Chapter 6

Generalized Quasi-complementarity Problems with Fuzzy Multivalued Maps

In this chapter, we study a class of generalized quasi-complementarity problems with fuzzy multivalued mappings and suggest a new algorithm for computing the approximate solutions of this class of generalized quasi-complementarity problems. We also discuss the existence of a solution of our problem without compactness assumption and the convergence of the iterative sequences generated by the algorithm. Some special cases are also given.

6.1 Introduction and Formulations

In an unpublished work, Ansari [4] introduced the concept of variational inequalities for fuzzy mappings in his Ph.D. thesis. Separately, Chang and Zhu [23] also studied a class of variational inequalities for fuzzy mappings. Several kind of variational inequalities and complementarity problems for fuzzy mappings were considered and studied by Chang [18], Chang and Huang [22], Noor [73] and Lee et al [62]. For a detail on Complementarity problems and their applications, we refer to Isac [57]. Motivated and inspired by the recent research work going on in this field, in this chapter, we study a class of generalized quasi-complementarity problems with fuzzy multivalued mappings. A new algorithm for computing the approximate solutions of the generalized quasi-complementarity problem with fuzzy multivalued mappings is suggested. We also discuss the existence of a solution of our problem without compactness assumption and the convergence of iterative sequences generated by our algorithm.

Let H be a real Hilbert space endowed with the norm $\|.\|$ and inner product $\langle ., . \rangle$. If K is a closed convex cone in H, we denote by K^* the polar cone of K, i.e.,

$$K^* = \{u \in H : \langle u, v \rangle \geq 0, \quad \forall\, v \in K\}.$$

Let $N : H \times H \to H$ and $m, g : H \to H$ be single valued mappings and $F, G, A : H \to \mathcal{F}(H)$ be fuzzy mappings. Let $a, b, c : H \to [0, 1]$ be given functions. We consider the following *generalized quasi-complementarity problem with fuzzy multivalued mappings*:

$$\text{(GQCPFM)} \quad \begin{cases} \text{Find } u, x, y, w \in H \text{ such that} \\ F_u(x) \geq a(u), \ G_u(y) \geq b(u), \ A_u(w) \geq c(u), \\ g(u) \in K(w), \ N(x, y) \in K^*(w) \text{ and} \\ \langle g(u) - m(w), N(x, y) \rangle = 0, \end{cases}$$

where $K(w) = m(w) + K$ and thus $K^*(w) = (m(w) + K)^* = m^*(w) \cap K^*$.

SPECIAL CASES:

(i) If $F, G, A : H \to CB(H)$ are classical multivalued mappings, we can define the fuzzy mappings $F, G, A : H \to \mathcal{F}(H)$ by

$$u \to \chi F(u), \ u \to \chi G(u), \ u \to \chi A(u),$$

where $\chi F(u), \ \chi G(u), \ \chi A(u)$ are the characteristic functions of $F(u), G(u)$ and $A(u)$, respectively. Taking $a(u) = b(u) = c(u) = 1, \ \forall\, u \in H$, (GQCPFM) is equivalent to the following problem:

$$\begin{cases} \text{Find } u \in H, x \in F(u), y \in G(u), w \in A(u) \text{ such that} \\ g(u) \in K(w), \ N(x, y) \in K^*(w) \text{ and} \\ \langle g(u) - m(w), N(x, y) \rangle = 0. \end{cases} \tag{6.1.1}$$

Problem (6.1.1) is a generalization of the problem considered in [45].

(ii) If $N(x, y) = x - y$, $\forall\, x, y \in H$ and $A(u) = u$, then the problem (6.1.1) is equivalent to the following *generalized strongly nonlinear quasi-complementarity problem*:

$$\text{(GSNQCP)} \quad \begin{cases} \text{Find } u \in H, x \in F(u), y \in G(u) \text{ such that} \\ g(u) \in K(u), x - y \in K^*(u) \text{ and} \\ \langle g(u) - m(u), x - y \rangle = 0. \end{cases}$$

It is considered and studied by Deing [34].

6.2 Existence and Convergence Theory

In this section, we first convert (GQCPFM) into a quasi-variational inequality problem for fuzzy mappings. By using this equivalence and projection operator, we suggest an iterative algorithm for finding the approximate solutions of (GQCPFM). The convergence of approximate solutions obtained by the suggested algorithm is also studied.

Theorem 6.2.1. *If $K \subset H$ is a closed convex cone and $K(w) = m(w) + K$, then the set of elements (u, x, y, w) is a solution of (GQCPFM) if and only if $u \in H, x \in \tilde{F}(u), y \in \tilde{G}(u)$ and $w \in \tilde{A}(u)$ such that*

$$g(u) \in K(w) \;:\; \langle N(x, y), g(v) - g(u) \rangle \geq 0, \;\forall\, g(v) \in K(w). \tag{6.2.1}$$

Proof. Let the set of elements $u \in H, x \in \tilde{F}(u), y \in \tilde{G}(u)$ and $w \in \tilde{A}(u)$ is a solution of (GQCPFM). Since $K(w) = m(w) + K$, if $g(v) \in K(w)$, it can be written as $g(v) = m(w) + z$ for some $z \in K$ and thus $\forall\, g(v) \in K(w)$, we have

$$\begin{aligned} \langle N(x, y), g(v) - g(u) \rangle &= \langle N(x, y), m(w) + z - g(u) \rangle \\ &= \langle N(x, y), m(w) - g(u) \rangle + \langle N(x, y), z \rangle \\ &= \langle N(x, y), z \rangle \geq 0. \end{aligned}$$

Therefore (u, x, y, w) is a solution of problem (6.2.1)

Conversely, Suppose that the set of elements $u \in H, x \in \tilde{F}(u)$, $y \in \tilde{G}(u)$ and

$w \in \tilde{A}(u)$ such that $g(u) \in K(w)$ satisfies the inequality (6.2.1). Since $g(u) \in K(w)$, we know that $g(u) - m(w) \in K$ and hence $2g(u) - m(w) \in K(w)$. From $0 \in K$, we get $m(w) \in K(w)$. Taking $g(v) = 2g(u) - m(w)$ and $g(v) = m(w)$ in (6.2.1), we obtain

$$\langle N(x,y), g(u) - m(w)\rangle \geq 0 \quad \text{and} \quad \langle N(x,y), m(w) - g(u)\rangle \geq 0.$$

It follows from the above inequalities that

$$\langle N(x,y), g(u) - m(w)\rangle = 0$$

It remains to prove that $N(x,y) \in K^*(w)$. Taking $g(v) = m(w) + z$ in (6.2.1), we obtain

$$\begin{aligned} 0 \leq \langle N(x,y), g(v) - g(u)\rangle &= \langle N(x,y), m(w) + z - g(u)\rangle \\ &= \langle N(x,y), z\rangle. \end{aligned}$$

This implies that $N(x,y) \in K^*(w)$. Therefore (u, x, y, w) is a solution of (GQCPFM). □

We need the following lemma whose proof is similar to the proof of Theorem 3.1 in [71].

Lemma 6.2.2. *If K is a closed convex cone in H and $K(w) = m(w) + K$, $\forall\ w \in K$. Then $u \in H, x \in \tilde{F}(u), y \in \tilde{G}(u)$ and $w \in \tilde{A}(u)$ satisfy (6.2.1) if and only if these satisfy the relation*

$$g(u) = m(w) + P_{K(w)}[g(u) - m(w) - \rho N(x,y)],$$

where $\rho > 0$ is a constant.

On the basis of Theorem 6.2.1 and Lemma 6.2.2, we propose the following algorithm.

Algorithm 6.2.1. *Suppose that K is a closed convex cone in H and $N : H \times H \to H$, $m, g : H \to H$. Let $F, G, A : H \to \mathcal{F}(H)$ be fuzzy mappings satisfying condition $(*)$ of section 4.4 and $\tilde{F}, \tilde{G}, \tilde{A} : H \to CB(H)$ be fuzzy mappings induced by F, G and A, respectively. For given $u_0 \in H$, we take $x_0 \in \tilde{F}(u_0)$, $y_0 \in \tilde{G}(u_0)$ and $w_0 \in \tilde{A}(u_0)$ and let*

$$g(u_1) = m(w_0) + P_{K(w_0)}[g(u_0) - m(w_0) - \rho N(x_0, y_0)],$$

where $\rho > 0$ is a constant.

Since $x_0 \in \tilde{F}(u_0) \in CB(H)$, $y_0 \in \tilde{G}(u_0) \in CB(H)$ and $w_0 \in \tilde{A}(u_0) \in CB(H)$, by Nadler [69], there exist $x_1 \in \tilde{F}(u_1), y_1 \in \tilde{G}(u_1)$ and $w_1 \in \tilde{A}(u_1)$ such that

$$\begin{aligned} \|x_0 - x_1\| &\leq (1+1)\hat{\mathcal{H}}(\tilde{F}(u_0), \tilde{F}(u_1)), \\ \|y_0 - y_1\| &\leq (1+1)\hat{\mathcal{H}}(\tilde{G}(u_0), \tilde{G}(u_1)), \\ \|w_0 - w_1\| &\leq (1+1)\hat{\mathcal{H}}(\tilde{A}(u_0), \tilde{A}(u_1)). \end{aligned}$$

Let

$$g(u_2) = m(w_1) + P_{K(w_1)}[g(u_1) - m(w_1) - \rho N(x_1, y_1)].$$

By induction, we can obtain sequences $\{x_n\}, \{y_n\}, \{w_n\}$ and $\{u_n\}$ such that

$$\begin{aligned} x_n \in \tilde{F}(u_n), \|x_n - x_{n+1}\| &\leq (1+(1+n)^{-1})\hat{\mathcal{H}}(\tilde{F}(u_n), \tilde{F}(u_{n+1})), \\ y_n \in \tilde{G}(u_n), \|y_n - y_{n+1}\| &\leq (1+(1+n)^{-1})\hat{\mathcal{H}}(\tilde{G}(u_n), \tilde{G}(u_{n+1})), \\ w_n \in \tilde{A}(u_n), \|w_n - w_{n+1}\| &\leq (1+(1+n)^{-1})\hat{\mathcal{H}}(\tilde{A}(u_n), \tilde{A}(u_{n+1})), \end{aligned}$$

$$g(u_{n+1}) = m(w_n) + P_{K(w_n)}[g(u_n) - m(w_n) - \rho N(x_n, y_n)], \quad n = 0, 1, 2, 3, \ldots.$$

where $\rho > 0$ is a constant.

We consider those conditions under which the solution of (GQCPFM) exists and the sequences of the approximate solutions obtained by Algorithm 6.2.1 converge strongly to the exact solution of (GQCPFM).

Theorem 6.2.3. *Let K is a closed convex cone in H, $F, G, A : H \to \mathcal{F}(H)$ be fuzzy mappings satisfying the condition $(*)$ of section 4.4 and let $\tilde{F}, \tilde{G}, \tilde{A} : H \to CB(H)$ be multivalued mappings induced by F, G and A, respectively. Let $m, g : H \to H$ be single valued mappings such that $g(H)$ is closed in H and m is Lipschitz continuous with constant t. Let $\tilde{F}, \tilde{G}$ and $\tilde{A}$ are $\hat{\mathcal{H}}$-Lipschitz continuous with respect to g with constants μ, σ and γ, respectively. Let $N : H \times H \longrightarrow H$ be relaxed monotone in the first argument with constant c and Lipschitz continuous in the first and second arguments with constants β and ξ, respectively. If the following condition holds:*

$$\left|\rho - \frac{\xi\sigma(4t\gamma - 1) - c}{\beta^2\mu^2 - \sigma^2\xi^2}\right| < \frac{\sqrt{(\xi\sigma(4t\gamma - 1) - c)^2 - (\beta^2\mu^2 - \sigma^2\xi^2)(1 - 2t\gamma)8t\gamma}}{\beta^2\mu^2 - \sigma^2\xi^2}$$

$$c > (1-4t\gamma)\sigma\xi + \sqrt{(\beta^2\mu^2 - \xi^2\sigma^2)(1-2t\gamma)(8t\gamma)}$$
$$\rho\xi\sigma < 4t\gamma - 1$$
$$\xi\sigma < \beta\mu, \tag{6.2.2}$$

then there exists a set of elements $u \in H, x \in \tilde{F}(u), y \in \tilde{G}(u), w \in \tilde{A}(u)$ *such that* (u, x, y, w) *is a solution of (GQCPFM) and*

$$g(u_n) \to g(u), x_n \to x, y_n \to y, w_n \to w \text{ as } n \to \infty,$$

where $\{u_n\}, \{x_n\}, \{y_n\}$ *and* $\{w_n\}$ *are sequences defined in Algorithm 6.2.1.*

Proof. Since m is Lipschitz continuous, it follows by Remark 4.1 in [65] that $\forall\, x, y, z \in H$,

$$\|P_{K(x)}(z) - P_{K(y)}(z)\| \le 2t\|x - y\|. \tag{6.2.3}$$

From Algorithm 6.2.1, Lemma 1.2.3 and (6.2.3),we have

$$\begin{aligned}
\|g(u_{n+1}) - g(u_n)\| &= \|m(w_n) + P_{K(w_n)}[g(u_n) - m(w_n) - \rho N(x_n, y_n)] \\
&\quad - m(w_{n-1}) - P_{K(w_{n-1})}[g(u_{n-1}) - m(w_{n-1}) - \rho N(x_{n-1}, y_{n-1})]\| \\
&\le \|m(w_n) - m(w_{n-1})\| + \|P_{K(w_n)}[g(u_n) - m(w_n) - \rho N(x_n, y_n)] \\
&\quad - P_{K(w_n)}[g(u_{n-1}) - m(w_{n-1}) - \rho N(x_{n-1}, y_{n-1})]\| \\
&\quad + \|P_{K(w_n)}[g(u_{n-1}) - m(w_{n-1}) - \rho N(x_{n-1}, y_{n-1})] \\
&\quad - P_{K(w_{n-1})}[g(u_{n-1}) - m(w_{n-1}) - \rho N(x_{n-1}, y_{n-1})]\| \\
&\le 2\|m(w_n) - m(w_{n-1})\| + \|g(u_n) - g(u_{n-1}) \\
&\quad - \rho(N(x_n, y_n) - N(x_{n-1}, y_{n-1})\| + 2t\|w_n - w_{n-1}\| \\
&\le 2t\|w_n - w_{n-1}\| + \rho\|N(x_{n-1}, y_n) - N(x_{n-1}, y_{n-1})\| \\
&\quad + \|g(u_n) - g(u_{n-1}) - \rho(N(x_n, y_n) - N(x_{n-1}, y_n)\| \\
&\quad + 2t\|w_n - w_{n-1}\|.
\end{aligned}$$

Since $N(.,.)$ is Lipschitz continuous in the second argument and m is Lipschitz continuous, we have

$$\begin{aligned}
\|g(u_{n+1}) - g(u_n)\| &\le \|g(u_n) - g(u_{n-1}) - \rho(N(x_n, y_n) - N(x_{n-1}, y_n)\| \\
&\quad + \rho\xi\|y_n - y_{n-1}\| + 4t\|w_n - w_{n-1}\|.
\end{aligned} \tag{6.2.4}$$

By $\hat{\mathcal{H}}$−Lipschitz continuity of $\tilde{F}$ with respect g, we have

$$\|x_n - x_{n-1}\| \le (1 + n^{-1})\mu\|g(u_n) - g(u_{n-1})\|. \tag{6.2.5}$$

Since $N(.,.)$ is relaxed monotone with respect to g in the first argument, Lipschitz continuous in the first argument and using (6.2.4), we have

$$\begin{aligned}
&\|g(u_n) - g(u_{n-1}) - \rho(N(x_n, y_n) - N(x_{n-1}, y_n)\|^2 \\
= \; & \|g(u_n) - g(u_{n-1})\|^2 - 2\rho\langle N(x_n, y_n) - N(x_{n-1}, y_n), g(u_n) - g(u_{n-1})\rangle \\
& +\rho^2\|N(x_n, y_n) - N(x_{n-1}, y_n)\|^2 \\
\leq \; & \|g(u_n) - g(u_{n-1})\|^2 + 2\rho c\|g(u_n) - g(u_{n-1})\|^2 + \rho^2\beta^2\|x_n - x_{n-1}\|^2 \\
\leq \; & [1 + 2\rho c + \rho^2\beta^2(1 + n^{-1})\mu^2]\|g(u_n) - g(u_{n-1})\|^2. \qquad (6.2.6)
\end{aligned}$$

Further, since $\tilde{G}$ and $\tilde{A}$ are $\hat{\mathcal{H}}$−Lipschitz continuous with respect to g, we have

$$\|y_n - y_{n-1}\| \leq (1 + n^{-1})\hat{\mathcal{H}}(\tilde{G}u_n, \tilde{G}u_{n-1}) \leq (1 + n^{-1})\sigma\|g(u_n) - g(u_{n-1})\| \qquad (6.2.7)$$

and

$$\|w_n - w_{n-1}\| \leq (1 + n^{-1})\hat{\mathcal{H}}(\tilde{A}u_n, \tilde{A}u_{n-1}) \leq (1 + n^{-1})\gamma\|g(u_n) - g(u_{n-1})\|. \qquad (6.2.8)$$

From (6.2.4) and (6.2.6) - (6.2.8), it follows that

$$\|g(u_{n+1}) - g(u_n)\| \leq \theta_n\|g(u_n) - g(u_{n-1})\|, \qquad (6.2.9)$$

where

$$\theta_n = \sqrt{1 + 2\rho c + \rho^2\beta^2(1 + n^{-1})\mu^2} + \rho\xi(1 + n^{-1})\sigma + 4t(1 + n^{-1})\gamma$$

Then $\theta_n \to \theta$ as $n \to \infty$. From (6.2.2), we have $\theta < 1$. Hence $\theta_n < 1$ for n sufficiently large. Therefore (6.2.8) implies that $\{g(u_n)\}$ is a Cauchy sequence in $g(H)$. Since $g(H)$ is closed in H, there exists $u \in H$ such that

$$g(u_n) \to g(u), \text{ as } n \to \infty.$$

By (6.2.5), (6.2.7) and (6.2.8), it is easy to see that $\{x_n\}, \{y_n\}$ and $\{w_n\}$ are Cauchy sequences in H. Since H is complete, we may let $x_n \to x, y_n \to y, w_n \to w$ as $n \to \infty$. Further, we have

$$\begin{aligned}
d(x, \tilde{F}(u)) &= \inf\{\|x - z\| : z \in \tilde{F}(u)\} \\
&\leq \|x - x_n\| + d(x_n, \tilde{F}(u)) \\
&\leq \|x - x_n\| + \hat{H}(\tilde{F}(u_n), \tilde{F}(u)) \\
&\leq \|x - x_n\| + \mu\|g(u_n) - g(u)\| \longrightarrow 0
\end{aligned}$$

and hence $x \in \tilde{F}(u)$. Similarly $y \in \tilde{G}(u)$ and $w \in \tilde{A}(u)$.

Now we prove that

$$g(u) = m(w) + P_{K(w)}[g(u) - m(w) - \rho N(x,y)]$$

In fact, from Lemma 1.2.3 and (6.2.3), we have

$$\begin{aligned}
&\|g(u_{n+1}) - m(w) - P_{K(w)}[g(u) - m(w) - \rho N(x,y)]\| \\
&= \|m(w_n) + P_{K(w_n)}[g(u_n) - m(w_n) - \rho N(x_n,y_n)] \\
&\quad -m(w) - P_{K(w)}[g(u) - m(w) - \rho N(x,y)]\| \\
&\leq \|m(w_n) - m(w)\| + \|P_{K(w_n)}[g(u_n) - m(w_n) - \rho N(x_n,y_n)] \\
&\quad -P_{K(w_n)}[g(u) - m(w) - \rho N(x,y)]\| \\
&\quad +\|P_{K(w_n)}[g(u) - m(w) - \rho N(x,y)] \\
&\quad -P_{K(w)}[g(u) - m(w) - \rho N(x,y)]\| \\
&\leq 2\|m(w_n) - m(w)\| + \|g(u_n) - g(u) - \rho(N(x_n,y_n) - N(x,y))\| \\
&\quad +2t\|w_n - w\| \\
&\leq \|g(u_n) - g(u)\| + \rho\|(N(x_n,y_n) - N(x,y_n))\| \\
&\quad +\rho\|N(x,y_n) - N(x,y)\| + 4t\|w_n - w\| \\
&\leq \|g(u_n) - g(u)\| + \rho\beta\|x_n - x\| + \rho\xi\|y_n - y\| + 4t\|w_n - w\|.
\end{aligned}$$

Since $g(u_n) \to g(u)$, $x_n \to x$, $y_n \to y$, $w_n \to w$, we have

$$g(u_{n+1}) \to m(w) + P_{K(w)}[g(u) - m(w) - \rho N(x,y)],$$

and therefore

$$g(u) = m(w) + P_{K(w)}[g(u) - m(w) - \rho N(x,y)]$$

This completes the proof. □