda[x - g(x) + m(w) + J_\eta^{\partial\phi(.,x)}[g(x) - \eta N(u,v) - m(w)]], \tag{3.2.2}$$

where $0 < \lambda < 1$ and $\eta > 0$ are both constants.

This fixed point formulation enables us to suggest the following algorithm.

Algorithm 3.2.1. *Let $g, m : H \to H$ be single valued mappings, $N : H \times H \to H$ be a bifunction and $A, S, T : H \to CB(H)$ be multivalued mappings. For given $x_0 \in H$, take $u_0 \in S(x_0)$, $v_0 \in T(x_0)$, $w_0 \in A(x_0)$ and for $\eta > 0$, and let*

$$x_1 = (1-\lambda)x_0 + \lambda[x_0 - g(x_0) + m(w_0) + J_\eta^{\partial\phi(.,x_0)}[g(x_0) - \eta N(u_0, v_0) - m(w_0)]],$$

where $0 < \lambda < 1$ is a constant.

Since $u_0 \in S(x_0) \in CB(H)$, $v_0 \in T(x_0) \in CB(H)$, and $w_0 \in A(x_0) \in CB(H)$, by Nadler [69], there exist $u_1 \in S(x_1)$, $v_1 \in T(x_1)$, and $w_1 \in A(x_1)$ such that

$$\begin{aligned} \|u_1 - u_0\| &\leq (1+1)\mathcal{H}(S(x_1), S(x_0)), \\ \|v_1 - v_0\| &\leq (1+1)\mathcal{H}(T(x_1), T(x_0)), \\ \|w_1 - w_0\| &\leq (1+1)\mathcal{H}(A(x_1), A(x_0)). \end{aligned}$$

Let

$$x_2 = (1-\lambda)x_1 + \lambda[x_1 - g(x_1) + m(w_1) + J_\eta^{\partial\phi(.,x_1)}[g(x_1) - \eta N(u_1, v_1) - m(w_1)]].$$

Since $u_1 \in S(x_1) \in CB(H)$, $v_1 \in T(x_1) \in CB(H)$, and $w_1 \in A(x_1) \in CB(H)$, there exist, $u_2 \in S(x_2)$, $v_2 \in T(x_2)$, and $w_2 \in A(x_2)$ such that

$$\begin{aligned} \|u_1 - u_2\| &\leq (1+2^{-1})\mathcal{H}(S(x_1), S(x_2)), \\ \|v_1 - v_2\| &\leq (1+2^{-1})\mathcal{H}(T(x_1), T(x_2)), \\ \|w_1 - w_2\| &\leq (1+2^{-1})\mathcal{H}(A(x_1), A(x_2)). \end{aligned}$$

By induction, we can obtain sequences $\{x_n\}$, $\{u_n\}$, $\{v_n\}$, and $\{w_n\}$ as

$$x_{n+1} = (1-\lambda)x_n + \lambda[x_n - g(x_n) + m(w_n) + J_\eta^{\partial\phi(.,x_n)}[g(x_n) - \eta N(u_n, v_n) - m(w_n)]], \quad (3.2.3)$$

$$\begin{aligned} u_n \in S(x_n), \quad \|u_n - u_{n+1}\| &\leq (1+(1+n)^{-1})\mathcal{H}(S(x_n), S(x_{n+1})), \\ v_n \in T(x_n), \quad \|v_n - v_{n+1}\| &\leq (1+(1+n)^{-1})\mathcal{H}(T(x_n), T(x_{n+1})), \\ w_n \in A(x_n), \quad \|w_n - w_{n+1}\| &\leq (1+(1+n)^{-1})\mathcal{H}(A(x_n), A(x_{n+1})), \end{aligned}$$

$n = 0, 1, 2, \ldots$.

Now we study the convergence of iterative sequences generated by Algorithm 3.2.1. and prove that existence of solutions of (GNQVIP)

Theorem 3.2.2. *Let $g : H \to H$ be a strongly monotone and Lipschitz continuous mapping with constants $\alpha > 0$ and $\beta > 0$, respectively, and $m : H \to H$ be a Lipschitz continuous mapping with constant $\gamma > 0$. Let $A, S, T : H \to CB(H)$ be $\mathcal{H}$-Lipschitz continuous mappings with constants $\sigma > 0$, $\xi > 0$ and $\rho > 0$, respectively. Let the bifunction $N : H \times H \to H$ be relaxed Lipschitz continuous with respect to S in first argument with constant $k \leq 0$, and relaxed monotone with respect to T in second argument with constant $c > 0$. Also, let the bifunction $N(.,.)$ be a Lipschitz continuous in first and second argument with constants $\delta > 0$ and $\omega > 0$, respectively. Let $\phi : H \times H \to \mathbb{R} \cup \{+\infty\}$ be such that for each fixed $y \in H, \phi(., y)$ is a proper, convex, lower semicontinuous function on H. For each $x, y, z \in H$ and $\eta > 0$, let*

$$\|J_\eta^{\partial\phi(.,x)}(w) - J_\eta^{\partial\phi(.,y)}(w)\| \leq \mu\|x - y\|$$

and if

$$\begin{aligned} \left|\eta - \frac{(k-c)}{(\delta\xi + \omega\rho)^2}\right| &< \frac{\sqrt{(k-c)^2 - q(2-q)(\delta\xi + \omega\rho)^2}}{(\delta\xi + \omega\rho)^2}, \\ (k-c) &> (\delta\xi + \omega\rho)\sqrt{q(2-q)}, \\ q &= 2\sqrt{1 - 2\alpha + \beta^2} + 2\gamma\sigma + \mu, \quad q < 1, \end{aligned} \tag{3.2.4}$$

then there exist $x \in H$, $u \in S(x)$, $v \in T(x)$, and $w \in A(x)$ satisfying the (GNQVIP). Moreover,

$$x_n \to x, \ u_n \to u \ v_n \to v, \ w_n \to w \quad as\ n \to \infty,$$

where the sequences $\{x_n\}$, $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are generated by Algorithm 3.2.1.

Proof. From Algorithm 3.2.1, we have

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|(1-\lambda)x_n - (1-\lambda)x_{n-1} + \lambda[x_n - x_{n-1} - (g(x_n) - g(x_{n-1})) \\ &\quad + m(w_n) - m(w_{n-1}) + J_\eta^{\partial\phi(.,x_n)}[g(x_n) - \eta N(u_n, v_n) - m(w_n)] \\ &\quad - J_\eta^{\partial\phi(.,x_{n-1})}[g(x_{n-1}) - \eta N(u_{n-1}, v_{n-1}) - m(w_{n-1})]]\| \end{aligned}$$

$$
\begin{aligned}
&\le (1-\lambda)\|x_n - x_{n-1}\| + \lambda\|x_n - x_{n-1} - (g(x_n) - g(x_{n-1}))\| \\
&\quad +\lambda\|m(w_n) - m(w_{n-1})\| + \lambda\|J_\eta^{\partial\phi(.,x_n)}[g(x_n) - \eta N(u_n, v_n) - m(w_n)] \\
&\quad -J_\eta^{\partial\phi(.,x_n)}[g(x_{n-1}) - \eta N(u_{n-1}, v_{n-1}) - m(w_{n-1})]\| \\
&\quad +\lambda\|J_\eta^{\partial\phi(.,x_n)}[g(x_{n-1}) - \eta N(u_{n-1}, v_{n-1}) - m(w_{n-1})] \\
&\quad -J_\eta^{\partial\phi(.,x_{n-1})}[g(x_{n-1}) - \eta N(u_{n-1}, v_{n-1}) - m(w_{n-1})]\| \\
&\le (1-\lambda)\|x_n - x_{n-1}\| + 2\lambda\|x_n - x_{n-1} - (g(x_n) - g(x_{n-1}))\| \\
&\quad +2\lambda\|m(w_n) - m(w_{n-1})\| + \mu\lambda\|x_n - x_{n-1}\| + \lambda\|x_n - x_{n-1} \\
&\quad -\eta(N(u_n, v_n) - N(u_{n-1}, v_n) + N(u_{n-1}, v_n) - N(u_{n-1}, v_{n-1}))\| \\
&\le (1-\lambda)\|x_n - x_{n-1}\| + 2\lambda\|x_n - x_{n-1} - (g(x_n) - g(x_{n-1}))\| \\
&\quad +2\gamma\lambda\|w_n - w_{n-1}\| + \mu\lambda\|x_n - x_{n-1}\| \\
&\quad +\lambda\|x_n - x_{n-1} - \eta(N(u_n, v_n) - N(u_{n-1}, v_n) \\
&\quad +N(u_{n-1}, v_n) - N(u_{n-1}, v_{n-1}))\|. \qquad (3.2.5)
\end{aligned}
$$

By the Lipschitz continuity and strong monotonicity of g, we obtain

$$
\|x_n - x_{n-1} - (g(x_n) - g(x_{n-1}))\|^2 \le (1 - 2\alpha + \beta^2)\|x_n - x_{n-1}\|^2. \qquad (3.2.6)
$$

Since A, S, and T are $\mathcal{H}$-Lipschitz continuous and N is Lipschitz continuous in both the arguments, we have

$$
\begin{aligned}
\|N(u_n, v_n) - N(u_{n-1}, v_n)\| &\le \delta\|u_n - u_{n-1}\| \\
&\le \delta\xi(1 + n^{-1})\|x_n - x_{n-1}\|. \qquad (3.2.7)
\end{aligned}
$$

$$
\begin{aligned}
\|N(u_{n-1}, v_n) - N(u_{n-1}, v_{n-1})\| &\le \omega\|v_n - v_{n-1}\| \\
&\le \omega\rho(1 + n^{-1})\|x_n - x_{n-1}\| \qquad (3.2.8)
\end{aligned}
$$

and

$$
\|w_n - w_{n-1}\| \le \sigma(1 + n^{-1})\|x_n - x_{n-1}\|. \qquad (3.2.9)
$$

Since N is relaxed Lipschitz continuous with respect to S in first argument and relaxed monotone with respect to T in second argument, we have

$$\begin{aligned}
&\|x_n - x_{n-1} - \eta(N(u_n, v_n) - N(u_{n-1}, v_n) + N(u_{n-1}, v_n) - N(u_{n-1}, v_{n-1}))\|^2 \\
\leq\ & \|x_n - x_{n-1}\|^2 - 2\eta\langle N(u_n, v_n) - N(u_{n-1}, v_n), x_n - x_{n-1}\rangle \\
& -2\eta\langle N(u_{n-1}, v_n) - N(u_{n-1}, v_{n-1}), x_n - x_{n-1}\rangle \\
& +\eta^2\|N(u_n, v_n) - N(u_{n-1}, v_n) + N(u_{n-1}, v_n) - N(u_{n-1}, v_{n-1})\|^2 \\
\leq\ & \|x_n - x_{n-1}\|^2 - 2\eta k\|x_n - x_{n-1}\|^2 + 2\eta c\|x_n - x_{n-1}\|^2 \\
& +\eta^2(\delta\xi + \omega\rho)^2(1 + n^{-1})^2\|x_n - x_{n-1}\|^2 \\
\leq\ & [1 - 2\eta(k - c) + \eta^2(\delta\xi + \omega\rho)^2(1 + n^{-1})^2]\|x_n - x_{n-1}\|^2. \qquad (3.2.10)
\end{aligned}$$

From (3.2.4) - (3.2.10), it follows that

$$\|x_{n+1} - x_n\| \ \leq\ \theta_n\|x_n - x_{n-1}\|, \qquad (3.2.11)$$

where

$$\theta_n = \lambda q_n + \lambda\sqrt{1 - 2\eta(k - c) + \eta^2(\delta\xi + \omega\rho)^2(1 + n^{-1})^2} + (1 - \lambda)$$

and

$$q_n = 2\sqrt{1 - 2\alpha + \beta^2} + 2\gamma\sigma(1 + n^{-1}) + \mu.$$

Letting

$$\theta = \lambda q + \lambda\sqrt{1 - 2\eta(k - c) + \eta^2(\delta\xi + \omega\rho)^2} + (1 - \lambda).$$

We know that $\theta_n \to \theta$ as $n \to \infty$. It follows from (3.2.4) that $\theta < 1$. Hence $\theta_n < 1$ for n sufficiently large. Therefore (3.2.11) implies that $\{x_n\}$ is a Cauchy sequence in H and we can suppose that $x_n \to x \in H$ as $n \to \infty$.

Now we prove that $u_n \to u \in S(x)$, $v_n \to v \in T(x)$, and $w_n \to w \in A(x)$. In fact, it follows from Algorithm 3.2.1, that

$$\|u_n - u_{n-1}\| \leq \xi(1 + n^{-1})\|x_n - x_{n-1}\|,$$

$$\|v_n - v_{n-1}\| \leq \lambda(1 + n^{-1})\|x_n - x_{n-1}\|,$$

$$\|w_n - w_{n-1}\| \leq \sigma(1 + n^{-1})\|x_n - x_{n-1}\|,$$

that is, $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are also Cauchy sequences in H. Let $u_n \to u$, $v_n \to v$, $w_n \to w$ as $n \to \infty$. Further, we have

$$\begin{aligned} d(u, S(x)) &= \inf\{\|u - y\| : y \in S(x)\} \\ &\leq \|u - u_n\| + d(u_n, S(x)) \\ &\leq \|u - u_n\| + \mathcal{H}(S(x_n), S(x)) \\ &\leq \|u - u_n\| + \xi\|x_n - x\| \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Hence $u \in S(x)$. Similarly $v \in T(x)$, and $w \in A(x)$. From (3.2.3), we have

$$g(x) = m(w) + J_\eta^{\partial\phi(.,x)}[g(x) - \eta N(u, v) - m(w)].$$

Therefore, it follows from Lemma 3.2.1 that the set of elements (x, u, v, w) is a solution of that (GNQVIP). □

3.3 An Ishikawa Type Perturbed Iterative Algorithm and a Convergence Result

In this section, we establish the equivalence of the generalized nonlinear quasi-variational inclusion (GNQVIP) to a nonlinear equation. Then we suggest an Ishikawa type perturbed iterative algorithm for finding the approximate solutions of (GNQVIP).

For finding the approximate solutions of (GNQVIP), we can apply a successive approximation method to the problem of solving

$$x \in \psi(x) \tag{3.3.1}$$

where

$$\psi(x) = x - g(x) + m(w) + J_\eta^{\partial\phi(.,x)}[g(x) - \eta N(u, v) - m(w)].$$

On the basis of Lemma 3.2.1, we suggest the following Ishikawa type perturbed iterative algorithm.

Algorithm 3.3.1. *(Ishikawa Type Perturbed Iterative Algorithm). Let $g, m : H \to H$ be single valued mappings, $N : H \times H \to H$ be a bifunction and $A, S, T : H \to CB(H)$ be multivalued mappings.*

For given $x_0 \in H$, we take $u_0 \in S(x_0), v_0 \in T(x_0)$, and $w_0 \in A(x_0)$, the iterative sequences $\{x_n\}, \{u_n\}, \{v_n\}$, and $\{w_n\}$ are defined by

$$x_{n+1} = (1-\lambda_n)x_n + \lambda_n[y_n - g(y_n) + m(\bar{w}_n) + J_\eta^{\partial\phi(.,x_n)}[g(y_n) - \eta N(\bar{u}_n, \bar{v}_n) - m(\bar{w}_n)]] + e_n,$$

$$y_n = (1-t_n)x_n + t_n[x_n - g(x_n) + m(w_n) + J_\eta^{\partial\phi(.,y_n)}[g(x_n) - \eta N(u_n, v_n) - m(w_n)]] + t_n r_n,$$

for $n \geq 0$, where e_n and r_n are the error terms which are taken into account for a possible inexact computation of the points $x_n \in H, u_n \in S(x_n), \bar{u}_n \in S(y_n), v_n \in T(x_n), \bar{v}_n \in T(y_n)$ and $w_n \in A(x_n), \bar{w}_n \in A(y_n), \rho > 0$ is a constant and $\{\lambda_n\}$ and $\{t_n\}$ are the sequences in $[0, 1]$ satisfy the following conditions :

(i) $\lambda_0 = 1, \quad 0 \leq \lambda_n, \quad t_n \leq 1, \ \forall\, n \geq 0,$

(ii) $\sum_{n=0}^{\infty} \lambda_n$ *diverges and*

(iii) $\sum_{i=0}^{n} \prod_{j=i+1} (1 - (1-c)\lambda_j)$ *converges, where* $0 \leq c \leq 1$.

Theorem 3.3.1. *Let $g : H \to H$ be a strongly monotone and Lipschitz continuous mapping with constant $\alpha > 0$ and $\beta > 0$, respectively and $m : H \to H$ be a Lipschitz continuous mapping with constant $\gamma > 0$. Let $A, S, T : H \to CB(H)$ be $\mathcal{H}$ Lipschitz continuous mappings with constant $\sigma > 0$, $\xi > 0$ and $\rho > 0$, respectively. Let the bifunction $N : H \times H \to H$ be relaxed Lipschitz continuous with respect to S in first argument with constant $k \leq 0$ and relaxed monotone with respect to T in second argument with constant $c > 0$. Also, let the bifunction $N(.,.)$ be Lipschitz continuous in first and second argument with constant $\delta > 0$ and $w > 0$, respectively. Let $\phi : H \times H \to \mathbb{R} \cup \{+\infty\}$ be such that for each fixed $y \in H$, $\phi(., y)$ is a proper, convex, lower semicontinuous function on H. For each $x, y, z \in H$ and $\eta > 0$, let*

$$||J_\eta^{\partial\phi(.,x)}(z) - J_\eta^{\partial\phi(.,y)}(z)|| \leq \mu||x - y||,$$

and if

$$\left|\eta - \frac{(k-c)}{(\delta\xi + w\rho)^2}\right| < \frac{\sqrt{(k-c)^2 - q(2-q)(\delta\xi + w\rho)^2}}{(\delta\xi + w\rho)^2}$$
$$(k-c) > (\delta\xi + w\rho)\sqrt{q(2-q)}$$
$$q = 2\sqrt{1 - 2\alpha + \beta^2} + 2\gamma\sigma + \mu, \qquad q < 1 \tag{3.3.2}$$

then (x, u, v, w) is a solution of (GNQVIP). Moreover if

$$\lim_{n\to\infty} \|J_\eta^{\partial\phi(.,x_n)}(y) - J_\eta^{\partial\phi(.,x)}(y)\| = 0, \qquad \forall\, y \in H,$$

and $\{x_n\}, \{u_n\}, \{v_n\}$, and $\{w_n\}$ are defined by ITPIA with conditions

(a) $\lim_{n\to\infty} \|e_n\| = 0 = \lim_{n\to\infty} \|r_n\|$ *and*

(b) $\sum_{i=0}^{n} \prod_{j=i+1}^{n} (1 - \lambda_j(1-c))$ *converges*, $0 \leq c < 1$.

Then $\{x_n\}, \{u_n\}, \{v_n\}$, and $\{w_n\}$ are strongly converge to x, u, v and w, respectively.

Proof. First we prove that (GNQVIP) has a solution (x, u, v, w). By Lemma (3.2.1) it is enough to show that mapping $\psi : H \to 2^H$ defined by (3.3.1) has a fixed point x. For any $x, y \in H, a \in \psi(x)$ and $b \in \psi(y)$, there exist $u_1 \in S(x), u_2 \in S(y), v_1 \in T(x), v_2 \in T(y), w_1 \in A(x)$, and $w_2 \in A(y)$ such that

$$a = x - g(x) + m(w_1) + J_\eta^{\partial\phi(.,x)}[g(x) - \eta N(u_1, v_1) - m(w_1)]$$

and

$$b = y - g(y) + m(w_2) + J_\eta^{\partial\phi(.,y)}[g(y) - \eta N(u_2, v_2) - m(w_2)].$$

By Definition 1.2.6, we have

$$\begin{aligned} \|a - b\| = {} & \|x - g(x) + m(w_1) + J_\eta^{\partial\phi(.,x)}[g(x) - \eta N(u_1, v_1) - m(w_1)] \\ & -(y - g(y) + m(w_2) + J_\eta^{\partial\phi(.,y)}[g(y) - \eta N(u_2, v_2) - m(w_2)])\| \end{aligned}$$

$$\begin{aligned}
&\le \|x-y-(g(x)-g(y))\| + \|m(w_1)-m(w_2)\| \\
&\quad + \|J_\eta^{\partial\phi(.,x)}[g(x)-\eta N(u_1,v_1)-m(w_1)] \\
&\quad - J_\eta^{\partial\phi(.,x)}[g(y)-\eta N(u_2,v_2)-m(w_2)]\| \\
&\quad + \|J_\eta^{\partial\phi(.,x)}[g(y)-\eta N(u_2,v_2)-m(w_2)] \\
&\quad - J_\eta^{\partial\phi(.,y)}[g(y)-\eta N(u_2,v_2)-m(w_2)]\| \\
&\le 2\|x-y-g(x)-g(y)\| + 2\gamma\|w_1-w_2\| + \mu\|x-y\| \\
&\quad + \|x-y-\eta(N(u_1,v_1)-N(u_2,v_1)+N(u_2,v_1)-N(u_2,v_2))\|. \qquad (3.3.3)
\end{aligned}$$

By Lipschitz continuity and strong monotonicity of g, we obtain

$$\|x-y-g(x)-g(y)\|^2 \le (1-2\alpha+\beta^2)\|x-y\|^2. \qquad (3.3.4)$$

Since A, S, T are $\mathcal{H}$-Lipschitz continuous and N is Lipschitz continuous in both the arguments, we have

$$\|N(u_1,v_1)-N(u_2,v_1)\| \le \delta\|u_1-u_2\| \le \delta\xi\|x-y\| \qquad (3.3.5)$$

$$\|N(u_2,v_1)-N(u_2,v_2)\| \le w\|v_1-v_2\| \le w\rho\|x-y\| \qquad (3.3.6)$$

and

$$\|w_1-w_2\| \le \sigma\|x-y\|. \qquad (3.3.7)$$

Since N is relaxed Lipschitz continuous with respect to S in the first argument and relaxed monotone with respect to T in the second argument and from (3.3.5), (3.3.6), we have

$$\begin{aligned}
&\|x-y-\eta(N(u_1,v_1)-N(u_2,v_1)+N(u_2,v_1)-N(u_2,v_2))\|^2 \\
&\le \|x-y\|^2 - 2\eta\langle N(u_1,v_1)-N(u_2,v_1), x-y\rangle - 2\eta\langle N(u_2,v_1)-N(u_2,v_2), x-y\rangle \\
&\quad + \eta^2\|N(u_1,v_1)-N(u_2,v_1)+N(u_2,v_1)-N(u_2,v_2)\|^2 \\
&\le \|x-y\|^2 - 2\eta k\|x-y\|^2 + 2\eta c\|x-y\|^2 + \eta^2(\delta\xi+w\rho)^2\|x-y\|^2 \\
&\le [1-2\eta(k-c)+\eta^2(\delta\xi+w\rho)^2]\|x-y\|^2. \qquad (3.3.8)
\end{aligned}$$

From (3.3.3) - (3.3.8), it follows that

$$\|a-b\| \le \theta\|x-y\|, \qquad (3.3.9)$$

where

$$\theta = q + \sqrt{1 - 2\eta(k-c) + \eta^2(\delta\xi + w\rho)^2}$$

and

$$q = 2\sqrt{1 - 2\alpha + \beta^2} + 2\gamma\sigma + \mu.$$

By condition (3.3.2), we have $0 < \theta_n < 1$. It follows from (3.3.1) that ψ has a fixed point $x \in H$. By Lemma (3.2.1), there exist $x \in H, u \in S(x), v \in T(x)$, and $w \in A(x)$ such that the set (x, u, v, w) is a solution of (GNQVIP).

Next, we prove that the iterative sequences $\{x_n\}, \{u_n\}, \{v_n\}$, and $\{w_n\}$ defined by ITPIA strongly converge to x, u, v, and w, respectively. Since (GNQVIP) has a solution (x, u, v, w), then by Lemma (3.2.1), we have

$$x = x - g(x) + m(w) + J_\eta^{\partial\phi(.,x)}[g(x) - \eta N(u,v) - m(w)].$$

By making use of the same argument used for obtaining (3.3.4) and (3.3.8), we get

$$\|x_n - x - g(x_n) - g(x)\| \leq \sqrt{1 - 2\alpha + \beta^2}\|x_n - x\|,$$

$$\|x_n - x - \eta N(u_1, v_1) - \eta N(u_2, v_2)\| \leq \sqrt{1 - 2\eta(k-c) + \eta^2(\delta\xi + w\rho)^2}\|x_n - x\|,$$

$$\|y_n - x - g(y_n) - g(x)\| \leq \sqrt{1 - 2\alpha + \beta^2}\|y_n - x\|,$$

$$\|y_n - x - \eta N(u_1, v_1) - \eta N(u_2, v_2)\| \leq \sqrt{1 - 2\eta(k-c) + \eta^2(\delta\xi + w\rho)^2}\|y_n - x\|.$$

By setting

$$h(x) = g(x) - \eta N(u,v) - m(w)$$

$$h(y_n) = g(y_n) - \eta N(\bar{u_n}, \bar{v_n}) - m(\bar{w_n}).$$

We have

$$\begin{aligned}
\|x_{n+1} - x\| &= \|(1-\lambda_n)x_n + \lambda_n[y_n - g(y_n) + m(\bar{w_n}) + J_\eta^{\partial\phi(.,x_n)}(h(y_n))] \\
&\quad + e_n - (1-\lambda_n)x + \lambda_n[x - g(x) + m(w) + J_\eta^{\partial\phi(.,x)}(h(x))\| \\
&\leq (1-\lambda_n)\|x_n - x\| + \lambda_n\|y_n - x - (g(y_n) - g(x))\| \\
&\quad + \lambda_n\|m(\bar{w_n}) - m(w)\| \\
&\quad + \lambda_n\|J_\eta^{\partial\phi(.,x_n)}(h(y_n)) - J_\eta^{\partial\phi(.,x)}(h(x))\| + \|e_n\|. \qquad (3.3.10)
\end{aligned}$$

Now since $J_\eta^{\partial\phi(.,x_n)}$ is nonexpansive, we have

$$\begin{aligned}
&||J_\eta^{\partial\phi(.,x_n)}(h(y_n)) - J_\eta^{\partial\phi(.,x)}(h(x))|| \\
&\leq ||h(y_n) - h(x)|| + ||J_\eta^{\partial\phi(.,x_n)}(h(x)) - J_\eta^{\partial\phi(.,x)}(h(x))|| \\
&\leq ||y_n - x - (g(y_n) - g(x))|| + ||y_n - x - \eta(N(\bar{u_n}, \bar{v_n}) - N(u,v))|| \\
&\quad +||m(\bar{w_n}) - m(w)|| + ||J_\eta^{\partial\phi(.,x_n)}(h(x)) - J_\eta^{\partial\phi(.,x)}(h(x))|| \\
&\leq \sqrt{1-2\alpha+\beta^2}||y_n - x|| + ||y_n - x - \eta(N(\bar{u_n}, \bar{v_n}) - N(u_{n-1}, v_n) \\
&\quad +N(u_{n-1}, v_n) - N(u,v))|| + \gamma||\bar{w_n} - w|| + ||J_\eta^{\partial\phi(.,x_n)}(h(x)) - J_\eta^{\partial\phi(.,x)}(h(x))|| \\
&\leq \sqrt{1-2\alpha+\beta^2}||y_n - x|| + \sqrt{1-2\eta(k-c)+\eta^2(\delta\xi + w\rho)^2}||y_n - x|| \\
&\quad +\gamma\sigma||y_n - x|| + \mu||y_n - x|| + ||J_\eta^{\partial\phi(.,x_n)}(h(x)) - J_\eta^{\partial\phi(.,y_n)}(h(x))||. \qquad (3.3.11)
\end{aligned}$$

By combining (3.3.10) and (3.3.11), we get

$$\begin{aligned}
||x_{n+1} - x|| &\leq (1-\lambda_n)||x_n - x|| + \lambda_n[2\sqrt{1-2\alpha+\beta^2} \\
&\quad +\sqrt{1-2\eta(k-c)+\eta^2(\delta\xi + w\rho)^2} + 2\gamma\sigma + \mu]||y_n - x|| \\
&\quad +\lambda_n||J_\eta^{\partial\phi(.,x_n)}(h(x)) - J_\eta^{\partial\phi(.,y_n)}(h(x))|| + ||e_n|| \\
&= (1-\lambda_n)||x_n - x|| + \lambda_n\theta||y_n - x|| + \lambda_n\varepsilon_n + ||e_n||, \qquad (3.3.12)
\end{aligned}$$

where

$$\theta = 2\sqrt{1-2\alpha+\beta^2} + \sqrt{1-2\eta(k-c)+\eta^2(\delta\xi + w\rho)^2} + 2\gamma\sigma + \mu$$

and

$$\varepsilon_n = ||J_\eta^{\partial\phi(.,x_n)}(h(x)) - J_\eta^{\partial\phi(.,y_n)}(h(x))||.$$

Next,

$$\begin{aligned}
||y_n - x|| &= ||(1-t_n)x_n + t_n[x_n - g(x_n) + m(w_n) + J_\eta^{\partial\phi(.,y_n)}(h(x_n)] + t_n r_n \\
&\quad -(1-t_n)x + t_n[x - g(x) + m(w) + J_\eta^{\partial\phi(.,x)}(h(x)]|| \\
&\leq (1-t_n)||x_n - x|| + t_n||x_n - x - (g(x_n) - g(x))|| + t_n||m(\bar{w_n}) - m(w)|| \\
&\quad +t_n||J_\eta^{\partial\phi(.,y_n)}(h(x_n)) - J_\eta^{\partial\phi(.,x)}(h(x))|| + t_n||r_n||. \qquad (3.3.13)
\end{aligned}$$

By making use of the same argument used for obtaining (3.3.11), we get

$$
\begin{aligned}
& \|J_\eta^{\partial\phi(.,y_n)}(h(x_n)) - J_\eta^{\partial\phi(.,x)}(h(x))\| \\
\leq\ & \{\sqrt{1-2\alpha+\beta^2} + \sqrt{1-2\eta(k-c)+\eta^2(\delta\xi+w\rho)^2} + \gamma\sigma + \mu\}\|x_n - x\| \\
& +\varepsilon_n. \qquad (3.3.14)
\end{aligned}
$$

On combining (3.3.13) and (3.3.14), we get

$$
\begin{aligned}
\|y_n - x\| &\leq (1-t_n)\|x_n - x\| + t_n\theta\|x_n - x\| + t_n(\varepsilon_n + \|r_n\|) \\
&\leq (1 - t_n(1-\theta))\|x_n - x\| + t_n(\varepsilon_n + \|r_n\|) \\
&\leq \|x_n - x\| + t_n(\varepsilon_n + \|r_n\|). \qquad (3.3.15)
\end{aligned}
$$

Since $(1 - t_n(1-\theta)) \leq 1$, by combining (3.3.12) and (3.3.15), we get

$$
\begin{aligned}
\|x_{n+1} - x\| &\leq (1-\lambda_n)\|x_n - x\| + \lambda_n\theta\|x_n - x\| + \theta\lambda_n t_n(\varepsilon_n + \|r_n\|) + \lambda_n\varepsilon_n + \|e_n\| \\
&\leq (1-\lambda_n(1-\theta))\|x_n - x\| + \theta\lambda_n t_n(\varepsilon_n + \|r_n\|) + \lambda_n\varepsilon_n + \|e_n\| \\
&\leq \prod_{i=1}^{n}(1-\lambda_i(1-\theta))\|x_0 - x\| + \sum_{j=0}^{n}\lambda_j \prod_{i=j+1}^{n}(1-\lambda_i(1-\theta))\varepsilon_j \\
&\quad +\theta\sum_{j=0}^{n}\lambda_j t_j \prod_{i=j+1}^{n}(1-\lambda_i(1-\theta))(\varepsilon_j + \|r_j\|) \\
&\quad +\sum_{j=0}^{n}\prod_{i=j+1}^{n}(1-\lambda_i(1-\theta))\|e_j\|, \qquad (3.3.16)
\end{aligned}
$$

where

$$\prod_{i=j+1}^{n}(1-\lambda_i(1-\theta)) = 1, \quad when\ j = n.$$

Now, let B denote the lower triangular matrix with entries

$$b_{nj} = \lambda_j \sum_{i=j+1}^{n}(1-\lambda_i(1-\theta)).$$

Then B is multiplicative, see Rhodaes [84], so that

$$\lim_{n\to\infty}\sum_{j=0}^{n}\lambda_j \prod_{i=j+1}^{n}(1-\alpha_i(1-\theta))\varepsilon_j = 0,$$

$$\lim_{n\to\infty}\sum_{j=0}^{n}\lambda_j t_j\prod_{i=j+1}^{n}(1-\lambda_i(1-\theta))(\varepsilon_j+\|r_j\|)=0.$$

Since

$$\lim_{n\to\infty}\|r_n\|=\lim_{n\to\infty}\varepsilon_n=0.$$

Let D denote the lower triangular matrix with entries

$$d_{nj}=\prod_{i=j+1}^{n}(1-\lambda_i(1-\theta)).$$

The condition (b) implies that D is multiplicative and hence

$$\lim_{n\to\infty}\sum_{j=0}^{n}\prod_{i=j+1}^{n}(1-\lambda_i(1-\theta))\|e_j\|=0,$$

since

$$\lim_{n\to\infty}\|e_n\|=0.$$

Also

$$\lim_{n\to\infty}\prod_{i=0}^{n}(1-\lambda_i(1-\theta))=0.$$

Since

$$\sum_{i=0}^{n}\lambda_i=\infty,$$

it follows that from inequality (3.3.16) that

$$\lim_{n\to\infty}\|x_{n+1}-x\|=0,$$

that is, the sequence $\{x_n\}$ strongly converges to x in H. The inequality (3.3.15) implies that the sequence $\{y_n\}$ also converges to y. Since $u_n\in S(x_n), u\in S(x), v_n\in T(x_n), v\in T(x)$ and $w_n\in A(x_n), w\in A(x)$ and S,T,A are $\mathcal{H}$ Lipschitz continuous, we have

$$\|u_n-u\|\le\xi\|x_n-x\|\to 0 \; as \; n\to\infty,$$

that is, $\{u_n\}$ strongly converges to u. Similarly $\{v_n\}$ and $\{w_n\}$ strongly converge to v and w, respectively. This complete the proof. □