CHAPTER I
INTRODUCTION

The purpose of this chapter is to give a brief resume of the results in the geometry of submanifolds, Homology, Morse theory and Total absolute curvature. Muchthough all these results are readily available in review articles and some even in standard books e.g. K. Nomizu and S. Kobayashi [17], S.T. Hu [19], J. Milnor [25], T.E. Cecil and P.J. Ryan [30] and many others, nevertheless we have collected them here to fix up our terminology. Only those definitions and results have been given which are relevant in the subsequent chapters.

1. GEOMETRY OF SUBMANIFOLDS:

To study the geometry of submanifolds, sometimes, it becomes more convenient to first embed it into a manifold of which the geometry is known and then study the geometry which is induced on it.

Let $\bar{M}$ be an $(n+N)$-dimensional Riemannian manifold. $M$ is said to be a Submanifold of $\bar{M}$, if there exists an immersion $F : M \rightarrow \bar{M}$. For the submanifold $M$ of $\bar{M}$, for convenience we denote $F(x) \in \bar{M}$ by the same letter $x$. 
Let $T(M)$ and $T(M)$ denote the tangent bundle of $M$ and $\bar{M}$ respectively. $F$ is said to be an isometric immersion if the differential map $F_* : T(M) \to T(\bar{M})$ preserves the metric i.e. for $X, Y \in T(M)$

$$g(F_*X, F_*Y) = g(X, Y)$$

where we denote by the same letter $g$ the Riemannian metric on $M$ and $\bar{M}$ respectively. For local calculations, we identify $T(M)$ and $F_*(T(M))$ through this isomorphism. Hence a tangent vector in $T(\bar{M})$ tangent to $M$ shall mean a tangent vector which is the image of an element in $T(M)$ under $F_*$. Those tangent vectors of $T(\bar{M})$ which are normal to $M$ form the normal bundle $\nu(M)$. The Riemannian connexion $\bar{\nabla}$ on $\bar{M}$ induces Riemannian connexions $\nabla$ and $\nabla^\perp$ in the tangent bundle of $M$ and the normal bundle $\nu(M)$ respectively and they are related by Gauss and Weingarten formulae.

$$\nabla_X Y = \nabla_X Y + T(X, Y)$$

$$\nabla_X N = -S_N X + \nabla^\perp_X N.$$
For $X, Y \in T(M)$ and $N \in \nu(M)$, $\nabla_X Y$ (resp. $-S_N X$) and $T(X, Y)$ (resp. $\nabla_X N$) are tangential and normal component of $\nabla_X Y$ (resp. $\nabla_X N$) where $T(X, Y)$ is a normal valued symmetric tensor called the Shape operator, $S_N X$ is a symmetric $(1,1)$ tensor on $M$ and they are related by

$$g(S_N X, Y) = g(T(X, Y), N)$$

If we denote the curvature tensor corresponding to $\nabla^\perp$, $\nabla$ and $\nabla^\parallel$ by $\overline{R}$, $R$ and $\overline{R}$ respectively, then the Gauss, Coddazi and Ricci Equations are given by [17]

$$\overline{R}(X, Y; Z, W) = R(X, Y; Z, W) + g(T(X, Z), T(X, W)) - g(T(Y, Z), T(X, W))$$

$$[\overline{R}(X, Y) Z]^\perp = (\nabla_X T)(Y, Z) - (\nabla_Y T)(X, Z)$$

$$\overline{R}(X, Y; N_1, N_2) = \overline{R}(X, Y; N_1, N_2) - g([S_{N_1}, S_{N_2}] X, Y).$$

Where in (1.1.6) $[\overline{R}]$ denotes the normal component of $[\overline{R}]$ and

$$(\nabla_X T)(Y, Z) = \overline{\nabla}_X T(Y, Z) - T(\overline{\nabla}_X Y, Z) - T(Y, \overline{\nabla}_X Z)$$

For each plane $p$ in the tangent space $T_x(M)$, the **Sectional Curvature** $K(p)$ for $p$ is defined by,
(1.1.9) \[ K(p) = R(X_1, X_2; X_2, X_1) = g(R(X_1, X_2)X_2, X_1). \]

Where \( \{X_1, X_2\} \) is the orthonormal basis for \( p \). This can be seen that \( K(p) \) is independent of the choice of an orthonormal basis \( \{X_1, X_2\} \). If \( K(p) \) is constant for all planes \( p \) in \( T_x(M) \) and for all points \( x \in M \), then \( M \) is called a **Space of constant curvature**. The following theorem is due to F. Schur [17].

**Theorem (1.1.1):** Let \( M \) be a connected Riemannian manifold of dimension \( \geq 3 \). If the sectional curvature \( K(p) \), where \( p \) is a plane in \( T_x(M) \), depends only on \( x \), then \( M \) is a space of constant curvature.

**Corollary (1.1.1):** For a space of constant curvature \( k \), we have,

(1.1.10) \[ R(X, Y)Z = k(g(Z, Y)X - g(Z, X)Y) \]

From (1.1.5) it follows that the sectional curvature \( \bar{K} \) and \( K \) of \( \bar{M} \) and \( M \) of plane section spanned by vectors \( X \) and \( Y \) satisfy

(1.1.11) \[ \bar{K}(X, Y) = K(X, Y) + g(T(X, Y), T(X, Y)) - g(T(X, X), T(Y, Y)). \]
Since $S^N$ is symmetric, there exists an orthonormal local frame $(e_1, \ldots, e_n)$ on $M$ such that $S_N e_i = \lambda_i e_i$
where $\lambda_i$ depends on $M$. Now let us consider the local formalism. For this we choose a local field of orthonormal frame $e_1, e_2, \ldots, e_{n+N}$ in $\bar{M}$ such that restricted to $M$, the vectors $e_1, e_2, \ldots, e_n$ are tangential to $M$ and the rest of the $N$ vectors $e_{n+1}, \ldots, e_{n+N}$ are normal to $M$. We shall make use of the following convention of the ranges of indices

1 $\leq A, B, C, \ldots, \leq n+N$
1 $\leq i, j, k, \ldots, \leq n$
n+1 $\leq r, s, t, \ldots, \leq n+N$. 

The structure equation of $\bar{M}$ are given by

\begin{align*}
(1.1.12) \quad d\omega_A &= -\Sigma_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0 \\
(1.1.13) \quad d\omega_{AB} &= -\Sigma_C \omega_{AC} \wedge \omega_{CB} + \Omega_{AB}
\end{align*}

where $-\Omega_{AB} = \frac{1}{2} \Sigma_{C,D} \bar{R}_{ABCD} \omega_C \wedge \omega_D$

where $\omega_1, \ldots, \omega_{n+N}$ is the dual frame to $e_1, \ldots, e_{n+N}$ and $\omega_{AB}$ are the connexion forms. If we restrict these forms to $M$, $\omega_r = 0$ and (1.1.12) gives
Using Cartan's lemma, we write
\[ \omega_{ri} = \sum \Lambda_{ij}^r \omega_j; \quad \Lambda_{ij}^r = \Lambda_{ji}^r. \]

From these formulas, we obtain
\[ d\omega_i = -\sum \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0 \]
\[ d\omega_{ij} = -\sum \omega_{ik} \wedge \omega_{kj} + \Omega_{ij} \]
where \[ \Omega_{ij} = \frac{1}{2} R_{ijkl} \omega_k \wedge \omega_l. \]

2. HOMOLOGY:

Let \( \mathbb{R}^n \) be an \( n \)-dimensional Euclidean space consider the following set
\[ \Delta_n = \{ (x_1, \ldots, x_n) \in \mathbb{R}^n/ \sum_{i=1}^n x_i \leq 1, \ x_i \geq 0 \} \]

Clearly \( \Delta_n \) is a compact subset of \( \mathbb{R}^n \), is called an \( n \)-simplex. \((0,0,\ldots,0) \in \Delta_n \) is called the 0-th vertex and the points \( v_i = (0, \ldots, 0, 1, 0, \ldots) \) with 1 at the i-th place is called the i-th vertex. The closed subspace
\[ \Delta_n^0 = \{ (x_1, x_2, \ldots, x_n) \in \Delta_n/ \sum_{i=1}^n x_i = 1 \} \]
is called the 0-th face of $\Delta_n$ and 

$$\Delta_n^{(i)} = \{(x_1, x_2, \ldots, x_n) \in \Delta_n : x_i = 0\}$$

is called the $i$-th face of $\Delta_n$. Clearly $v_i \not\in \Delta_n^{(i)}$. For completeness, we define $\Delta_0 = \{0\}$. Assume $n > 0$ and define the map

$$k_i : \Delta_{n-1} \longrightarrow \Delta_n$$

by $k_i(x_1, x_2, \ldots, x_{n-1}) = (1 - \sum_{i=1}^{n-1} x_i, x_1, \ldots, x_{n-1})$

and $k_i(x_1, x_2, \ldots, x_{n-1}) = (x_1, \ldots, x_{i-1}, 0, x_i, \ldots, x_{n-1})$.

It is easy to verify that

$$k_i : \Delta_{n-1} \longrightarrow \Delta_n$$

is an embedding and 

$$k_i(\Delta_{n-1}) = \Delta_n^{(i)}.$$

Next assume $n > 1$ and consider the map

$$\Delta_{n-2} \xrightarrow{k_i} \Delta_{n-1} \xrightarrow{k_i} \Delta_n$$

Then one can prove, $k_i \circ k_j = k_j \circ k_{i-1}$.

**DEFINITION (1.2.1):** Let $X$ be a topological space. A continuous map

$$\sigma : \Delta_n \longrightarrow X$$
is called a **Singular p-simplex**.

Since $\Delta$ is fixed for each $p$, the function $\sigma$ carries the same information as does the set $\sigma(\Delta_p)$

Let $S_n(X)$ be the set of all n-Singular Simplexes.

Clearly for $m \neq n$,

$$S_m(X) \cap S_n(X) = \emptyset$$

Assume $n > 1$ and let $\sigma \in S_n(X)$. For $i = 0, 1, \ldots, n$,

Consider the composition

$$\sigma \circ k_i : \Delta_{n-1} \rightarrow X$$

Clearly $\sigma \circ k_i \in S_{n-1}(X)$. We put $\sigma \circ k_i = \sigma(i)$ which is called the $i$-th face of $\sigma$. We have clearly,

$$[\sigma(i)]^{(j)} = [\sigma(j)]^{(i-1)}$$

for $n > 1$

for all non negative integers $n$, let $C_n(X)$ be the free abelian group generated by $S_n(X)$. Now we define

$$\partial : C_n(X) \rightarrow C_{n-1}(X)$$

by

$$\partial(\sigma) = \sum_{i=1}^{n} (-1)^i \sigma(i).$$

For completeness, we define

$$C_n(X) = 0, \quad n < 0.$$
Then $\delta$ defines a homomorphism, and

$$\delta \circ \delta = 0$$

Thus we obtain a semi exact sequence

$$\ldots \delta \rightarrow C_{n+1}(X) \delta \rightarrow C_n(X) \delta \rightarrow C_{n-1}(X) \delta \rightarrow \ldots$$

The elements $\gamma \in C_n(X)$ are called Singular Chain and $\delta \gamma \in C_n(X)$ is called the boundary of chain $\gamma$. Define

$$Z_n(X) = \{\ker \delta/\delta : C_n(X) \rightarrow C_{n-1}(X)\}$$

and

$$B_n(X) = \{\text{Im} \delta/\delta : C_{n+1}(X) \rightarrow C_n(X)\}$$

The elements of $Z_n$ are called Cycles and that of $B_n$ are called boundaries. We define

$$H_n(X) = \frac{Z_n(X)}{B_n(X)}$$

**Definition (1.2.2):** $H_n(X)$ is called the $n$-dimensional Singular homology group.

**Definition (1.2.3):** Let $C^n(X, R) = \text{Hom}(C_n(X), R)$. The elements of $C^n(X, R)$ are called real singular cochains.

For $n < 0$, we have

$$C^n(X, R) = 0,$$

as $C_n(X) = 0$. 
For every $n$, define homomorphism

$$\delta : C^n(X, R) \longrightarrow C^{n+1}(X, R)$$

as follows:

If $n < 0$, $C^n(X, R) = 0$ and $\delta = 0$.

Assume $n \geq 0$, let $\phi \in C^n(X, R)$

$$\delta(\phi) = \phi \circ \delta : C_{n+1}(X) \longrightarrow R$$

Clearly $\delta$ is a homomorphism. We call $\delta$ the coboundary operator.

**Lemma (1.2.1):** $\delta \circ \delta = 0$.

As in the previous case, we define $Z^n(X, R)$ to be the Ker $\delta$ and $B^n(X, R)$ as the image of $\delta$ and call them cocycles and coboundaries respectively.

**Definition (1.2.4):** $H^n(X, R) = Z^n(X, R)/B^n(X, R)$

is called the $n$-dimensional real Singular Cohomology group.

**Cohomology of Forms:** Let $M$ be an $n$-dimensional differentiable manifold. Then we know the following algebra of differential forms

$$C(M) = \sum_{k=0}^{n} C^k(M).$$
Where \( C^k(M) \) is the set of differential \( k \)-forms and is associated with \( M \), called the Exterior algebra of differential forms. There is a unique exterior differential operator

\[
d : C(M) \rightarrow C(M)
\]

which maps a \( k \)-form into a \((k+1)\)-form and satisfies \( d \circ d = 0 \). Also, \( C^k(M) = 0 \), if \( k > n \) and \( k < 0 \) and hence we have the following semi exact sequence:

\[
\cdots \xrightarrow{d} C^{k-1}(M) \xrightarrow{d} C^k(M) \xrightarrow{d} C^{k+1}(M) \xrightarrow{d} \cdots
\]

let \( Z^k(M) = \{ \eta \in C^k(M) / d\eta = 0 \} \)

\[
= \{ \text{Ker} \ d / d : C^k(M) \rightarrow C^{k+1}(M) \}.
\]

i.e. set of all closed forms. Define

\[
B^k(M) = d(C^{k-1}(M))
\]

\[
= \{ d\eta : \eta \in C^{k-1}(M) \}
\]

Now as before we have the following lemma

**Lemma** (1.2.2): For each \( k \), \( B^k(M) \subseteq Z^k(M) \). Infact it is a linear subspace.
**DEFINITION (1.2.5):** $H^k(M) = Z^k(M)/B^k(M)$ called the $k$-dimensional De-Rham cohomology group of differentiable manifold $M$.

**REMARK:**

1. $H^0(X) = Z^0(X)/B^0(X) = Z^0(X)$
2. $H^n(X) = C^n(X)/B^n(X)$
3. $H^{n+1}(X) = 0$.

$H^k(M)$ consist of equivalence classes of the type $[\eta]$, where $\eta$ is a $k$-form which is closed and not exact. Also we observe that

$$\xi \in [\eta]$$

if $\xi = \eta + d\alpha$ for some $(k-1)$-form $\alpha$

i.e. $\xi - \eta$ is cohomologous to zero.

From Poincare's lemma it is known that every closed form in $\mathbb{R}^n$ is exact. Therefore we have,

**LEMMA (1.2.3):** $H^k(\mathbb{R}^n) = 0$

**LEMMA (1.2.4):** If $[\xi] \in H^k(M)$ and $[\eta] \in H^l(M)$, then $[\xi \wedge \eta] \in H^{k+l}(M)$.

**LEMMA (1.2.5):** For sphere $S^n$, we have,

$$H^0(S^n) = H^n(S^n) \cong \mathbb{R} \quad \text{and} \quad H^p(S^n) = 0, \quad 1 \leq p \leq n-1$$
Now we can state the famous De-Rham's theorem.

**THEOREM (1.2.1):** For every integer $k$, the De-Rham's Cohomology group $H^k(M)$ is isomorphic to $H^k(M, \mathbb{R})$.

**DEFINITION (1.2.6):** $\dim H^k(M)$ is defined to be $b_k(M)$, the $k$-th Betti number of the manifold, and

$$
\chi(M) = \sum_{k=0}^{n} (-1)^k b_k(M)
$$

is called the Euler Characteristic.

3. **MORSE THEORY:**

The underlying idea of Morse theory is that a considerable knowledge of the topological properties of a manifold can be obtained by studying the number and the nature of the critical points of a real valued function defined over $M$.

**DEFINITION (1.3.1):** A smooth function $\phi$ defined over a manifold $M$ has a critical point at $x$ if $\phi$ vanishes at $x$. In terms of local coordinates $(x^i)$ this means that at $x$,

$$
\left(\frac{\partial \phi}{\partial x^i}\right) = 0, \quad \text{for all } i
$$

**DEFINITION (1.3.2):** Let $\phi : M \to \mathbb{R}$ be a differentiable function. The **hessian** of $\phi$ at $x \in M$ is an $(n \times n)$-matrix
$$\left(\frac{\partial^2 \phi}{\partial x^i \partial x^j}\right)$$ where \((x^1, x^2, \ldots, x^n)\) are local coordinates at \(x\).

**DEFINITION (1.3.3):** Let \(\phi : M \rightarrow \mathbb{R}\) be a differentiable function. A critical point \(x\) of \(\phi\) is said to be non-degenerate if the hessian of \(\phi\) at \(x\) is non-singular otherwise the critical point is said to be degenerate.

**REMARK:** The degeneracy or non-degeneracy is independent of the local coordinates system around the critical point.

**DEFINITION (1.3.4):** Let \(x \in M\) be the non-degenerate critical point of the function \(\phi : M \rightarrow \mathbb{R}\). Then the maximal dimension of the subspace of \(T_x(M)\) on which the hessian of \(\phi\) is negative definite is called the index of \(x\).

The behaviour of \(\phi\) in a neighbourhood of a non-degenerate critical point is determined by the index as follows:

**LEMMA (1.3.1):** Let \(x \in M\) be a non-degenerate critical point of \(\phi : M \rightarrow \mathbb{R}\) of index \(k\). Then there is a local coordinate system \(x^1, x^2, \ldots, x^n\) in a neighbourhood \(U\) with origin at \(x\) such that the identity

\[
\phi = \phi(x) - (x^1)^2 - (x^2)^2 - \ldots - (x^k)^2 + (x^{k+1})^2 + \ldots + (x^n)^2
\]

holds throughout \(U\).
**Lemma (1.3.2):** The non-degenerate critical point of a function are isolated. Thus a function with compact domain has only finitely many non-degenerate critical points provided it has no degenerate critical point.

**Definition (1.3.5):** A function \( \phi \) which has only non-degenerate critical points is called a non-degenerate function or a Morse function.

Let \( \mu_k(\phi; M) \) be the number of critical points of \( \phi \) of index \( k \). Let

\[
\mu(\phi; M) = \sum_{k=0}^{n} \mu_k(\phi; M), \quad \bar{\mu}(\phi; M) = \sum_{k=1}^{n-1} \mu_k(\phi; M)
\]

\[
\mu_k(M) = \inf \{ \mu_k(\phi; M)/\phi : M \to \mathbb{R} \}
\]

and \( \mu(M) = \inf \{ \mu(\phi; M)/\phi : M \to \mathbb{R} \}, \quad \bar{\mu}(M) = \inf \{ \bar{\mu}(\phi; M)/\phi : M \to \mathbb{R} \} \).

where infimum is taken over all the non-degenerate functions on \( M \).

The Morse inequalities which hold for a non-degenerate function \( \phi \) on a compact manifold \( M \) give

\[
\mu_k(\phi; M) \geq \mu_k(M) \geq b_k(M, F).
\]

Also \( \mu(\phi; M) \geq \mu(M) \geq \sum_{i} \mu_i(M) \).
4. **TOTAL ABSOLUTE CURVATURE:**

The theory of total absolute curvature can be considered to have been originated in 1929 with a paper of W. Fenchel [12]. In that paper he proved that for a closed space curve \( C \) in \( \mathbb{R}^3 \) of class \( C^3 \)

\[
\frac{1}{\pi} \int_C |k(S)| \, ds \geq 2.
\]

(1.4.1)

Where \( k(S) \) is the ordinary curvature of the curve \( C \).

Equality holds if and only if \( C \) is a plane convex curve. This idea was further generalized by K. Borsuk for a curve in \( \mathbb{R}^n \), \( n \geq 3 \).

Now before coming to the general case i.e. the total absolute curvature of a compact manifold in Euclidean space, let us consider the Gauss map of \( M^n \) immersed in \( \mathbb{R}^{n+N} \). In the classical theory of surfaces \( \Sigma \) in \( \mathbb{R}^3 \), the Gauss map is defined by

\[
\nu : \Sigma \longrightarrow S^2_0
\]

Such that \( \nu (p) = P^* \)

Where the unit normal to the surface \( \Sigma \) at \( P \) is parallel to the position vector \( P^* \) of the sphere \( S^2_0 \) (i.e. the unit vector originating from origin and with end point as \( P^* \))
The idea of Gauss map and Gaussian curvature is generalized to differentiably immersed submanifolds of Euclidean spaces by S.S. Chern [9] and thereby gave the notion of total absolute curvature. Let

\[ F : M \rightarrow \mathbb{R}^{n+N} \]

be an immersion of an \( n \)-dimensional \( C^\infty \)-manifold into \( (n+N) \)-dimensional Euclidean space. The immersion \( F \) gives rise to a unit normal bundle \( \gamma(M) \) over \( M \) whose bundle space consists of all the pairs \( (p,e) \) where \( p \in M \) and \( e \) is the unit normal vector at \( F(p) \). If \( S_{0}^{n+N-1} \) is the unit sphere in \( \mathbb{R}^{n+N} \) centred at origin, we define the Gauss map \( \nu(M) \) as follows:

\[ \nu : \nu(M) \rightarrow S_{0}^{n+N-1} \]

Such that \( \nu(p,e) = e_{o} \)

that is under Gauss map every unit normal vector at \( F(p) \) is mapped to the end point of a unit vector through origin parallel to it.

If \( dy \) is the volume element of \( M \) and \( d\sigma \) be \((N-1)\)-differential form on \( \nu(M) \) such that its restriction to each
fibre is the volume element of $v(M)$. Let $d\Sigma$ be the volume element of $S^{n+N-1}_0$, then we define $G(p,e)$ on $v(M)$ by

$$(1.4.2) \quad v^* d\Sigma = G(p,e) dv \wedge d\sigma.$$ 

Where $v^*$ is the dual mapping on differential forms induced by $v$. $G(p,e)$ is called the Lipshitz Killing Curvature.

**Definition (1.4.1):** The total absolute curvature $K^*(p)$ at $p \in M$ of the immersion $F : M^n \rightarrow R^{n+N}$ is defined by

$$(1.4.3) \quad K^*(p) = \frac{1}{|S(1)|} \int_{\Sigma \in v(M)} |G(p,e)| \ d\sigma.$$ 

Where $|S(1)|$ denotes the volume of $(n+N-1)$-dimensional unit sphere. The total absolute curvature of $M$ is defined by

$$(1.4.4) \quad \tau(M,F) = \int_M K^*(p) \ dv.$$ 

In the case of a 2-dimensional (orientable) closed surfaces $M^2$ immersed in $R^3$, the Lipschitz curvature becomes the Gaussian curvature $K$ and $(1.4.4)$ becomes

$$\tau(M^2,F) = \frac{1}{2\pi} \int_M |K| ds.$$ 

The total absolute curvature of the immersion remains unchanged if $F(M^n)$ is considered as a submanifold of a higher
dimensional Euclidean space which contains $\mathbb{R}^{n+N}$ as a linear subspace.

Following are the main results on total curvature by S.S. Chern and R.K. Lashoff [9].

**THEOREM (1.4.1):** Let $M^n$ be a compact, oriented $C^\infty$-manifold immersed in $\mathbb{R}^{n+N}$. Its total curvature satisfies the inequality

$$\frac{1}{n+N-1} \int_M K^*(p) \, dv \geq 2.$$ 

**THEOREM (1.4.2):** Under the hypothesis of the theorem (1.4.1), if

$$\frac{1}{n+N-1} \int_M K^*(p) \, dv < 3$$

then $M^n$ is homeomorphic to a sphere of $n$-dimension.

**THEOREM (1.4.3):** Under the same hypothesis, if

$$\frac{1}{n+N-1} \int_M K^*(p) \, dv = 2,$$

then $M^n$ belongs to a linear subvariety $\mathbb{R}^{n+1}$ of dimension $n+1$, and is imbedded as a convex hypersurface in $\mathbb{R}^{n+1}$. The converse of this is also true.