Chapter 1

Preliminaries

In this chapter, we first present some basic definitions and results from Functional Analysis which will be used in the sequel. Then we give a brief introduction of variational inequalities, variational-like inequalities, variational inclusions and complementarity problems.

1.1 Introduction

As long as a branch of knowledge offers an abundance of problems, it is full of vitality.

David Hilbert

Because of the applications of Functional Analysis in Sciences, Engineering and Social Sciences, a great deal of work has been done in this area. Specially, the Nonlinear Analysis, a branch of Functional Analysis, has grown very rapidly and has many interesting applications in partial differential equations, mechanics, optimization, game theory, economics, engineering, etc. The theory of variational inequalities is one of the fields of applications of Nonlinear Analysis. It was introduced in sixties by the Italian and French Schools as a joint and concerted efforts of two leading mathematicians of this period, Guido Stampacchia and Jacques-Louis Lions. This theory has many applications in different branches of Sciences. In the last four decades, variational inequalities have been extended and generalized in different directions.

In the second section, we present some definitions and results from Functional Analysis which will be used in the sequel. The third section deals with the brief introduction of variational inequalities. Section 4 is devoted to the variational-like inequalities which are generalizations of variational inequalities. We also present different kinds of variational-like inequalities in this section. In section 5, we present the generalization of variational inequalities, known as variational inclusions. A brief introduction of complementarity problems is given in the last section of this chapter.

1.2 Tools from Functional Analysis

In this section, we present some basic definitions and results from Functional Analysis which will be used in the subsequent chapters.

Throughout the thesis, we shall use the following notations. For any nonempty set X, we denote by $2^X, CB(X)$ and $C(X)$, the family of all nonempty subsets of X, the family of all nonempty, closed and bounded subsets of X, and family of all nonempty compact subsets of X, respectively. If A is a nonempty subset of a topological vector space Y then we denote by co(A), int(A) and $\partial\phi$, the convex hull, the interior of A and the subdifferential of ϕ, respectively.

Let H be a Hilbert space with its dual H^* whose norm and inner product are denoted by $\|.\|$ and $\langle .,. \rangle$, respectively.

Definition 1.2.1. A mapping $g : H \to H$ is called:

(i) *monotone* if $\langle g(x) - g(y), x - y \rangle \geq 0, \quad \forall\, x, y \in H;$ (1.2.1)

(ii) *strictly monotone* if equality holds in (1.2.1) only if $x = y$;

(iii) *strongly monotone* if there exists a constant $r > 0$ such that

$$\langle g(x) - g(y), x - y \rangle \geq r\|x - y\|^2, \quad \forall\, x, y \in H;$$

(iv) *Lipschitz continuous* if there exists a constant $s > 0$ such that

$$\|g(x) - g(y)\| \leq s\|x - y\|, \quad \forall\, x, y \in H.$$

Theorem 1.2.1. *[86] Let K be a nonempty, closed and convex subset of Hilbert space H. Then $\forall\ z \in H$, $\exists$ unique $u \in K$ such that*

$$||z - u|| = \ inf_{v \in K}||z - v||. \tag{1.2.2}$$

Definition 1.2.2. The point u satisfying (1.2.2) is called the *projection of z onto K* and we write

$$u = P_K(z).$$

Lemma 1.2.2. *[68]. If K is a nonempty, closed and convex subset of H and z is a given point in H, then $u \in K$ satisfies the inequality*

$$\langle u - z, v - u \rangle \geq 0, \quad \forall\ v \in K,$$

if and only if

$$u = P_K(z), \tag{1.2.3}$$

where P_K is the projection of H onto K.

Lemma 1.2.3. *[68]. The mapping P_K defined by (1.2.3) is nonexpansive, i.e.,*

$$||P_K(u) - P_K(v)|| \leq ||u - v||, \quad \forall\ u, v \in H.$$

Lemma 1.2.4. *[71]. If $K(u) = m(u) + K$ and K is a nonempty, closed and convex subset of H, then $\forall\ u, v \in H$,*

$$P_{K(u)}v = m(u) + P_K(v - m(u)), \tag{1.2.4}$$

where $m : H \rightarrow H$ is a mapping.

Theorem 1.2.5. *[86] (Riesz representation theorem) If f is bounded linear functional on a Hilbert space H, there exists a unique vector $v \in H$ such that*

$$f(u) = \langle u, v \rangle, \quad \forall\ u \in H \text{ and } ||f|| = ||v||.$$

Now, we present some basic definitions and results from the Set-Valued Analysis which will be used in the sequel.

Definition 1.2.3. Let X and Y be topological vector space. A multivalued mapping $P : X \rightarrow 2^Y$ is called

(i) *upper semicontinuous* at $x_0 \in X$ if for every open set V in Y containing $P(x_0)$, there exists an open neighborhood U of x_0 in X such that $P(x) \subseteq V, \ \forall\, x \in U$,

(ii) *closed* if for every net $\{x_\lambda\}$ converges to x_* and $\{y_\lambda\}$ converges to y_* such that $\forall\ \lambda,\ y_\lambda \in P(x_\lambda)$ implies that $y_* \in P(x_*)$,

(iii) *graph* of P, denoted by $G(P)$ is

$$G(P) = \{(x, z) \in X \times Y : x \in X, z \in P(x)\}.$$

Remark 1.2.1. (i) Every upper semicontinuous multivalued map is closed.
(ii) A multivalued map is closed if its graph is closed.

Theorem 1.2.6. *[7] Let K be a nonempty convex subset of a Hausdorff topological vector space X, and let $P, Q : K \to 2^K$ be two multivalued maps. Assume that the following conditions hold.*

(i) For each $x \in K$, $co(P(x)) \subseteq Q(x)$ and $P(x)$ is nonempty.

(ii) $K = \bigcup\{int_K P^{-1}(y) : y \in K\}$.

(iii) If K is not compact, assume that there exist a nonempty compact convex subset B of K and a nonempty compact subset D of K such that for each $x \in K \setminus D$ there exists $\tilde{y} \in B$ such that $x \in int_K P^{-1}(\tilde{y})$.

Then there exists $\bar{x} \in K$ such that $\bar{x} \in Q(\bar{x})$.

Definition 1.2.4. [27] A multivalued mapping $Q : H \to 2^H$ is said to be *$\mathcal{H}$-Lipschitz continuous* if there exist a constant $\gamma > 0$ such that

$$\mathcal{H}(Q(x), Q(y)) \leq \gamma\|x - y\|, \quad \forall\ x, y \in H,$$

where $\mathcal{H}(.,.)$ is a Hausdorff metric on H.

Definition 1.2.5. A multivalued mapping $Q : H \to 2^H$ is called:

(i) *monotone* if $\langle u-v, x-y\rangle \geq 0, \quad \forall\ x, y \in H,\ u \in Q(x), v \in Q(y);$ (1.2.5)

(ii) *strictly monotone* if equality holds in (1.2.5) only if $x = y$;

(iii) *strongly monotone* if there exists a constant $r > 0$ such that

$$\langle u - v, x - y\rangle \geq r\|x - y\|^2, \quad \forall\ x, y \in H,\ u \in Q(x), v \in Q(y);$$

(iv) *maximal monotone* if and only if Q is monotone and there is no other monotone mapping whose graph contains strictly the graph of Q.

Lemma 1.2.7. *[12] Let B be a reflexive Banach space endowed with strictly convex norm and $\phi : B \to \mathbb{R} \cup \{+\infty\}$ be proper, convex, lower semicontinuous function. Then the subdifferential map $\partial\phi : B \to 2^{B^*}$ is maximal monotone.*

Definition 1.2.6. [11, 105] Let $Q : H \to 2^H$ be a maximal monotone mapping. For any fixed $\alpha > 0$, the mapping $J_\alpha^Q(x) : H \to H$ defined by

$$J_\alpha^Q(x) = (I + \alpha Q)^{-1}(x), \quad \forall\, x \in H,$$

is called *resolvent operator* of Q, where I stands for the identity mapping on H.

The resolvent operator J_α^Q is single valued and nonexpansive, that is,

$$||J_\alpha^Q(x) - J_\alpha^Q(y)|| \leq ||x - y|| \quad \forall\, x, y \in H.$$

Since the sudifferential $\partial\phi$ of a proper, convex, lower semicontinuous function $\phi : H \to \mathbb{R} \cup \{+\infty\}$ is a maximal monotone multivalued map, it follows that the resolvent operator $J_\alpha^{\partial\phi}$ of index α of $\partial\phi$ is given by

$$J_\alpha^{\partial\phi}(x) = (I + \alpha\partial\phi)^{-1}(x), \quad \forall\, x \in H.$$

Definition 1.2.7. A mapping $J : B \to B^*$ is called *normalized duality mapping* if

$$||J(x)||_* = ||x|| \text{ and } \langle x, J(x)\rangle \;=\; ||x||^2, \quad \forall\, x \in B.$$

The *uniform convexity* of the space B means that for any given $\epsilon > 0$ there exists $\delta > 0$ such that $\forall\, x, y \in B$, $||x|| \leq 1$, $||y|| \leq 1$, $||x - y|| = \epsilon$ ensure the following inequality

$$||x + y|| \leq 2(1 - \delta).$$

The function

$$\delta_B(\epsilon) = \inf\left\{1 - \frac{||x + y||}{2} : ||x|| = 1,\ ||y|| = 1,\ ||x - y|| = \epsilon\right\}$$

is called the *modulus of the convexity* of the space B.

The *uniform smoothness* of the space B means that for any given $\epsilon > 0$, there exists $\delta > 0$ such that

$$\frac{||x + y|| + ||x - y||}{2} - 1 \leq \epsilon||y||$$

holds.

The function

$$\rho_B(t) = \sup\left\{\frac{||x+y|| + ||x-y||}{2} - 1 : ||x|| = 1,\ ||y|| = t\right\}$$

is called the *modulus of the smoothness* of space B.

Remark 1.2.2. The space B is uniformly convex if and only if $\delta_B(\epsilon) > 0$ for all $\epsilon > 0$, and it is uniformly smooth if and only if $\lim\limits_{t\to 0} t^{-1}\rho_B(t) = 0$.

Proposition 1.2.8. *[2] Let B be a uniformly smooth Banach space and J be a normalized duality mapping from B to B^*. Then, $\forall\ x, y \in B$, we have*

(i) $||x+y||^2 \ \leq\ ||x||^2 + 2\langle y, J(x+y)\rangle$,

(ii) $\langle x-y, J(x) - J(y)\rangle \ \leq\ 2d^2\rho_B(4||x-y||/d)$,

where $d = \sqrt{(||x||^2 + ||y||^2)/2}$.

Definition 1.2.8. The mapping $G : B \to B$ is said to be *strongly accretive* if there exist a constant $\gamma > 0$ such that

$$\langle G(x) - G(y), J(x-y)\rangle \geq \gamma||x-y||^2, \quad \forall\ x, y \in B.$$

Definition 1.2.9. [13, 39] A mapping $Q_\Omega : B \to \Omega$, where Ω be a nonempty, closed and convex subset of B, is said to be

(i) *retraction* on Ω if $Q_\Omega^2 = Q_\Omega$,

(ii) *nonexpansive retraction* if it satisfies the inequality

$$||Q_\Omega x - Q_\Omega y|| \leq ||x-y||, \quad \forall\ x, y \in B,$$

(iii) *sunny retraction* if for all $x \in B$ and for all $-\infty < t < \infty$

$$Q_\Omega(Q_\Omega x + t(x - Q_\Omega x)) = Q_\Omega x.$$

Proposition 1.2.9. *[39] Q_Ω is a sunny nonexpansive retraction if and only if* $\forall\ x, y \in B$

$$\langle x - Q_\Omega x,\ J(Q_\Omega x - y)\rangle \geq 0.$$

Proposition 1.2.10. *[3] Let Ω be a nonempty, closed and convex subset of a Banach space B and let $m : B \to B$ be a mapping. Then $\forall\ x \in B$, we have*

$$Q_{\Omega+m(x)}x = m(x) + Q_\Omega(x - m(x)).$$

Definition 1.2.10. [14] A multivalued map $A : D(A) \subset B \to 2^B$ is said to be

(i) *accretive* if $\forall\, x, y \in D(A)$, $u \in A(x)$, $v \in A(y)$, $\exists\, j(x-y) \in J(x-y)$ such that

$$\langle u - v, j(x-y) \rangle \geq 0,$$

(ii) *k-strongly accretive*, $k \in (0,1)$ if $\forall\, x, y \in D(A)$, $\exists\, j(x-y) \in J(x-y)$ such that $\forall\, u \in A(x)$, $v \in A(y)$,

$$\langle u - v, j(x-y) \rangle \geq k\|x-y\|^2,$$

(iii) *m-accretive* if A is accretive and $(I + \rho A)(D(A)) = B$ for every $\rho > 0$, where I is the identity mapping.

Remark 1.2.3. [29] If $B = B^* = H$, Hilbert space, then $A : D(A) \subset H \to 2^H$ is an m-accretive mapping if and only if $A : D(A) \subset H \to 2^H$ is a maximal monotone mapping.

1.3 Variational Inequalities

In this section, we present a brief introduction of variational inequalities.

The theory of variational inequalities was introduced in connection of a mechanical problem by Fichera [36], and Lions and Stampacchia [64]. Many problems of elasticity and fluid mechanics can be expressed in terms of an unknown u, representing the displacement of a mechanical system satisfying

$$a(u, v-u) \geq f(v-u), \quad \forall\, v \in K, \tag{1.3.1}$$

where the set K of admissible displacements is a nonempty, closed and convex subset of a Hilbert space H, $a(.,.)$ is a bilinear form and f is a bounded linear functional on H. The inequalities of type (1.3.1) are called *variational inequalities.*

If the bilinear form $a(.,.)$ is continuous, then by Riesz-representation Theorem 1.2.6, we have

$$a(u, v) = \langle A(u), v \rangle, \quad \forall\, u, v \in H, \tag{1.3.2}$$

where A is a continuous linear operator on H. Then the inequality (1.3.1) is equivalent to find $u \in K$ such that

$$\langle A(u), v-u\rangle \geq \langle f, v-u\rangle, \quad \forall\, v \in K. \tag{1.3.3}$$

If the operator A and f are nonlinear, the variational inequality (1.3.3) is known as *strongly nonlinear variational inequality*, introduced and studied by Noor [70].

If $f \equiv 0$, then (1.3.3) is equivalent to find $u \in K$ such that

$$\langle A(u), v-u\rangle \geq 0, \quad \forall\, v \in K. \tag{1.3.4}$$

For further detail on variational inequalities, see [12, 24, 25, 37, 38, 57, 60, 83].

It is worth mentioning that the unilateral contact problem involving contact laws of monotone nature do not lead to the formulation of variational inequalities directly. However, it has been shown by Panagiotopoulus [80], using the notions of Clarke's generalized gradient and Rockafeller's upper subderivative, that the nonconvex unilateral contact problems can only be characterized by a class of strongly nonlinear variational inequalities (1.3.3).

If in a variational inequality formulation, the convex set K does depend upon its solution, then this class of variational inequalities is called a quasi-variational inequality. This class of inequalities is introduced and studied by Bensoussan, Goursat, and Lions [15]. For further detail, we refer to Bansoussan and Lions [16], Baiocchi and Capelo [12], and Mosco [68].

1.4 Variational-like Inequalities

The theory of variational inequalities has been extended and generalized in many different directions because of the applications in different branches of sciences, engineering, optimization, economics, equilibrium theory etc. The variational-like inequality is one of the generalized form of variational inequality, which was initially studied by Parida, Sahoo and Kumar [82] in 1989. They developed a theory for existence of its solutions in finite dimension Euclidean space $\mathbb{R}^n$ by using Kakutani fixed point theorem and they have also shown its relationship with mathematical programming. In 1992, Yang and Chen [99] have introduced a new class of nonconvex nonsmooth function (semi-preinvex functions). They derived the Fritz-John condition by using an alternative theorem for semi-preinvex program and studied the variational-like inequality and shown that it is a necessary condition for the optimal solutions. Some existence theorems are also proved. In 1994, Siddiqi, Khaliq and Ansari [92] studied variational-like inequalities in the setting of reflexive Banach spaces and topological vector spaces with or without convexity assumptions. Recently, Noor [74] has suggested an algorithm to find the approximate solutions of variational-like inequality and prove that the minimum of the arcwise directional differentiable semiconvex function [10] can be characterized by a class of variational-like inequalities. By using η-subdifferentiability, Lee, Ansari and Yao [63] suggested an perturbed iterative algorithm for finding the approximate solutions of variational-like inequalities. Dien [30] has studied a more general class of variational-like and quasivariational-like inequalities. Ansari and Yao [9] studied the existence of solutions of general variational-like inequalities and also proposed auxiliary principle technique to compute the approximate solutions.

In this section, we present different kinds of scaler-valued variational-like inequalities studied till now. These inequalities have not been much studied, therefore, there is a lot of scope to do research in this area.

Let $\langle E, E^* \rangle$ be a dual system of locally convex spaces and K be a nonempty, closed and convex subset of E. Given two mappings $f : E \to E^*$ and $\eta(.,.) : K \times K \to E$, then the *variational-like inequality problem* is the following:

$$\text{(VLIP)} \quad \begin{cases} \text{Find } x \in K \text{ such that} \\ \langle f(x), \eta(y,x) \rangle \geq 0, \quad \forall\, y \in K. \end{cases} \tag{1.4.1}$$

Inequality (1.4.1) is called *variational-like inequality.* It is first studied in 1989 by Parida, Sahoo and Kumar [82] in the setting of n-dimensional Euclidean space $\mathbb{R}^n$ in the study of mathematical programming problem. It has been further studied by Yang and Chen [99] in the study of economic equilibrium problems, and Siddiqi, Khaliq and Ansari [92] in the setting of reflexive Banach spaces and topological vector spaces.

Recently, Noor [74], Ansari and Yao [9] have studied such type of problems by using the auxiliary variational principle technique and suggested an iterative algorithms.

Dien [30] has considered the following general variational-like inequality problem in $\mathbb{R}^n$. Given $\phi : K \to \mathbb{R}$ find $x \in K$ such that

$$\text{(gVLIP)} \quad \langle f(x), \eta(y,x)\rangle \geq \phi(x) - \phi(y), \quad \forall\, y \in K. \tag{1.4.2}$$

It has been further studied by Siddiqi, Ansari and Ahmad [90] in the setting of reflexive Banach spaces and topological vector spaces with or without convexity assumptions.

Remark 1.4.1. If $\eta(y,x) = y - x$, then variational-like inequality problem and general variational-like inequality problem become the following variational inequality problem.

In many applications, the convex set in the formulation of (VLIP) also depends upon the solution itself. In this case (VLIP) is called quasi-variational-like inequality problem. More precisely, for a given a multivalued map $Q : K \to 2^K$, the *general quasi-variational-like inequality problem* [30] is the following:

$$\text{(GQVLIP)} \quad \begin{cases} \text{Find } x \in K \text{ such that } x \in Q(x) \text{ and} \\ \langle f(x), \eta(y,x)\rangle \geq \phi(x) - \phi(y), \quad \forall\, y \in Q(x). \end{cases} \tag{1.4.3}$$

Let $T : E \to 2^{E^*}$ be a multivalued map, then the *variational-like inequality problem for multivalued map* defined as follows:

$$\begin{cases} \text{Find } x \in K \text{ such that } u \in T(x) \text{ and} \\ \langle u, \eta(y,x)\rangle \geq 0, \quad \forall\, y \in K. \end{cases} \tag{1.4.4}$$

In the same way, we define *general variational-like inequality problem for multivalued maps* as follows:

$$\begin{cases} \text{Find } x \in K \text{ such that } u \in T(x) \text{ and} \\ \langle u, \eta(y,x)\rangle \geq \phi(x) - \phi(y), \quad \forall\, y \in K. \end{cases} \tag{1.4.5}$$

More general, we consider the *generalized variational-like inequality problem.* Let C be a nonempty subset of E^* and $N : K \times C \to E^*$ and $\eta : K \times K \to E$ be two single valued mappings. Let $T : K \to 2^C$ be a multivalued mapping. Then the *generalized variational-like inequality problem* is the following:

$$\text{(GVLIP)} \quad \begin{cases} \text{Find } x \in K, \text{ and } u \in T(x) \text{ such that} \\ \langle N(x,u), \eta(y,x) \rangle \geq 0, \quad \forall\, y \in K. \end{cases}$$

Such problem is studied by Parida and Sen [81] in $\mathbb{R}^n$. They have also shown its relationship with nonlinear programming problem and saddle point problem. It has been further studied by Yao [101] with applications in generalized complementarity problems see also Cubiotti and Yao [28]

Again, if the convex set K involved in the formulation of problem (1.4.4) depends upon the solution itself then we have the *generalized quasi-variational-like inequality problem* [102] as follows: Given a multivalued mapping $Q : K \to 2^K$,

$$\text{(GQVLIP)} \quad \begin{cases} \text{Find } x \in Q(x), \text{ and } u \in T(x) \text{ such that} \\ \langle N(x,u), \eta(y,x) \rangle \geq 0, \quad \forall\, y \in Q(x). \end{cases}$$

It is first studied by Yao [102] with applications in economic equilibrium problem. It has been further studied by Tian [97] Kum [61].

The following *generalized quasi-variational-like inequality* is considered by Ben-El-Mechaickh and Isac [67].

$$\text{(GQVLIP)(I)} \quad \begin{cases} \text{Find } x \in Q(x), u \in T(x) \text{ such that} \\ \langle N(x,u), \eta(y,x) \rangle \geq \phi(x) - \phi(y), \quad \forall\, y \in Q(x). \end{cases}$$

Now, we consider a more general form of generalized variational-like inequalities which includes all above mentioned variational-like inequalities as special case.

Given $b(.,.) : K \times K \to \mathbb{R}$, which is not necessarily differentiable mapping. Then we have the following problem:

$$\begin{cases} \text{Find } x \in K, u \in T(x) \text{ such that} \\ \langle N(x,u), \eta(y,x) + b(x,y) - b(x,x) \rangle \geq 0, \quad \forall\, y \in K. \end{cases} \tag{1.4.6}$$

It is studied by Siddiqi Ansari and Khan [91].

1.5 Variational Inclusions and Quasi-variational Inclusions

A useful and important generalization of variational inequalities is a mixed type variational inequality containing nonlinear term. Due to the presence of the nonlinear term, the projection method can not be used to study the existence and algorithm of solutions for the mixed type variational inequalities. In 1994, Hassouni and Moudafi [41] used the resolvent operator technique for maximal monotone mappings to study mixed type variational inequalities with single valued mappings which are called *variational inclusions* and developed a perturbed algorithm for finding approximate solutions of mixed variational inequalities.

Let H be a real Hilbert space endowed with a norm $\|.\|$ and inner product $\langle .,. \rangle$ and given continuous mappings $T, g : H \to H$, with $\text{Img} \cap \text{dom}\ \partial\phi \neq \phi$, consider the following problem:

$$\text{(GSNVIP)} \quad \begin{cases} \text{Find } u \in H \text{ such that} \\ g(u) \cap \ \text{dom}\ \partial\phi \neq \phi \\ \langle T(u) - A(u), v - g(u) \rangle \geq \phi(g(u)) - \phi(v), \ \forall\ v \in H. \end{cases}$$

In (GSNVIP), A is nonlinear continuous mapping on H, $\partial\phi$ denotes the subdifferential of proper, convex and lower semicontinuous function $\phi : H \to \mathbb{R} \cup \{+\infty\}$.

The class of variational inclusions considered in [41] is more general than the class of variational and quasi-variational inequalities studied by Noor [70, 72], Isac [55] and Siddiqi and Ansari [87, 88]. More precisely, with the choice $\phi = \delta_K$ the indicator function of closed convex set K, the class of strongly nonlinear variational inequality problem given by

$$\langle T(u) - A(u), v - g(u) \rangle \geq 0, \quad \forall\ v \in K, \tag{1.5.1}$$

is recovered.

For the case when $\phi(.,.) = \delta_K(. - m(u)) = \delta_{K-m(u)}(.)$ is single valued mapping on H, the problem (GSNVIP) reduces to *general strongly quasi-variational inequality problem* given by

$$\langle T(u) - A(u), v - g(u) \rangle \geq 0, \quad \forall\ v \in K. \tag{1.5.2}$$

where the set $K(u)$ is equal to $K + m(u)$.

In 1998, Huang [48] introduced and studied the following class of important generalized set-valued implicit variational inclusion problems in a Hilbert space H.

Let $M : H \to 2^H$ is a maximal monotone mapping and Range (g) and dom $M \neq \phi$, where $g : H \to H$ is a single valued mapping. For given mapping $f, p : H \to H$ and $A, S : H \to CB(H)$, find $u \in H, w \in A(u),$ *and* $y \in S(u)$ such that $g(u) \in$ dom M *and*

$$0 \in f(w) - p(y) + M(g(u)). \tag{1.5.3}$$

In 2000, Shim et al [85], introduce and study a new class of quasi-variational inclusion, which is called *generalized set-valued strongly nonlinear quasi-variational inclusion.*

Given set-valued mappings $G, S, T : H \to 2^H$, single valued mappings $p, m : H \to H$, and $N : H \times H \to H$. Suppose that $M : H \to 2^H$ is a maximal monotone mapping.

Find $u \in H,\ x \in S(u), y \in T(u)$ and $z \in G(u)$ such that $P(u) - m(z) \in$ dom M,

$$0 \in N(x, y) + M(P(u) - m(z)). \tag{1.5.4}$$

1.6 Complementarity Problems

Due to the applications in area such as, optimization theory, structural engineering, mechanics, elasticity, lubrication theory, economics, equilibrium theory on networks, stochastic optimal control, etc., the complementarity problem is one of the interesting and important problems which was introduced by G. E. Lemke in 1964, but Cottle [26] and Cottle and Dantzig [27] formally defined the linear complementarity problem and called it the fundamental problem. In recent years, it has been generalized and extended to many different directions. The implicit complementarity problem which is one of the generalizations of complementarity problems, is introduced by Isac [53, 54]; See also [56, 57] and references therein. In 1987, Noor [72] considered and studied an important and useful generalization of complementarity problem which is called *mildly nonlinear complementarity problem.* Siddiqi and Ansari [89] considered a new class of implicit complementarity problem and studied the existence of its solution. An iterative algorithm is also given to find the approximate solutions of the new problem and proved that the approximate solutions converge to the exact solution. They also mentioned several special cases.

Let H be a Hilbert spaces whose norm and inner product are denoted by $\|.\|$ and $\langle .,. \rangle$, respectively. Let K be a closed convex cone in H and K^* be the polar cone of K defined by

$$K^* = \{u \in H : \langle u, v \rangle \geq 0, \quad \forall\, v \in K\}.$$

Let $T : H \to H^*$ be a given nonlinear operator. Then the problem of finding $u \in K$ such that

$$T(u) \in K^* \text{ and } \langle u, T(u) \rangle = 0, \tag{1.6.1}$$

is known as *nonlinear complementarity problem* (for short, NCP).

Let $G : K \to K$ be a nonlinear map. Isac [57] extended (NCP) to the following problem which is known as *implicit complementarity problem*:

$$\text{(ICP)} \quad \begin{cases} \text{Find } u \in K \text{ such that} \\ G(u) \in K, T(u) \in K^* \text{ and} \\ \langle G(u), T(u) \rangle = 0. \end{cases}$$

The class of (ICP) is considered and extensively studied by Isac; See for example, [57] and references therein.

Further (ICP) is generalized by Siddiqi and Ansari [89] in the following form.

If $K \subseteq H$ and $T, A : H \to H$ are nonlinear mappings and $G : K \to H$ is a mapping, then the *strongly nonlinear implicit complementarity problem* is the following:

$$\text{(SNICP)} \quad \begin{cases} \text{Find } u \in K \text{ such that} \\ G(u) \in K, \ T(u) + A(u) \in K^* \text{ and} \\ \langle G(u), T(u) + A(u) \rangle = 0, \quad \forall\, v \in K. \end{cases}$$

By considering A is a multivalued mapping, Chang and Huang [21] considered and studied a complementarity problem, known as *generalized multivalued implicit* *complementarity problem*, that is, given a subset $K \subseteq H$ and $T,\ G : K \to H$ are single valued mappings and $A : K \to 2^H$ is a multivalued mapping. Then the problem of finding $u \in K$, and $y \in A(u)$ such that

$$G(u) \in K, \quad T(u) + y \in K^*, \quad \langle G(u), T(u) + y \rangle = 0. \tag{1.6.2}$$

Of course, if A is a single valued map, then (1.6.2) reduces to (SNICP).

Recently, several kind of complementarity problems for fuzzy mappings are considered and studied by Chang [18], Chang and Huang [22], Noor [73] and Lee et al [62]. For a detail on Complementarity problems and their applications, we refer to Isac [57] and references therein.