Chapter 2

Completely Generalized Nonlinear Variational-like Inclusions

In this chapter, we consider the completely generalized nonlinear variational-like inequalities/inclusions with or without compact valued mappings. We first prove the existence of weak solutions of completely generalized nonlinear variational-like inequality problems in the setting of locally convex Hausdorff topological vector spaces. Secondly, we propose an iterative algorithm for computing the approximate solutions of completely generalized nonlinear variational-like inclusions with noncompact valued mappings in the setting of Hilbert spaces. We prove that the approximate solutions obtained by the proposed algorithm converge to the exact solution of our variational-like inclusion. We also prove that the existence of a solution of our problem. Some special cases are also discussed. In the last of this chapter, we consider the random generalization of completely generalized nonlinear variational-like inclusions. The iterative algorithm for finding the approximate solutions, their convergence and existence of a solution are also discussed.

2.1 Introduction and Formulations

The variational-like inequality, also known as pre-variational inequality, is one of the generalized form of variational inequality; See, for example, [6, 9, 63, 74, 82, 92, 99] and references therein. The variational-like inequality and generalized variational-like inequality problems are the powerful tool to study the nonconvex optimization problems, and nonconvex and nondifferentiable optimization problems, respectively; See, for instant, [6, 8, 63, 82, 99] and references therein. The resolvent operator technique initially used by Hassouni and Moudafi [41] and Adly [1] to suggest an iterative algorithm for finding the approximate solutions of variational inequality problems. In most of the papers on variational-like inequalities/inclusions and their generalizations appeared in the literature, only variational principle technique is used to find the approximate solutions of these problems. Recently, Lee, Ansari and Yao [63] developed the proximal map technique, which is an extension of the resolvent operator technique, to propose an iterative algorithm for computing the approximate solutions of variational-like inequality problems. In the resolvent operator technique, it is necessary to assume that the functional involved in the formulation of variational inequality problems to be convex and lower semicontinuous, but in [63] these conditions are not required.

Let H be a real Hilbert space whose norm and inner product are denoted by $\|\cdot\|$ and $\langle.,.\rangle$, respectively. Given single valued mappings $f, g, p : H \to H$, a bifunction $\eta : H \times H \to H$, and multivalued mappings $M, S, T : H \to 2^H$, we consider the following *completely generalized nonlinear variational-like inclusion problem*:

$$\text{(CGNVLIP)} \quad \begin{cases} \text{Find } x \in H, u \in M(x), v \in S(x), \text{ and } w \in T(x) \text{ such that} \\ x \in \text{dom } \phi \text{ and} \\ \langle p(u) - (f(v) - g(w)), \eta(y,x)\rangle \geq \phi(x) - \phi(y), \quad \forall\, y \in H, \end{cases}$$

where $\phi : H \to \mathbb{R} \cup \{+\infty\}$ and dom $\phi = \{x \in H : \phi(x) < \infty\}$.

SPECIAL CASES:

(i) If $p \equiv 0$, f, g and M are identity mappings, and S and T are single valued mappings, then (CGNVLIP) reduces to the problem of finding $x \in H$ such that $x \in$ dom ϕ and

$$\langle T(x) - S(x), \eta(y,x)\rangle \geq \phi(x) - \phi(y), \quad \forall\, y \in H. \tag{2.1.1}$$

It is considered and studied by Lee, Ansari and Yao [63].

(ii) If $\phi \equiv \delta_K$, the indicator function of the closed convex set K in H defined by

$$\delta_K(x) = \begin{cases} 0, & x \in K \\ +\infty, & \text{otherwise,} \end{cases}$$

f, g and M are identity mappings and $\eta(y, x) = y - p(x)$, then (CGNVLIP) becomes the problem of finding $x \in K$, $v \in S(x)$, and $w \in T(x)$ such that

$$\langle p(x) - (v - w), y - p(x) \rangle \geq 0, \quad \forall\, y \in K. \tag{2.1.2}$$

Such a problem is considered and studied by Verma [98]. (CGNVLIP) also contains many other kinds of variational inequalities and variational-like inequalities and inclusions studied in [63, 98] and references therein.

2.2 Existence Theory in Topological Vector Spaces

In this section, we consider the weak formulation of completely generalized nonlinear variational-like inequality problem in the setting of locally convex topological vector spaces and prove the existence of its solution.

Let $\langle E, E^* \rangle$ be a dual system of locally convex spaces and K be a nonempty convex subset of E. Given single valued mappings $f, g, p : E^* \to E^*$, a bifunction $\eta : K \times K \to E$, multivalued maps $M, S, T : K \to E^*$, and a functional $\phi : K \to \mathbb{R}$, we consider the following *weak formulation of completely generalized nonlinear variational-like inequality problem*:

$$\text{(WCGNVLIP)} \quad \begin{cases} \text{Find } x \in K \text{ such that } \forall\, y \in K, \\ \exists\, u \in M(x),\ v \in S(x),\ \text{and } w \in T(x) \text{ satisfying} \\ \langle p(u) - (f(v) - g(w)), \eta(y, x) \rangle \geq \phi(x) - \phi(y). \end{cases}$$

The solution of (WCGNVLIP) is called the *weak solution* of (CGNVLIP). We notice that u, v and w depend on y and therefore the above mentioned problem is called the weak formulation of completely generalized nonlinear variational-like inequality problem. While every solution of completely generalized nonlinear variational-like inequality problem is a weak solution but the converse is not true in general.

We also remark that the existence of a weak solution of a generalized nonlinear variational-like inequality problem implies the existence of a nondifferentiable and nonconvex optimization problem under certain conditions; See for example [5, 8] and references therein.

If f, g and M are identity mappings, $\phi \equiv 0$ and $\eta(y,x) = y - q(x)$, where $q : K \to K$ is a nonlinear operator, then (WCGNVLIP) reduces to the problem of finding $x \in K$ such that $\forall\, y \in K$, $\exists\, v \in S(x)$ and $w \in T(x)$ satisfying

$$\langle p(x) - (v - w), y - q(x)\rangle \geq 0. \tag{2.2.1}$$

Throughout this section, we assume that the pairing $\langle .,. \rangle$ is continuous.

Theorem 2.2.1. *Let E be a locally convex Hausdorff topological vector space and K be a nonempty convex subset of E. Let $M, S, T : K \to 2^{E^*}$ be upper semicontinuous multivalued maps with nonempty and compact values and $f, g, p : E^* \to E^*$ be continuous. Assume that the following conditions hold.*

(i) $\phi : K \to \mathbb{R}$ *is continuous on* K*;*

(ii) $\eta : K \times K \to E$ *is continuous in the second argument such that* $\eta(x,x) = 0,\ \forall\, x \in K$*;*

(iii) The set $\{y \in K : \forall\, u \in M(x),\ v \in S(x)$ *and* $w \in T(x)$ *s.t.* $\langle p(u) - (f(v) - g(w)), \eta(y,x)\rangle < \phi(x) - \phi(y)\}$ *is convex,* $\forall\, x \in K$*;*

(iv) If K is not compact, assume that there exist a nonempty compact convex subset B of K and a nonempty compact subset D of K such that $\forall\, x \in K \setminus D$, $\exists\, \tilde{y} \in B$ *satisfying* $\langle p(u) - (f(v) - g(w)), \eta(\tilde{y},x)\rangle < \phi(x) - \phi(\tilde{y})$, $\forall\, u \in M(x),\ v \in S(x)$ *and* $w \in T(x)$.

Then (WCGNVLIP) has a solution.

Proof. Assume that the conclusion of this theorem does not hold. Then $\forall\, x \in K$, the set

$\{y \in K : \forall\, u \in M(x),\ v \in S(x)$ and $w \in T(x)$ satisfying

$$\langle p(u) - (f(v) - g(w)), \eta(y,x)\rangle < \phi(x) - \phi(y)\} \neq \emptyset.$$

We define a multivalued map $Q : K \to 2^K$ by

$Q(x) = \{y \in K : \forall\, u \in M(x),\ v \in S(x)$ and $w \in T(x)$ satisfying

$$\langle p(u) - (f(v) - g(w)), \eta(y,x)\rangle < \phi(x) - \phi(y)\}, \quad \forall\, x \in K.$$

Then clearly $Q(x) \neq \emptyset$, $\forall\, x \in K$.

We prove that $Q^{-1}(y)$ is open in K. For that it is sufficient to show that $[Q^{-1}(y)]^c$, the complement of $Q^{-1}(y)$ in K, is closed in K.

Let $\{x_\lambda\}_{\lambda\in\Lambda}$ be a net in $[Q^{-1}(y)]^c$ such that $\{x_\lambda\}$ converges to $x^* \in K$. Then $\exists\, u_\lambda \in M(x_\lambda),\ v_\lambda \in S(x_\lambda)$ and $w_\lambda \in T(x_\lambda)$ such that

$$\langle p(u_\lambda) - (f(v_\lambda) - g(w_\lambda)), \eta(y,x_\lambda)\rangle - \phi(x_\lambda) + \phi(y) \geq 0, \quad \forall\, \lambda.$$

Let $\mathscr{A} = \{x_\lambda\}\bigcup\{x^*\}$. Then $\mathscr{A}$ is compact and $u_\lambda \in M(\mathscr{A})$, $v_\lambda \in S(\mathscr{A})$ and $w_\lambda \in T(\mathscr{A})$ which are compact. Therefore $\{u_\lambda\}$, $\{v_\lambda\}$ and $\{w_\lambda\}$ have convergent subnet with limit, say, u_*, v_* and w_*, respectively. Without loss of generality we may assume that $\{u_\lambda\}$ converges to u_*, $\{v_\lambda\}$ converges to v_*, and $\{w_\lambda\}$ converges to w_*. Then by the upper semicontinuity of M, S and T, we have $u_* \in M(x^*)$, $v_* \in S(x^*)$ and $w_* \in T(x^*)$. By the continuity of f, g, p and (i) and (ii), we have

$$\langle p(u_\lambda) - (f(v_\lambda) - g(w_\lambda)), \eta(y,x_\lambda)\rangle - \phi(x_\lambda) + \phi(y)$$

converges to

$$\langle p(u_*) - (f(v_*) - g(w_*)), \eta(y,x_*)\rangle - \phi(x_*) + \phi(y).$$

Hence

$$\langle p(u_*) - (f(v_*) - g(w_*)), \eta(y,x_*)\rangle \geq \phi(x_*) - \phi(y),$$

and therefore $x^* \in [Q^{-1}(y)]^c$. Thus $[Q^{-1}(y)]^c$ is closed in K.

Since $Q(x)$, $\forall\, x \in K$ is nonempty and open in K, we have

$$K = \bigcup_{y\in K} Q^{-1}(y) = \bigcup_{y\in K} \mathrm{int}_K Q^{-1}(y).$$

By (iii), $Q(x)$, $\forall\, x \in K$ is convex. Since $\forall\, x \in K \setminus D$, $\exists\, \tilde{y} \in B$ satisfying $\langle p(u) - (f(v) - g(w)), \eta(\tilde{y},x)\rangle < \phi(x) - \phi(\tilde{y})$, $\forall\, u \in M(x)$ $v \in S(x)$ and $w \in T(x)$, we have $x \in \mathrm{int}_K Q^{-1}(\tilde{y})$. Hence Q satisfies all the conditions of Theorem 2.2.1. Therefore by Theorem 1.2.6 with $P \equiv Q$, $\exists\ \bar{x} \in K$ such that $\bar{x} \in Q(\bar{x})$, that is, $\forall\ \bar{u} \in M(\bar{x})$, $\bar{v} \in S(\bar{x})$ and $\bar{w} \in T(\bar{x})$, we have

$$\langle p(\bar{u}) - (f(\bar{v}) - g(\bar{w})), \eta(\bar{x}, \bar{x})\rangle < \phi(\bar{x}) - \phi(\bar{x}).$$

Since $\eta(x, x) = 0, \forall\ x \in K$, we reach to a contradiction. This completes the proof. □

Corollary 2.2.2. *Let E be a locally convex Hausdorff topological vector space and K be a nonempty convex subset of E. Let $M, S, T : K \to 2^{E^*}$ be upper semicontinuous multivalued maps with nonempty and compact values and $f, g, p : E^* \to E^*$ and $\eta : K \times K \to E$ be functions such that $\liminf_{x \to x^*, u \to u^*, v \to v^*, w \to w^*} \langle p(u) - (f(v) - g(w)), \eta(y, x)\rangle \leq \langle p(u^*) - (f(v^*) - g(w^*)), \eta(y, x^*)\rangle$. Also, let $\phi : K \to \mathbb{R}$ be continuous on K and the set $\{y \in K : \forall\ u \in M(x),\ v \in S(x)$ and $w \in T(x)$ s.t. $\langle p(u) - (f(v) - g(w)), \eta(y, x)\rangle < \phi(x) - \phi(y)\}$ is convex, $\forall\ x \in K$. If K is not compact, assume that there exist a nonempty compact convex subset B of K and a nonempty compact subset D of K such that $\forall\ x \in K \setminus D$, $\exists\ \tilde{y} \in B$ satisfying $\langle p(u) - (f(v) - g(w)), \eta(\tilde{y}, x)\rangle < \phi(x) - \phi(\tilde{y})$, $\forall\ u \in M(x)$ $v \in S(x)$ and $w \in T(x)$. Then (WCGNVLIP) has a solution.*

2.3 Iterative Algorithms and Convergence Results

In this section, we consider the completely generalized nonlinear variational-like inclusions with noncompact valued mappings which include many known variational-like inclusions and variational inclusions as special cases. We propose an iterative algorithm for computing the approximate solutions of this class of variational-like inclusions. We prove that the approximate solutions obtained by the proposed algorithm converge to the exact solution of our variational-like inclusion. The existence of a solution of our problem is also studied.

Recently, Lee, Ansari and Yao [63] introduced the following concept of η-subdifferential in a more general setting than that given in [98].

Let $\eta : H \times H \to H$ and $\phi : H \to \mathbb{R} \cup \{+\infty\}$. A vector $v \in H$ is called an *η-subgradient of ϕ* at $x \in \text{dom } \phi$ if

$$\langle v, \eta(y,x) \rangle \leq \phi(y) - \phi(x), \quad \forall\, y \in H. \tag{2.3.1}$$

Each ϕ can be associated with the following η-subdifferential map $\partial_\eta \phi$ defined by

$$\partial_\eta \phi(x) = \begin{cases} \{v \in H : \langle v, \eta(y,x) \rangle \leq \phi(y) - \phi(x), \forall\, y \in H\}, & x \in \text{dom } \phi \\ \emptyset, & x \notin \text{dom } \phi. \end{cases}$$

For $x \in \text{dom } \phi$, $\partial_\eta \phi(x)$ is called *η-subdifferential* of ϕ at x.

We require the following definitions to achieve the main result of this section.

Definition 2.3.1. [63] A mapping $\eta : H \times H \to H$ is called:

(i) *monotone* if

$$\langle x - y, \eta(x,y) \rangle \geq 0, \quad \forall\, x, y \in H; \tag{2.3.2}$$

(ii) *strictly monotone* if the equality holds in (2.3.2) only when $x = y$;

(iii) *strongly monotone* if there exists a constant $\sigma > 0$ such that

$$\langle x - y, \eta(x,y) \rangle \geq \sigma \|x - y\|^2, \quad \forall\, x, y \in H;$$

(iv) *Lipschitz continuous* if there exists a constant $\delta > 0$ such that

$$\|\eta(x,y)\| \leq \delta \|x - y\|, \quad \forall\, x, y \in H.$$

Definition 2.3.2. [98] Let $f : H \to H$ be a mapping. A multivalued mapping $Q : H \to 2^H$ is said to be:

(i) *relaxed Lipschitz with respect to f* if there exists a constant $k \geq 0$ such that

$$\langle f(u) - f(v), x - y \rangle \leq -k\|x - y\|^2, \quad \forall\, x, y \in H,\ u \in Q(x),\ v \in Q(y);$$

(ii) *relaxed monotone with respect to f* if there exists a constant $c > 0$ such that

$$\langle f(u) - f(v), x - y \rangle \geq -c\|x - y\|^2, \quad \forall\, x, y \in H,\ u \in Q(x),\ v \in Q(y).$$

Definition 2.3.3. Let $\eta : H \times H \to H$ be a given map. A multivalued map $Q : H \to 2^H$ is called *η-monotone* if for all $x, y \in H$,

$$\langle u - v, \eta(x, y) \rangle \geq 0, \quad \forall\, u \in Q(x),\ v \in Q(y).$$

Q is called *maximal η-monotone* if and only if it is η-monotone and there is no other η-monotone multivalued map whose graph strictly contains the graph of Q.

Assumption 2.3.1. *The operator $\eta : H \times H \to H$ satisfies the condition*

$$\eta(y, x) + \eta(x, y) = 0, \quad \forall\, x, y \in H.$$

Remark 2.3.1. If $\eta : H \times H \to H$ satisfies Assumption 2.3.1 and $\phi : H \to \mathbb{R} \cup \{+\infty\}$, then it is easy to see that the multivalued map $\partial_\eta \phi : H \to 2^H$ is η-monotone.

We need the following result due to Lee, Ansari and Yao [63], to transform (CGN-VLIP) into a fixed point problem.

Proposition 2.3.1. *[63] Let $\eta : H \times H \to H$ be strictly monotone and $Q : H \to 2^H$ be an η-monotone multivalued map. If, the range of $(I + \lambda Q)$, $R(I + \lambda Q) = H$, for $\lambda > 0$ and I is the identity operator, then Q is maximal η-monotone. Furthermore, the inverse operator $(I + \lambda Q)^{-1}$ is single-valued.*

Throughout this section, we will assume that $\eta : H \times H \to H$ is strictly monotone and satisfies Assumption 2.3.1 and $\phi : H \to \mathbb{R} \cup \{+\infty\}$ is a map such that $R(I + \lambda \partial_\eta \phi) = H$ for $\lambda > 0$.

From Proposition 2.3.1, we note that the mapping

$$J_\lambda^\phi(x) = (I + \lambda \partial_\eta \phi)^{-1}(x), \quad \forall\, x \in H,$$

is single valued.

Let us transform (CGNLVLIP) into a fixed point problem.

Lemma 2.3.2. *The set of elements (x, u, v, w) such that $x \in H, u \in M(x), v \in S(x)$ and $w \in T(x)$, is a solution of (CGNVLIP) if and only if it satisfies the following relation:*

$$x = J_\lambda^\phi[x - \lambda(p(u) - (f(v) - g(w)))], \tag{2.3.3}$$

where $\lambda > 0$ is a constant, $J_\lambda^\phi = (I + \lambda\partial_\eta\phi)^{-1}$ is so-called proximal map and I stands for the identity operator on H.

Proof. From the definition of J_λ^ϕ, we have

$$x - \lambda(p(u) - (f(v) - g(w))) \in x + \lambda\partial_\eta\phi(x)$$

and hence

$$(f(v) - g(w)) - p(u) \in \partial_\eta\phi(x).$$

By using the definition of η-subdifferential, we have

$$\langle (f(v) - g(w)) - p(u), \eta(y, x)\rangle \leq \phi(y) - \phi(x), \quad \forall\, y \in H.$$

Thus (x, u, v, w) is a solution of (CGNVLIP). □

From Lemma 2.3.2, we see that (CGNVLIP) is equivalent to the fixed point problem of type

$$x = q(x) \tag{2.3.4}$$

where $q(x) = J_\lambda^\phi[x - \lambda(p(u) - (f(v) - g(w)))]$,
which implies that

$$x = (1 - \rho)x + \rho J_\lambda^\phi[x - \lambda(p(u) - (f(v) - g(w)))],$$

where $0 < \rho < 1$ is a parameter and $\lambda > 0$ is a constant.

On the basis of the above mentioned observations, we propose the following iterative algorithm to compute the approximate solutions of (CGNVLIP).

Algorithm 2.3.1. *Assume that $\eta : H \times H \to H$ is a map and $f, g, p : H \to H$ are single valued mappings. Let $S, M, T : H \to CB(H)$ be multivalued mappings. For given $x_0 \in H$, take $u_0 \in M(x_0), v_0 \in S(x_0)$ and $w_0 \in T(x_0)$, and let*

$$x_1 = (1 - \rho)x_0 + \rho J_\lambda^{\phi_0}[x_0 - \lambda(p(u_0) - (f(v_0) - g(w_0)))].$$

Since $u_0 \in M(x_0) \in CB(H)$, $v_0 \in S(x_0) \in CB(H)$, $w_0 \in T(x_0) \in CB(H)$, *by Nadler [69], there exist* $u_1 \in M(x_1)$, $v_1 \in S(x_1)$, $w_1 \in T(x_1)$ *such that*

$$\begin{aligned} \|u_0 - u_1\| &\leq (1+1)\ \mathcal{H}(M(x_0), M(x_1)), \\ \|v_0 - v_1\| &\leq (1+1)\ \mathcal{H}(S(x_0), S(x_1)), \\ \|w_0 - w_1\| &\leq (1+1)\ \mathcal{H}(T(x_0), T(x_1)). \end{aligned}$$

Let

$$x_2 = (1-\rho)x_1 + \rho J_\lambda^{\phi_1}[x_1 - \lambda(p(u_1) - (f(v_1) - g(w_1)))].$$

By induction, we can obtain sequences $\{x_n\}, \{u_n\}, \{v_n\}$ *and* $\{w_n\}$ *as*

$$x_{n+1} = (1-\rho)x_n + \rho\ J_\lambda^{\phi_n}[x_n - \lambda(p(u_n) - (f(v_n) - g(w_n)))], \tag{2.3.5}$$

$$u_n \in M(x_n), \quad \|u_n - u_{n+1}\| \leq (1 + (n+1)^{-1})\ \mathcal{H}(M(x_n), M(x_{n+1})),$$

$$v_n \in S(x_n), \quad \|v_n - v_{n+1}\| \leq (1 + (n+1)^{-1})\ \mathcal{H}(S(x_n), S(x_{n+1})),$$

$$w_n \in T(x_n), \quad \|w_n - w_{n+1}\| \leq (1 + (n+1)^{-1})\ \mathcal{H}(T(x_n), T(x_{n+1})),$$

$n = 0, 1, 2, 3, \ldots$.

We need the following lemma due to Lee, Ansari and Yao [63].

Lemma 2.3.3. *[63] Let* $\eta : H \times H \to H$ *be strongly monotone and Lipschitz continuous with constants* $t > 0$ *and* $s > 0$*, respectively, and satisfy Assumption 2.3.1. Then*

$$\|J_\lambda^\phi(x) - J_\lambda^\phi(y)\| \leq \tau\|x - y\|, \quad \forall\ x, y \in H,$$

where $\tau = \frac{s}{t}$.

Now we are ready to prove the main result of this section.

Theorem 2.3.4. *Let* $\eta : H \times H \to H$ *be strongly monotone and Lipschitz continuous with constants* $t > 0$ *and* $s > 0$*, respectively, and satisfy Assumption 2.3.1. Let* $f, g, p : H \to H$ *be Lipschitz continuous with corresponding constants* ξ, r *and* σ*, respectively. Let* $M, S, T : H \to CB(H)$ *be* $\mathcal{H}$*-Lipschitz continuous with corresponding* *constants* γ, h *and* d*, respectively, and* S *be relaxed Lipschitz with respect to* f *with constant* k *and* T *be relaxed monotone with respect to* g *with constant* c*. For each* n*, let* $\phi_n : H \to \mathbb{R}\cup\{+\infty\}$ *and* $\phi : H \to \mathbb{R}\cup\{+\infty\}$ *be mappings such that* $R(I+\lambda\partial_\eta\phi_n) = R(I+\lambda\partial_\eta\phi) = H$ *for* $\lambda > 0$*. Assume that*

$$\lim_{n\to+\infty} \|J_\lambda^{\phi_n}(z) - J_\lambda^{\phi_{n-1}}(z)\| = 0, \ \forall\ z \in H$$

and if

$$\left|\lambda - \frac{(k-c) - a\sigma\nu}{(\xi h + rd)^2 - \sigma^2\nu^2}\right| < \frac{\sqrt{(a\sigma\nu - (k-c))^2 - ((\xi h + rd)^2 - \sigma^2\nu^2)(1-a^2)}}{(\xi h + rd)^2 - \sigma^2\nu^2},$$

$$(a\sigma\nu - (k-c)) > \sqrt{((\xi h + rd)^2 - \sigma^2\nu^2)(1-a^2)}, \tag{2.3.6}$$

$$\xi h + rd > \sigma\nu, s > t, \sigma\nu > 1 \text{ and } t > 1,$$

where $a = \frac{1}{\tau}$*.*

Then, there exists a set (x, u, v, w) *such that* $x \in H, u \in M(x), v \in S(x)$ *and* $w \in T(x)$*, which is a solution of (CGNVLIP), and* $x_n \to x$*,* $u_n \to u$*,* $v_n \to v$*,* $w_n \to w$ *as* $n \to \infty$*, where* $\{x_n\}, \{u_n\}, \{v_n\}$ *and* $\{w_n\}$ *are the sequences obtained by Algorithm 2.3.1.*

Proof. From (2.3.5), we have

$$\|x_{n+1} - x_n\| = \|(1-\rho)(x_n - x_{n-1}) + \rho J_\lambda^{\phi_n}(h(x_n)) - \rho J_\lambda^{\phi_{n-1}}(h(x_{n-1}))\|, \tag{2.3.7}$$

where $h(x_n) = x_n - \lambda(p(u_n) - (f(v_n) - g(w_n)))$.
By introducing the term $J_\lambda^{\phi_n}(h(x_{n-1}))$, we get

$$\begin{aligned} \|J_\lambda^{\phi_n}(h(x_n)) - J_\lambda^{\phi_{n-1}}(h(x_{n-1}))\| \ &\leq\ \|J_\lambda^{\phi_n}(h(x_n)) - J_\lambda^{\phi_n}(h(x_{n-1}))\| \\ &\quad + \|J_\lambda^{\phi_n}(h(x_{n-1})) - J_\lambda^{\phi_{n-1}}(h(x_{n-1}))\|. \end{aligned} \tag{2.3.8}$$

By Lemma 2.3.2, we have

$$\begin{aligned} \|J_\lambda^{\phi_n}(h(x_n)) - J_\lambda^{\phi_{n-1}}(h(x_{n-1}))\| \ &\leq\ \tau\|h(x_n) - h(x_{n-1})\| \\ &\quad + \|J_\lambda^{\phi_n}(h(x_{n-1})) - J_\lambda^{\phi_{n-1}}(h(x_{n-1}))\|, \end{aligned} \tag{2.3.9}$$

where $\tau = \frac{s}{t}$, and

$$
\begin{aligned}
\|h(x_n) - h(x_{n-1})\| &= \|x_n - \lambda(p(u_n) - (f(v_n) - g(w_n))) \\
&\quad -x_{n-1} + \lambda(p(u_{n-1}) - (f(v_{n-1}) - g(w_{n-1})))\| \\
&\leq \|x_n - x_{n-1} + \lambda(f(v_n) - f(v_{n-1})) - \lambda(g(w_n) - g(w_{n-1}))\| \\
&\quad +\lambda\tau\|p(u_n) - p(u_{n-1}))\|. \qquad (2.3.10)
\end{aligned}
$$

From (2.3.7) - (2.3.10), we get

$$
\begin{aligned}
\|x_{n+1} - x_n\| &\leq (1-\rho)\|x_n - x_{n-1}\| \\
&\quad +\rho\tau\|x_n - x_{n-1} + \lambda(f(v_n) - f(v_{n-1})) - \lambda(g(w_n) - g(w_{n-1}))\| \\
&\quad +\lambda\rho\tau\|p(u_n) - p(u_{n-1})\| \\
&\quad +\rho\|J_\lambda^{\phi_n}(h(x_{n-1})) - J_\lambda^{\phi_{n-1}}(h(x_{n-1}))\|. \qquad (2.3.11)
\end{aligned}
$$

Since M, S and T are $\mathcal{H}$-Lipschitz continuous, and f, g and p are Lipschitz continuous, we have

$$\|p(u_n) - p(u_{n-1})\| \leq \sigma\|u_n - u_{n-1}\| \leq \sigma\gamma(1+1/n)\|x_n - x_{n-1}\|, \qquad (2.3.12)$$

$$\|f(v_n) - f(v_{n-1})\| \leq \xi\|v_n - v_{n-1}\| \leq \xi h(1+1/n)\|x_n - x_{n-1}\|, \qquad (2.3.13)$$

$$\|g(w_n) - g(w_{n-1})\| \leq r\|w_n - w_{n-1}\| \leq rd(1+1/n)\|x_n - x_{n-1}\|. \qquad (2.3.14)$$

Further, since S is relaxed Lipschitz and T is relaxed monotone, we have
$\|x_n - x_{n-1} + \lambda(f(v_n) - f(v_{n-1})) - \lambda(g(w_n) - g(w_{n-1}))\|^2$

$$
\begin{aligned}
&= \|x_n - x_{n-1}\|^2 + 2\lambda\langle f(v_n) - f(v_{n-1}), x_n - x_{n-1}\rangle \\
&\quad -2\lambda\langle g(w_n) - g(w_{n-1}), x_n - x_{n-1}\rangle + \lambda^2\|f(v_n) - f(v_{n-1}) - (g(w_n) - g(w_{n-1}))\|^2 \\
&\leq [1 - 2\lambda(k-c) + \lambda^2(1+1/n)^2(\xi h + rd)^2]\|x_n - x_{n-1}\|^2. \qquad (2.3.15)
\end{aligned}
$$

From (2.3.11) - (2.3.15), it follows that

$$\|x_{n+1} - x_n\| \leq \theta_n\|x_n - x_{n-1}\| + \rho\|J_\lambda^{\phi_n}(h(x_{n-1})) - J_\lambda^{\phi_{n-1}}(h(x_{n-1}))\|, \qquad (2.3.16)$$

where $\theta_n = (1-\rho) + \rho[\tau\sqrt{(1 - 2\lambda(k-c) + \lambda^2(1+\frac{1}{n})^2(\xi h + rd)^2)} + \lambda\sigma\gamma\tau(1+\frac{1}{n})]$.
Let $\theta = (1-\rho) + \rho[\tau\sqrt{(1 - 2\lambda(k-c) + \lambda^2(\xi h + rd)^2)} + \lambda\sigma\gamma\tau]$, then $\theta_n \to \theta$ as

$n \to \infty$. It follows from (2.3.6) that $\theta < 1$. Hence $\theta_n < 1$ for n sufficiently large. Since $\lim_{n\to+\infty} \|J_\lambda^{\phi_n}(z) - J_\lambda^{\phi_{n-1}}(z)\| = 0$, it follows from (2.3.16) that $\{x_n\}$ is a Cauchy sequence in H. Since H is complete we may suppose that $x_n \to x \in H$.

Now we prove that $u_n \to u \in M(x), v_n \to v \in S(x)$ and $w_n \to w \in T(x)$. In fact, it follows from Algorithm 2.3.1 that

$$\|u_n - u_{n-1}\| \leq (1 + 1/n)\ \gamma\|x_n - x_{n-1}\|,$$

$$\|v_n - v_{n-1}\| \leq (1 + 1/n)\ h\|x_n - x_{n-1}\|,$$

$$\|w_n - w_{n-1}\| \leq (1 + 1/n)\ d\|x_n - x_{n-1}\|,$$

which implies that $\{u_n\}, \{v_n\}$ and $\{w_n\}$ are also Cauchy sequences in H. Let $u_n \to u,\ v_n \to v,\ w_n \to w$ as $n \to \infty$. We have

$$\begin{aligned} d(v, S(x)) &= \inf\{\|v - y\| : y \in S(x)\} \\ &\leq \|v - v_n\| + d(v_n, S(x)) \\ &\leq \|v - v_n\| + \mathcal{H}(S(x_n), S(x)) \\ &\leq \|v - v_n\| + h\|x_n - x\| \to 0 \text{ as } n \to \infty\,. \end{aligned}$$

Hence $v \in S(x)$. Similarly we can prove that $u \in M(x), w \in T(x)$. This completes the proof. □

2.4 Iterative Algorithms and Convergence Results for Random Completely Generalized Nonlinear Variational-like Inclusions

The variational inequalities and quasi-variational inequalities for the random operators are called *random variational inequalities* and *random quasi-variational inequalities*; See for example [17, 32, 35, 43, 44, 50, 52, 94, 95, 96, 103] and references therein. Tan et al [94] studied random variational inequalities with applications to random minimization and nonlinear boundary problems, while Tarafdar and Yuan [96] gave the applications of random variational inequalities to random best approximation and fixed point theorems. In [52] and [103], random quasi-variational inequalities are studied with applications to random generalized games.

In this section, we consider the completely generalized nonlinear variational-like inclusions for noncompact valued random mappings and suggest new iterative algorithm to compute the approximate solutions of our problem. We prove the existence of a random solution of our random completely generalized nonlinear variational-like inclusion and we study the convergence of random iterative sequences generated by the suggested algorithm. Several special cases are also considered.

Throughout this section, let (Ω, Σ) be a measurable space, where Ω is a set and Σ is a σ-algebra of subsets of Ω. Let H be a real separable Hilbert space whose norm and inner product are denoted by $\|.\|$ and $\langle .,. \rangle$, respectively. We denote by $\mathcal{B}(H)$ the class of Boral σ-field in H.

Definition 2.4.1. A mapping $x : \Omega \to H$ is said to be *measurable* if for any $B \in \mathcal{B}(H)$, $\{t \in \Omega : x(t) \in B\} \in \Sigma$.

Definition 2.4.2. A mapping $f : \Omega \times H \to H$ is called a *random operator* if for any $x \in H$, $f(t,x) = x(t)$ is measurable. A random operator f is said to be *continuous* if for any $t \in \Omega$, the mapping $f(t,.) : H \to H$ is continuous.

Definition 2.4.3. A multivalued map $T : \Omega \to 2^H$ is said to be *measurable* if for any $B \in \mathcal{B}(H)$, $T^{-1}(B) = \{t \in \Omega : T(t) \cap B \neq \emptyset\} \in \Sigma$.

Definition 2.4.4. A mapping $u : \Omega \to H$ is called a *measurable selection* of a multivalued measurable map $T : \Omega \to 2^H$ if u is measurable and for any $t \in \Omega$, $u(t) \in T(t)$.

Definition 2.4.5. A map $T : \Omega \to 2^H$ is called a *random multivalued map* if for any $x \in H$, $T(.,x)$ is measurable. A random multivalued map $T : \Omega \times H \to CB(H)$ is said to be *$\mathcal{H}$-continuous* if for any $t \in \Omega$, $T(t,.)$ is continuous in the Hausdorff metric.

Given random multivalued maps $M, S, T : \Omega \times H \to 2^H$, random operators $f, g, p : \Omega \times H \to H$ with $Im(g) \cap dom\ \partial\phi \neq \emptyset$ and the random map $\eta : \Omega \times H \times H \to H$, we consider the following *random completely generalized nonlinear variational-like inclusion problem*:

$$\text{(CGNRVLIP)} \quad \begin{cases} \text{Find measuable mappings } x,\ u,\ v,\ w : \Omega \to H \text{ such that} \\ \forall\ t \in \Omega \text{ and } y(t) \in H \\ x(t) \in H,\ u(t) \in M(t, x(t)),\ v(t) \in S(t, x(t)), \\ w(t) \in T(t, x(t)),\ g(t, u(t)) \cap dom\ \partial\phi \neq \emptyset \text{ and} \\ \langle p(t, u(t)) - (f(t, v(t)) - g(t, w(t))), \eta(t, y(t), x(t))\rangle \geq \phi(x(t)) - \phi(y(t)), \end{cases}$$

where $\partial\phi$ is the subdifferential of a proper, convex and lower semicontinuous function $\phi : H \to \mathbb{R} \cup \{+\infty\}$. The set of measurable mappings (x, u, v, w) is called a *random solution* of (CGNRVLIP).

If $p \equiv 0$, f, g and M are identity maps, and S and T are single valued mappings, then (CGNRVLIP) reduces to the problem of finding a measurable mapping $x : \Omega \to H$ such that $\forall\ t \in \Omega$ and $\forall\ y(t) \in H$,

$$\langle T(t, x(t)) - S(t, x(t)), \eta(t, y(t), x(t))\rangle \geq 0. \tag{2.4.1}$$

It is considered and studied by Ding [35] in the setting of Banach spaces. He applied a random minimax inequality due to Tarafdar and Yuan [96] to prove existence and uniqueness theorems for the random solution of (2.4.1). By using the auxiliary principle technique, he suggested and analyzed an algorithm to compute the approximate random solutions of (2.4.1) in the setting of Banach spaces. He also discussed the convergence criteria.

We now give the following lemmas.

Lemma 2.4.1. *[17] Let $T : \Omega \times H \to CB(H)$ be a $\mathcal{H}$-continuous random multivalued map. Then for any measurable mapping $w : \Omega \to H$, the multivalued map $T(., w(.)) : \Omega \to CB(H)$ is measurable.*

Lemma 2.4.2. *[17] Let $S, T : \Omega \to CB(H)$ be two measurable multivalued maps, $\epsilon > 0$ be constant and $v : \Omega \to H$ be a measurable selection of S. Then there exists a measurable selection $w : \Omega \to H$ of T such that $\forall\ t \in \Omega$,*

$$||v(t) - w(t)|| \leq (1 + \epsilon)\mathcal{H}(S(t), T(t)).$$

Lemma 2.4.3. *The set of measurable mappings $x,\ u,\ v,\ w : \Omega \to H$ is a random solution of (CGNRVLIP) if and only if $\forall\ t \in \Omega,\ x(t) \in H,\ u(t) \in M(t, x(t)),\ v(t) \in S(t, x(t)),\ w(t) \in T(t, x(t))$ satisfy the following relation:*

$$x(t) = J^{\phi}_{\lambda(t)}[x(t) - \lambda(t)(p(t, u(t)) - (f(t, v(t)) - g(t, w(t)))], \tag{2.4.2}$$

where $\lambda : \Omega \to (0, \infty)$ is a measurable mapping and $J^{\phi}_{\lambda(t)} = [I + \lambda(t)\partial\phi]^{-1}$ is so called proximal map on H and I stands for the identity operator on H.

Proof. From the definition of $J^{\phi}_{\lambda(t)}$, it follows that

$$x(t) - \lambda(t)[p(t, u(t)) - (f(t, v(t)) - g(t, w(t)))] \in x(t) + \lambda(t)\partial_{\eta}\phi x(t)$$

and hence

$$[f(t, v(t)) - g(t, w(t))] - p(t, u(t)) \in \lambda(t)\partial_{\eta}\phi x(t).$$

By using the definition of η-subdifferential, we have

$$\langle (f(t, v(t)) - g(t, w(t))) - p(t, u(t)), \eta(t, y(t), x(t)) \rangle \leq \phi(y(t)) - \phi(x(t))\ \forall\, y(t) \in H,\ t \in \Omega.$$

Thus (x, u, v, w) is a random solution of (CGNRVLIP). □

We need the following definitions and results.

Definition 2.4.6. A random mapping $\eta : \Omega \times H \times H \to H$ is called:

(i) *monotone* if

$$\langle x(t) - y(t), \eta(t, x(t), y(t)) \rangle \geq 0, \quad \forall\ x(t), y(t) \in H,\ t \in \Omega; \tag{2.4.3}$$

(ii) *strictly monotone* if the equality holds in (2.4.3) only when $x(t) = y(t)$;

(iii) *strongly monotone* if there exists a measurable function $q : \Omega \to (0, \infty)$ such that

$$\langle x(t) - y(t), \eta(t, x(t), y(t)) \rangle \geq q(t)\|x(t) - y(t)\|^2, \quad \forall\ x(t), y(t) \in H,\ t \in \Omega;$$

(iv) *Lipschitz continuous* if there exists a measurable function $s : \Omega \to (0, \infty)$ such that

$$||\eta(t, x(t), y(t))|| \leq \delta||x(t) - y(t)||, \quad \forall\ x(t), y(t) \in H\ , t \in \Omega.$$

Definition 2.4.7. A random operator $g : \Omega \times H \to H$ is said to be *Lipschtiz continuous* if there exists a measurable function $r : \Omega \to (0, \infty)$ such that

$$||g(t, w_1(t)) - g(t, w_2(t))|| \leq r(t)||w_1(t) - w_2(t)||, \quad \forall\ w_1(t), w_2(t) \in H\ , t \in \Omega.$$

Definition 2.4.8. A random multivalued map $S : \Omega \times H \to CB(H)$ is said to be *$\mathcal{H}$-Lipschitz continuous* if there exists a measurable function $d : \Omega \to (0, \infty)$ such that

$$\mathcal{H}(S(t, x(t)), S(t, y(t))) \leq d(t)||x(t) - y(t)||, \quad \forall\ x(t), y(t) \in H\ t \in \Omega.$$

Definition 2.4.9. Let $f : H \to H$ be a random operator. A random multivalued map $S : H \to 2^H$ is said to be:

(i) *relaxed Lipschitz with respect to f* if there exists a measurable function $k : \Omega \to (0, \infty)$ such that

$$\langle f(t, u(t)) - f(t, v(t)), x - y\rangle \leq -k(t)||x(t) - y(t)||^2,$$
$$\forall\ x(t), y(t) \in H,\ u(t) \in S(t, x(t)),\ v(t) \in S(t, y(t)),\ t \in \Omega;$$

(ii) *relaxed monotone with respect to f* if there exists a measurable function $c : \Omega \to (0, \infty)$ such that

$$\langle f(t, u(t)) - f(t, v(t)), x(t) - y(t)\rangle \geq -c(t)||x(t) - y(t)||^2,$$
$$\forall\ x(t), y(t) \in H,\ u(t) \in S(t, x(t)),\ v(t) \in S(t, y(t)),\ t \in \Omega.$$

Definition 2.4.10. Let $\eta : \Omega \times H \times H \to H$ be a given random map. A random multivalued map $Q : \Omega \times H \to 2^H$ is called *η-monotone* if $\forall\ x(t), y(t) \in H$ and $t \in \Omega$,

$$\langle u(t) - v(t), \eta(t, x(t), y(t))\rangle \geq 0, \quad \forall\ u(t) \in Q(t, x(t)),\ v(t) \in Q(t, y(t)).$$

Q is called *maximal η-monotone* if and only if it is η-monotone and there is no other η-monotone random multivalued map whose graph strictly contains the graph of Q.

Assumption 2.4.1. *The random operator* $\eta : \Omega \times H \times H \to H$ *satisfies the condition*

$$\eta(t, y(t), x(t)) + \eta(t, x(t), y(t)) = 0, \quad \forall\ x(t), y(t) \in H,\ t \in \Omega.$$

Remark 2.4.1. If $\eta : \Omega \times H \times H \to H$ satisfies Assumption 2.4.1 and $\phi : H \to \mathbb{R} \cup \{+\infty\}$, then it is easy to see that the random multivalued map $\partial_\eta \phi : H \to 2^H$ is η-monotone.

The following result is the random version of Proposition 2.3.1 due to Lee, Ansari and Yao [63].

Proposition 2.4.4. *Let* $\eta : \Omega \times H \times H \to H$ *be strictly monotone random map and* $Q : \Omega \times H \to 2^H$ *an* η*-monotone random multivalued map. If, the range of* $(I + \lambda Q)$, $R(I + \lambda Q) = H$, *for* $\lambda > 0$ *and* I *is the identity operator, then* Q *is maximal* η*-monotone. Furthermore, the inverse operator* $(I + \lambda Q)^{-1}$ *is single valued.*

Throughout this section, we will assume that $\eta : \Omega \times H \times H \to H$ is strictly monotone and satisfies Assumption 2.4.1 and $\phi : H \to \mathbb{R} \cup \{+\infty\}$ is a functional such that $R(I + \lambda \partial_\eta \phi) = H$ for $\lambda > 0$.

From Proposition 2.4.4, we note that the mapping

$$J_\lambda^\phi(x(t)) = (I + \lambda \partial_\eta \phi)^{-1}(x(t)), \quad \forall\ x(t) \in H,\ t \in \Omega$$

is single valued.

To obtain the approximate solutions of (CGNRVLIP) we can apply a successive approximation method to the problem of solving

$$x(t) \in Q(t, x(t)) \tag{2.4.4}$$

for all $t \in \Omega$, where

$$Q(t, x(t)) = x(t) + J_{\lambda(t)}^\phi[x(t) - \lambda(t)((p(t, u(t)) - (f(t, S(t, v(t))) - g(t, T(t, w(t))))].$$

Based on (2.4.2) and (2.4.4), we propose the following algorithm to compute the approximate solutions of (CGNRVLIP).

Algorithm 2.4.1. *Let $M, S, T : \Omega \times H \to CB(H)$ be $\mathcal{H}$-continuous random multivalued maps and $f, g, p : \Omega \times H \to H$ be continuous random operators. For any given measurable mapping $x_0 : \Omega \to H$, the multivalued mappings $M(., x_0(.))$, $S(., x_0(.))$, $T(., x_0(.)) : \Omega \to CB(H)$ are measurable by Lemma 2.4.1. Hence there exist measurable selection $u_0 : \Omega \to H$ of $M(., x_0(.))$, measurable selection $v_0 : \Omega \to H$ of $S(., x_0(.))$ and measurable selection $w_0 : \Omega \to H$ of $T(., x_0(.))$ by Himmelberg [42]. Let*

$$x_1(t) = x_0(t) + J_{\lambda(t)}^{\phi_0}[x_0(t) - \lambda(t)((p(t, u_0(t)) - (f(t, v_0(t))) - g(t, w_0(t)))].$$

It is easy to see that $x_1 : \Omega \to H$ is measurable. By Lemma 2.4.2, there exist measurable selections $u_1 : \Omega \to H$ of $M(., x_1(.))$, measurable selection $v_1 : \Omega \to H$ of $S(., x_1(.))$ and measurable selection $w_1 : \Omega \to H$ of $T(., x_1(.))$ such that $\forall\ t \in \Omega$,

$$\begin{aligned} \|u_0(t) - u_1(t)\| &\leq (1+1)\ \mathcal{H}(M(t, x_0(t)), M(t, x_1(t))), \\ \|v_0(t) - v_1(t)\| &\leq (1+1)\ \mathcal{H}(S(t, x_0(t)), S(t, x_1(t))), \\ \|w_0(t) - w_1(t)\| &\leq (1+1)\ \mathcal{H}(T(t, x_0(t)), T(t, x_1(t))). \end{aligned}$$

Let

$$x_2(t) = x_1(t) + J_{\lambda(t)}^{\phi_1}[x_1(t) - \lambda(t)((p(t, u_1(t)) - (f(t, v_1(t))) - g(t, w_1(t)))],$$

then $x_2 : \Omega \to H$ is measurable.

By induction, we can obtain sequences $\{x_n(t)\}, \{u_n(t)\}, \{v_n(t)\}$ and $\{w_n(t)\}$ as follows:

$$x_{n+1}(t) = x_n(t) + J_{\lambda(t)}^{\phi_n}[x_n(t) - \lambda(t)((p(t, u_n(t)) - (f(t, v_n(t))) - g(t, w_n(t)))], \quad (2.4.5)$$

$$\|u_n(t) - u_{n+1}(t)\| \leq (1 + (n+1)^{-1})\ \mathcal{H}(M(t, x_n(t)), M(t, x_{n+1}(t))),$$

$$\|v_n(t) - v_{n+1}(t)\| \leq (1 + (n+1)^{-1})\ \mathcal{H}(S(t, x_n(t)), S(t, x_{n+1}(t))),$$

$$\|w_n(t) - w_{n+1}(t)\| \leq (1 + (n+1)^{-1})\ \mathcal{H}(T(t, x_n(t)), T(t, x_{n+1}(t))),$$

for any $t \in \Omega$ and $n = 0, 1, 2, \ldots$.

Lemma 2.4.5. *Let $\eta : \Omega \times H \times H \to H$ be a strongly monotone and Lipschitz continuous random map with constants $q(t) > 0$ and $s(t) > 0$, respectively , and satisfy Assumption (2.4.1). Then*

$$\|J^{\phi}_{\lambda(t)}x(t) - J^{\phi}_{\lambda(t)}y(t)\| \leq \tau(t)\|x(t) - y(t)\| \quad \forall\ x(t), y(t) \in H,$$

where $\tau(t) = \frac{s(t)}{q(t)}$.

Theorem 2.4.6. *Let $\eta : \Omega \times H \times H \to H$ be strongly monotone and Lipschitz continuous random map with constants $q(t) > 0$ and $s(t) > 0$, respectively, and satisfy Assumption (2.4.1). Let $f, g, p : \Omega \times H \to H$ be Lipschitz continuous with corresponding constants $\xi(t), r(t)$ and $\sigma(t)$, respectively. Let $M, S, T : \Omega \times H \to CB(H)$ be $\mathcal{H}$ -Lipschtiz continuous with corresponding constants $\gamma(t), h(t)$ and $d(t)$, respectively and S be the relaxed Lipschtiz with respect to f with constant $k(t)$ and T be relaxed monotone with respect to g with constant $c(t)$. For each n, let $\phi_n : H \to \mathbb{R} \cup \{+\infty\}$ and $\phi : H \to \mathbb{R} \cup \{+\infty\}$ be mappings such that $R(I + \lambda(t)\partial_n\phi_n) = R(I + \lambda(t)\partial_n\phi) = H$ for $\lambda(t) > 0$. Assume that*

$$\lim_{n\to+\infty} \|J^{\phi_n}_{\lambda(t)}z(t) - J^{\phi_{n-1}}_{\lambda(t)}z(t)\| = 0, \quad \forall\ z(t) \in H$$

and if

$$\left|\lambda(t) - \frac{k(t) - c(t)}{[\xi(t)h(t) + r(t)d(t)]^2 - \sigma^2(t)\gamma^2(t)}\right|$$
$$< \frac{\sqrt{(c(t) - k(t))^2 - [(\xi(t)h(t) + r(t)d(t))^2 - \sigma^2(t)\gamma^2(t)]}}{[\xi(t)h(t) + r(t)d(t)]^2 - \sigma^2(t)\gamma^2(t)}$$
$$c(t) - k(t) > \sqrt{[\xi(t)h(t) + \gamma(t)d(t)]^2 - \sigma^2(t)\gamma^2(t)} \tag{2.4.6}$$
$$\xi(t)h(t) + r(t)d(t) > \sigma(t)\gamma(t).$$

Then there exists a set of elements $x(t) \in H, u(t) \in M(t, x(t)), v(t) \in S(t, x(t))$ and $w(t) \in T(t, x(t))$ which is a solution of (CGNRVLIP) and $x_n(t) \to x(t), u_n(t) \to u(t), v_n(t) \to v(t), w_n(t) \to w(t)$ as $n \to \infty$, where $\{x_n(t)\}, \{u_n(t)\}, \{v_n(t)\}$ and $\{w_n(t)\}$ are the random sequences obtained by Algorithm (2.4.1).

Proof. From (2.4.5), we have

$$\|x_{n+1}(t)-x_n(t)\| = \|x_n(t)-x_{n-1}(t)+J^{\phi_n}_{\lambda(t)}(h(x_n(t))-J^{\phi_{n-1}}_{\lambda(t)}(h(x_{n-1}(t)))\|, \qquad (2.4.7)$$

where

$$h(x_n(t)) = x_n(t) - \lambda(t)[(p(t,u_n(t)) - (f(t,v_n(t)) - g(t,w_n(t)))].$$

By introducing the term $J^{\phi_n}_{\lambda(t)}(h(x_{n-1}(t))$, we get

$$\|J^{\phi_n}_{\lambda}(h(x_n(t))) - J^{\phi_{n-1}}_{\lambda}(h(x_{n-1}(t)))\|$$
$$\leq \|J^{\phi_n}_{\lambda}(h(x_n(t)))-J^{\phi_n}_{\lambda}(h(x_{n-1}(t)))\|+\|J^{\phi_n}_{\lambda}(h(x_{n-1}(t)))-J^{\phi_{n-1}}_{\lambda}(h(x_{n-1}(t)))\|. \qquad (2.4.8)$$

By Lemma (2.4.5), we have

$$\|J^{\phi_n}_{\lambda}(h(x_n(t))) - J^{\phi_{n-1}}_{\lambda}(h(x_{n-1}(t)))\|$$
$$\leq \tau(t)\|h(x_n(t))-h(x_{n-1}(t))\|+\|J^{\phi_n}_{\lambda}(h(x_{n-1}(t)))-J^{\phi_{n-1}}_{\lambda}(h(x_{n-1}(t)))\|, \qquad (2.4.9)$$

where $\tau(t) = \frac{s(t)}{q(t)}$, and

$$\begin{aligned}\|h(x_n(t)) - h(x_{n-1}(t))\| &= \|x_n(t) - \lambda(t)(p(t,u_n(t)) - (f(t,v_n(t)) \\ &\quad -g(t,w_n(t)))) - x_{n-1}(t) + \lambda(t)(p(t,u_{n-1}(t)) \\ &\quad -(f(t,v_{n-1}(t)) - g(t,w_{n-1}(t)))\| \\ &\leq \|x_n(t) - x_{n-1}(t) + \lambda(t)(f(t,v_n(t) - f(t,v_{n-1}(t)) \\ &\quad -\lambda(t)(g(t,w_n(t) - g(t,w_{n-1}(t))\| \\ &\quad +\lambda(t)\tau(t)\|(p(t,u_n(t) - p(t,u_{n-1}(t))\|. \qquad (2.4.10)\end{aligned}$$

From (2.4.7) - (2.4.10), we get

$$\begin{aligned}\|x_{n+1}(t) - x_n(t)\| &\leq \|x_n(t) - x_{n-1}(t)\| \\ &\quad +\tau(t)\|x_n(t) - x_{n-1}(t) + \lambda(t)(f(t,v_n(t) \\ &\quad -f(t,v_{n-1}(t)) - \lambda(t)(g(t,w_n(t) - g(t,w_{n-1}(t))\| \\ &\quad +\lambda(t)\tau(t)\|p(t,u_n(t) - p(t,u_{n-1}(t))\| \\ &\quad +\|J^{\phi_n}_{\lambda(t)}(h(x_{n-1}(t)) - J^{\phi_n}_{\lambda(t)}(h(x_{n-1}(t))\|. \qquad (2.4.11)\end{aligned}$$

Since M, S and T are $\mathcal{H}$- Lipschitz continuous, and f, g and p are Lipschtiz continuous, we have

$$\begin{aligned}\|p(t,u_n(t)) - p(t,u_{n-1}(t))\| &\leq \sigma(t)\|u_n(t) - u_{n-1}(t)\| \\ &\leq \sigma(t)\gamma(t)(1+1/n)\|x_n(t) - x_{n-1}(t)\| \qquad (2.4.12)\end{aligned}$$

$$\begin{aligned} \|f(t, v_n(t)) - f(t, v_{n-1}(t))\| &\leq \xi(t)\|v_n(t) - v_{n-1}(t)\| \\ &\leq \xi(t)h(t)(1 + 1/n)\|x_n(t) - x_{n-1}(t)\| \end{aligned} \tag{2.4.13}$$

$$\begin{aligned} \|g(t, w_n(t)) - g(t, w_{n-1}(t))\| &\leq r(t)\|w_n(t) - w_{n-1}(t)\| \\ &\leq r(t)d(t)(1 + 1/n)\|x_n(t) - x_{n-1}(t)\|. \end{aligned} \tag{2.4.14}$$

Further , since S is relaxed Lipschitz and T is relaxed monotone , we have

$$\begin{aligned} &\|x_n(t) - x_{n-1}(t) + \lambda(t)(f(t, v_n(t) - f(t, v_{n-1}(t)) - \lambda(t)(g(t, w_n(t) - g(t, w_{n-1}(t))\|^2 \\ = &\ \|x_n(t) - x_{n-1}(t)\|^2 + 2\lambda(t)\langle f(t, v_n(t)) - f(t, v_{n-1}(t)), x_n(t) \\ &-x_{n-1}(t)\rangle - 2\lambda(t)\langle g(t, w_n(t)) - g(t, w_{n-1}(t)), x_n(t) - x_{n-1}(t)\rangle \\ &+\lambda^2(t)\|f(t, v_n(t)) - f(t, v_{n-1}(t)) - (g(t, w_n(t)) - g(t, w_{n-1}(t))\|^2 \\ \leq &\ [1 - 2\lambda(t)(k(t) - c(t)) + \lambda^2(t)(1 + 1/n)^2(\xi(t)h(t) \\ &+r(t)d(t))^2]\|x_n(t) - x_{n-1}(t)\|^2. \end{aligned} \tag{2.4.15}$$

From (2.4.11) - (2.4.15) , it follows that

$$\begin{aligned} \|x_{n+1}(t) - x_n(t)\| &= \theta_n\|x_n(t) - x_{n-1}(t)\| \\ &+\|J^{\phi_n}_{\lambda(t)}(h(x_{n-1}(t)) - J^{\phi_{n-1}}_{\lambda(t)}(h(x_{n-1}(t)))\|, \end{aligned} \tag{2.4.16}$$

where

$$\begin{aligned} \theta_n(t) &= 1 + \tau(t)\sqrt{(1 - 2\lambda(t))(k(t) - c(t)) + \lambda^2(t)(1 + 1/n)^2[\xi(t)h(t) + r(t)d(t)]^2} \\ &+\lambda(t)\sigma(t)\gamma(t)\tau(t)(1 + 1/n). \end{aligned}$$

Let,

$$\begin{aligned} \theta(t) &= 1 + \tau(t)\sqrt{(1 - 2\lambda(t))(k(t) - c(t)) + \lambda^2(t)[\xi(t)h(t) + r(t)d(t)]^2} \\ &+\lambda(t)\sigma(t)\gamma(t)\tau(t). \end{aligned}$$

Then $\theta_n(t) \to \theta(t)$ as $n \to \infty$. It follows from (2.4.6) that $\theta(t) < 1$.

Since $\lim_{n\to+\infty} \|J^{\phi_n}_{\lambda(t)}z(t) - J^{\phi_{n-1}}_{\lambda(t)}z(t)\| = 0$, It follows from (2.4.16) that $\{x_n(t)\}$ is a Cauchy sequence in H. Since H is complete, then there exists a measurable map

$x : \Omega \to H$ such that $x_n(t) \to x(t)$, for all $t \in \Omega$. Now we prove that $u_n(t) \to u(t) \in M(t, x(t)), v_n(t) \to v(t) \in S(t, x(t))$ and $w_n(t) \to w(t) \in T(t, x(t))$. In fact , It follows from Algorithm (2.4.1) that

$$\|u_n(t) - u_{n-1}(t)\| \leq (1 + 1/n)\gamma(t)\|x_n(t) - x_{n-1}(t)\|,$$

$$\|v_n(t) - v_{n-1}(t)\| \leq (1 + 1/n)h(t)\|x_n(t) - x_{n-1}(t)\|,$$

$$\|w_n(t) - w_{n-1}(t)\| \leq (1 + 1/n)d(t)\|x_n(t) - x_{n-1}(t)\|,$$

which implies that $\{u_n(t)\}, \{v_n(t)\}$ and $\{w_n(t)\}$ are also Cauchy sequences in H. Let $u_n(t) \to u(t), v_n(t) \to v(t), w_n(t) \to w(t)$ as $n \to \infty$. We have

$$\begin{aligned} d(v(t), S(t, x(t))) &= inf\ \{\|v(t) - y(t)\| : y \in S(t, x(t)\} \\ &\leq \|v(t) - v_n(t)\| + d(v_n(t), S(t, x(t)) \\ &\leq \|v(t) - v_n(t)\| + \mathcal{H}(S(t, x_n(t)), S(t, x(t)) \\ &\leq \|v(t) - v_n(t)\| + h\|x_n(t) - x(t)\| \to 0 \text{ as } n \to \infty \end{aligned}$$

Hence $v(t) \in S(t, x(t))$. Similarly we can prove that $u(t) \in M(t, x(t)), w(t) \in T(t, x(t))$. This complete the proof. □