CHAPTER 8

AUXILIARY PROBLEM AND ALGORITHM FOR A GENERALIZED MIXED EQUILIBRIUM PROBLEM

8.1. INTRODUCTION

One of the most important and interesting problems in the theories of equilibrium problems and variational inequalities is to develop the methods which give efficient and implementable algorithms for solving equilibrium problems and variational inequalities. These methods include projection method and its variant forms, linear approximation, descent and Newton’s methods, and the method based on auxiliary principle technique.

It is well known that the projection method and its variants can not be extended for mixed equilibrium problems involving non-differentiable term. To overcome this drawback, one uses usually the auxiliary principle technique. This technique deals with finding a suitable auxiliary problem and prove that the solution of an auxiliary problem is the solution of original problem by using fixed-point approach. Glowinski, Lions and Tremoliers [72] used this technique to study the existence of solutions of mixed variational inequalities. Noor [115-117,121], Huang and Deng [74], Chidume et al. [40], Zeng et al. [142], Kazmi [82] and Kazmi and Khan [91] extended this technique to suggest and analyze a number of iterative methods for solving various classes of variational inequalities and equilibrium problems.

Motivated by recent work going in this direction, in this chapter, we extend auxiliary principle technique to a generalized mixed equilibrium problem (for short, GMEP) in Hilbert space. We prove existence of the unique solution of an auxiliary problem related to GMEP, which enable us to construct an algorithm for finding the approximate solution of GMEP. Further we prove that the approximate solution is strongly convergent to the unique solution of GMEP. The algorithms and results of this chapter are new and different from the algorithms and results of Noor [121]. The results presented here generalize the techniques and results of [82,115,116].

The remaining part of this chapter is organized as follows:
In Section 8.2, we consider GMEP in Hilbert space and discuss some of its special cases. In Section 8.3, we consider an auxiliary problem related to GMEP and give some concepts. Further, we establish an existence and uniqueness theorem for the auxiliary problem. Furthermore, using this theorem we construct an algorithm for GMEP. In Section 8.4, we discuss the convergence analysis of the algorithm and existence of solution of GMEP.

8.2. PRELIMINARIES

Let $H$ be a real Hilbert space whose inner product and norm are denoted by $(\cdot, \cdot)$ and $\|\cdot\|$, respectively, and let $K$ be nonempty, closed and convex subset of $H$. Given the single-valued mappings $T, S : H \to H$, $N, n : H \times H \to H$ and a bifunction $f : H \times H \to \mathbb{R}$ such that $f(x, x) = 0 \ \forall x \in H$, then we consider the generalized mixed equilibrium problem (GMEP) of finding $x \in K$ such that

$$f(x, y) + \langle N(Tx, Sx), \eta(y, x) \rangle + b(x, y) - b(x, x) \geq 0, \ \forall y \in K,$$

(8.2.1)

where the bifunction $b : H \times H \to \mathbb{R}$, which is not necessarily differentiable, satisfies the following properties:

(i) $b$ is linear in the first argument;

(ii) $b$ is bounded, that is, there exists a constant $\gamma > 0$ such that

$$b(x, y) \leq \gamma \|x\| \|y\|, \ \forall x, y \in H;$$

(iii) $b(x, y) - b(x, z) \leq b(x, y - z), \ \forall x, y, z \in H;$

(iv) $b$ is convex in the second argument.

Some special cases:

(I) If $f(x, y) = 0; b(x, y) = 0$ and $N(Tx, Sx) = B(x) \ \forall x, y \in K$, where $B : K \to H$, then GMEP (8.2.1) reduces to the variational-like inequality problem of finding $x \in K$ such that

$$\langle Bx, \eta(y, x) \rangle \geq 0, \ \forall y \in K.$$

(8.2.2)

This problem has been studied by Parida et al. [125].
(II) If $N(Tx, Sx) = 0 \forall x \in K$, then GMEP (8.2.1) reduces to the equilibrium problem of finding $x \in K$ such that

$$f(x, y) + b(x, y) - b(x, x) \geq 0, \forall y \in K. \quad (8.2.3)$$

This problem has been studied by Noor [121].

(III) If $N(Tx, Sx) = Bx$, $B : K \to H; b(x, y) = \phi(y) - \phi(x) \forall x, y \in K$, where $\phi : K \to \mathbb{R}$ and $f(x, x) = 0 \forall x \in K$, then GMEP (8.2.1) reduces to the problem of finding $x \in K$ such that

$$\langle Bx, \eta(y, x) \rangle + \phi(y) - \phi(x) \geq 0, \forall y \in K. \quad (8.2.4)$$

This problem has been studied by Dien [48] in $\mathbb{R}^n$.

(IV) If, in (III), $\eta(y, x) = y - x \forall x, y \in K$, then GMEP (8.2.1) reduces to the variational inequality problem of finding $x \in K$ such that

$$\langle Bx, y - x \rangle + \phi(y) - \phi(x) \geq 0, \forall y \in K. \quad (8.2.5)$$

This problem has been studied by Cohen [43].

8.3. AUXILIARY PROBLEM AND EXISTENCE OF SOLUTIONS

First related to GMEP (8.2.1), we consider the auxiliary problem and then establish an existence theorem for the auxiliary problem:

Auxiliary problem (AP). Given $x \in K$, find $z \in K$ such that

$$\rho f(z, y) + \langle Az - Ax + \rho N(Tx, Sx), \eta(y, z) \rangle + \rho [b(x, y) - b(x, z)] \geq 0, \forall y \in K, \quad (8.3.1)$$

where $\rho > 0$ is a constant and $A : K \to H$ is not necessarily a linear mapping.

We observe that if $z = x$, clearly $z$ is a solution of GMEP (8.2.1).

Now, we define the following concepts:

Definition 8.3.1. Let $f : K \times K \to \mathbb{R}; N : H \times H \to H; T, S : K \to K$ and $\eta : H \times H \to H$. Then:
(a) $T$ is said to be $\alpha$-strongly monotone if there exists a constant $\alpha > 0$ such that

$$f(x, y) + f(y, x) + \alpha \|x - y\|^2 \leq 0, \ \forall x, y \in H;$$

(b) $\eta$ is said to be $\delta$-Lipschitz continuous if there exists a constant $\delta > 0$ such that

$$\|\eta(x, y)\| \leq \delta \|x - y\|, \ \forall x, y \in H;$$

(c) $A$ is said to be $\tau$-strongly $\eta$-monotone if there exists a constant $\tau > 0$ such that

$$\langle Ax - Ay, \eta(x, y) \rangle \geq \tau \|x - y\|^2, \ \forall x, y \in H;$$

(d) $N$ is said to be $\epsilon$-strongly mixed $\eta$-monotone with respect to $T$ and $S$, if there exists a constant $\epsilon > 0$ such that

$$\langle N(Tx, Sx) - N(Ty, Sy), \eta(x, y) \rangle \geq \epsilon \|x - y\|^2, \ \forall x, y \in H;$$

(e) $N$ is said to be $(\beta_1, \beta_2)$-Lipschitz continuous if there exist constants $\beta_1, \beta_2 > 0$ such that

$$\|N(x_1, y_1) - N(x_2, y_2)\| \leq \beta_1 \|x_1 - x_2\| + \beta_2 \|y_1 - y_2\|, \ \forall x_1, x_2, y_1, y_2 \in H;$$

(f) $f$ and $A$ are said to be simultaneously hemicontinuous if for $\lambda \in [0, 1]$, $y_\lambda := \lambda y + (1 - \lambda)z$, $y, z \in K$, we have

$$f(y_\lambda, p) + \langle A(y_\lambda), p \rangle \rightarrow F(z, p) + \langle A(z), p \rangle$$

as $\lambda \rightarrow 0^+$ for any $p \in K$.

**Theorem 8.3.1.** Let $K$ be a nonempty, closed and convex subset in $H$. Let $\eta : K \times K \rightarrow K$ be affine in the first argument and continuous in the second argument such that $\eta(y, x) + \eta(x, y) = 0 \ \forall x, y \in K$; let $b : K \times K \rightarrow \mathbb{R}$ be convex in the second argument and continuous; let $f : K \times K \rightarrow \mathbb{R}$ be convex and lower semicontinuous in the second argument and $f(x, x) = 0 \ \forall x \in K$; let $A : K \rightarrow H$ be $\eta$-monotone and let $f$ and $A$ be simultaneously hemicontinuous. If there exists a
nonempty compact subset $D$ of $H$ and $z_0 \in D \cap K$ such that for any $z \in K \setminus D$, we have

$$\rho f(z_0, z) + \langle Az - Ax + \rho N(Tx, Sx), \eta(z_0, z) \rangle + \rho [b(x, z_0) - b(x, z)] < 0, \quad (8.3.2)$$

for given $x \in K$. Then AP (8.3.1) has a solution. Moreover, in addition, if $A$ is $\tau$-strongly $\eta$-monotone then the solution is unique.

**Proof.** Define the set-valued mappings $P, Q : K \to 2^K$ as

$$P(y) = \{z \in K : \rho f(z, y) + \langle Az - Ax + \rho N(Tx, Sx), \eta(y, z) \rangle + \rho [b(x, y) - b(x, z)] \geq 0\}, \quad (8.3.3)$$

and

$$Q(y) = \{z \in K : -\rho f(y, z) + \langle Ay - Ax + \rho N(Tx, Sx), \eta(y, z) \rangle + \rho [b(x, y) - b(x, z)] \geq 0\}, \quad (8.3.4)$$

for $y \in K$, respectively.

We claim that $P$ is a KKM-mapping. Indeed, let $\{z_1, z_2, ..., z_m\}$ be a finite subset of $K$ and let $\lambda_i \geq 0$, $1 \leq i \leq m$ with $\sum_{i=1}^{m} \lambda_i = 1$. Suppose that $z = \sum_{i=1}^{m} \lambda_i z_i \notin \bigcup_{i=1}^{m} P(z_i)$. Then

$$\rho f(z, z_i) + \langle Az - Ax + \rho N(Tx, Sx), \eta(z_i, z) \rangle + \rho [b(x, z_i) - b(x, z)] < 0, \quad \forall i.$$

Since $f$ and $b$ is convex in the second argument and $\eta$ is affine in the first argument, using above inequality we have

$$0 = \rho f(z, z_i) + \langle Az - Ax + \rho N(Tx, Sx), \eta(z_i, z) \rangle + \rho [b(x, z_i) - b(x, z)]$$

$$= \rho f(z, \sum_{i=1}^{m} \lambda_i z_i) + \langle Az - Ax + \rho N(Tx, Sx), \eta(\sum_{i=1}^{m} \lambda_i z_i, z) \rangle$$

$$+ \rho [b(x, \sum_{i=1}^{m} \lambda_i z_i) - \sum_{i=1}^{m} \lambda_i b(x, z_i)]$$

$$\leq \sum_{i=1}^{m} \lambda_i [\rho f(z, z_i) + \langle Az - Ax + \rho N(Tx, Sx), \eta(z_i, z) \rangle + \rho (b(x, z_i) - b(x, z))]$$

$$< 0,$$

which is absurd. Thus $z \in \bigcup_{i=1}^{m} P(z_i)$. Since $z$ was an arbitrary element of $\text{Co}\{z_1, z_2, ..., z_m\}$,
hence \( \text{Co}\{z_1, z_2, \ldots, z_m\} \subset \bigcup_{i=1}^{m} P(z_i) \). Thus \( P \) is a KKM mapping. Now, we claim that \( P(y) \subset Q(y) \) for every \( y \in K \). Indeed, let \( z \in P(y) \), we have

\[
\rho f(z, y) + \langle A z, \eta(y, z) \rangle \geq \langle A x - \rho N(Tx, Sx), \eta(y, z) \rangle - \rho [b(x, y) - b(x, z)]. \tag{8.3.5}
\]

Since \( f \) and \( A \) are monotone, then we have

\[
-\rho f(y, z) + \langle A y, \eta(y, z) \rangle \geq \rho f(z, y) + \langle A z, \eta(y, z) \rangle. \tag{8.3.6}
\]

From (8.3.5) and (8.3.6), we have

\[
-\rho f(y, z) + \langle A y - A x + \rho N(Tx, Sx), \eta(y, z) \rangle + \rho [b(x, y) - b(x, z)] \geq 0,
\]

that is, \( z \in Q(y) \). Thus \( Q \) is also a KKM mapping.

Since \( f \) is lower semicontinuous and \( \eta \) is continuous in the second argument, and \( b \) is continuous, it follows that \( Q(y) \) is closed for each \( y \in K \).

Finally, we claim that, for \( z_0 \in D \cap K \), \( Q(z_0) \) is compact. Indeed suppose that there exists \( \hat{z} \in Q(z_0) \) such that \( \hat{z} \not\in D \). Since \( z_0 \in D \cap K \) and \( \hat{z} \in Q(z_0) \), we have

\[
-\rho f(z_0, \hat{z}) + \langle A z_0 - A x + \rho N(Tx, Sx), \eta(z_0, \hat{z}) \rangle + \rho [b(x, z_0) - b(x, \hat{z})] \geq 0. \tag{8.3.7}
\]

Since \( \hat{z} \not\in D \), by hypothesis (8.3.2), we have

\[
-\rho f(z_0, \hat{z}) + \langle A z_0 - A x + \rho N(Tx, Sx), \eta(z_0, \hat{z}) \rangle + \rho [b(x, z_0) - b(x, \hat{z})] < 0,
\]

which is contradiction to (8.3.7). Hence \( Q(z_0) \subset D \). Since \( D \) is compact and \( Q(z_0) \) is closed, \( Q(z_0) \) is compact.

Hence by Theorem 1.2.7, it follows that \( \bigcap_{z \in K} Q(y) \neq \emptyset \), that is, there exists a \( z \in K \) such that

\[
-\rho f(y, z) + \langle A y - A x + \rho N(Tx, Sx), \eta(y, z) \rangle + \rho [b(x, y) - b(x, z)] \geq 0, \quad \forall y \in K.
\]

Since \( K \) is convex, for any \( \lambda \in (0, 1] \) and any \( y, z \in K \) we have \( y_\lambda := \lambda y + (1 - \lambda)z \in K \). Hence for given \( x \in K \), we have

\[
-\rho f(y_\lambda, z) + \langle A y_\lambda - A x + \rho N(Tx, Sx), \eta(y_\lambda, z) \rangle + \rho [b(x, y_\lambda) - b(x, z)] \geq 0.
\]
Since $b$ is convex in the second argument and $\eta$ is affine in the first argument, preceding inequality reduces to

$$-\rho f(y, z) + \lambda \langle Ay - Ax + \rho N(Tx, Sx), \eta(y, z) \rangle + \rho \lambda [b(x, y) - b(x, z)] \geq 0, \quad (8.3.8)$$

where we have used $\eta(z, z) = 0$.

Now, using (8.3.8), we have

$$0 = \rho f(y, y) \leq \rho \lambda f(y, y) + \rho (1 - \lambda) f(y, z)$$

$$\leq \rho \lambda f(y, y) + (1 - \lambda) [\lambda (Ay - Ax + \rho N(Tx, Sx), \eta(y, z))$$

$$+ \rho \lambda (b(x, y) - b(x, z))] \]$$

Dividing by $\lambda$, we have

$$\rho \lambda f(y, y) + (1 - \lambda) [\lambda (Ay - Ax + \rho N(Tx, Sx), \eta(y, z)) + (1 - \lambda) \rho [(b(x, y) - b(x, z))] \geq 0.$$

Since $f$ and $A$ are simultaneous hemicontinuous, then letting $\lambda \to 0^+$, we have

$$\rho f(z, y) + (A z - A x + \rho N(Tx, Sx), \eta(y, z)) + \rho [(b(x, y) - b(x, z))] \geq 0, \quad \forall y \in K.$$

Therefore $z \in K$ is a solution of AP (8.3.1).

**Uniqueness of solution** Let $z_1$ and $z_2$ be two solutions of AP (8.3.1). Then, for all $y \in K$,

$$\rho f(z_1, y) + (A z_1 - A x + \rho N(Tx, Sx), \eta(y, z_1)) + \rho [(b(x, y) - b(x, z_1))] \geq 0, \quad (8.3.9)$$

$$\rho f(z_2, y) + (A z_2 - A x + \rho N(Tx, Sx), \eta(y, z_2)) + \rho [(b(x, y) - b(x, z_2))] \geq 0. \quad (8.3.10)$$

Taking $y = z_2$ in (8.3.9), $y = z_1$ in (8.3.10) and adding these inequalities, we have

$$\rho (f(z_1, z_2) + f(z_2, z_1)) - (A z_1 - A z_2, \eta(z_1, z_2)) \geq 0,$$

since $\eta(x, y) + \eta(y, x) = 0$ for all $x, y \in K$.

Since $f$ is monotone and $A$ is $\tau$-strongly $\eta$-monotone, then it follows from preceding inequality that

$$\tau \|z_1 - z_2\|^2 \leq 0.$$
Since \( r > 0 \), we have \( z_1 = z_2 \). This completes the proof.

### 8.4. ALGORITHMS AND CONVERGENCE ANALYSIS

Based on Theorem 8.3.1, we construct an algorithm for GMEP (8.2.1). Further, we prove the existence of solutions for GMEP (8.2.1) and discuss the convergence criteria for the sequence generated by our algorithm.

For given \( x_0 \in K \), we know from Theorem 8.3.1 that the AP (8.3.1) has a solution, say, \( x_1 \in K \), that is,

\[
\rho f(x_1, y) + \langle Ax_1 - Ax_0 + \rho N(Tx_0, Sx_0), \eta(y, x_1) \rangle + \rho [(b(x_0, y) - b(x_0, x_1)] \geq 0, \quad \forall y \in K.
\]

By Theorem 8.3.1, again, for \( x_2 \in K \), AP (8.3.1) has a solution \( x_2 \), that is,

\[
\rho f(x_2, y) + \langle Ax_2 - Ax_1 + \rho N(Tx_1, Sx_1), \eta(y, x_2) \rangle + \rho [(b(x_1, y) - b(x_1, x_2)] \geq 0, \quad \forall y \in K.
\]

Hence by induction, we have:

**Algorithm 8.4.1.** For a given \( x_0 \in K \), compute an approximate solution \( x_n \in K \) satisfies the following condition:

\[
\rho f(x_{n+1}, y) + \langle Ax_{n+1} - Ax_n + \rho N(Tx_n, Sx_n), \eta(y, x_{n+1}) \rangle + \rho [(b(x_n, y) - b(x_n, x_{n+1})] \geq 0, \quad \forall y \in K, \quad n = 0, 1, 2, \ldots, \tag{8.4.1}
\]

where \( \rho > 0 \) is a constant and \( A : K \rightarrow H \) is not necessarily a linear mapping.

**Some special cases:**

1. If \( \eta(y, x) = y - x \) and \( b(x, y) = 0 \), \( N(Tx, Sx) = B(x) \forall x, y \in K \), where \( B : K \rightarrow H \), then Algorithm 8.4.1 reduces to the following algorithm for problem (7.2.2).

**Algorithm 8.4.2.** For a given \( x_0 \in K \), compute an approximate solution \( x_n \in K \) satisfy

\[
\rho f(x_{n+1}, y) + \langle Ax_{n+1} - Ax_n + \rho Bx_n, y - x_{n+1}] \rangle \geq 0, \quad \forall y \in K,
\]

\( n = 0, 1, 2, \ldots, \) where \( \rho > 0 \) is a constant and \( A : K \rightarrow H \) is not necessarily a linear mapping.
If $A = h'$, where $h'$ is the derivative of a given strictly convex function $h$ on $K$, then Algorithm 8.4.2 reduces to the algorithm studied by Moudafi and Thera [112].

(II) If $N(Tx, Sx) = 0$ and $\eta(y, x) = y - x \, \forall x, y \in K$, then Algorithm 8.4.1 reduces to the following algorithm for problem (8.2.3).

**Algorithm 8.4.3.** For a given $x_0 \in K$, compute an approximate solution $x_n \in K$ satisfy

$$\rho f(x_{n+1}, y) + \langle Ax_{n+1} - Ax_n, y - x_n \rangle + \rho \|b(x_n, y) - b(x_n, x_{n+1})\| \geq 0, \quad \forall y \in K,$$

$n = 0, 1, 2, ..., $ where $\rho > 0$ is a constant and $A : K \to H$ is not necessarily a linear mapping. Algorithm 8.4.3 is different from one considered by Noor [121].

(III) If $N(Tx, Sx) = 0$, $b(x, y) = 0$ and $\eta(y, x) = y - x \, \forall x, y \in K$, then Algorithm 8.4.1 reduces to the following algorithm for Problem 1.3.1.

**Algorithm 8.4.4.** For a given $x_0 \in K$, compute an approximate solution $x_n \in K$ satisfy

$$\rho f(x_{n+1}, y) + \langle Ax_{n+1} - Ax_n, y - x_n \rangle \geq 0, \quad \forall y \in K, \quad n = 0, 1, 2, ...,$$

where $\rho > 0$ is a constant and $A : K \to H$ is not necessarily a linear mapping.

**Theorem 8.4.1.** Let $K$ be a nonempty, closed and convex subset in $H$. Let $\eta : K \times K \to K$ be $\delta$-Lipschitz continuous and be such that $\eta$ is affine in the second argument and $\eta(x, y) + \eta(y, x) = 0$ for all $x, y \in K$. Let $T, S : K \to H$ be $t$-Lipschitz continuous and $s$-Lipschitz continuous mappings, respectively; let $N : H \times H \to H$ be $e$-strongly mixed $\eta$-monotone with respect to $T$ and $S$, and $(\beta_1, \beta_2)$-Lipschitz continuous; let $f : K \times K \to \mathbb{R}$ be convex and lower semicontinuous in the second argument and $\alpha$-strongly monotone; let $A : K \to H$ be $\tau$-strongly $\eta$-monotone and $\sigma$-Lipschitz continuous; let $f$ and $A$ be simultaneously hemicontinuous and let $b : H \times H \to \mathbb{R}$ satisfies properties (i)-(iv). If hypothesis (8.3.2) of Theorem 8.3.1 holds and $\rho > 0$ satisfy
Then the sequence \( \{x_n\} \) generated by Algorithm 8.4.1 converges strongly to \( x \in K \), where \( x \) is the unique solution of GMEP (8.2.1).

**Proof.** For any \( y \in K \), it follows from Algorithm 8.4.1 that

\[
\rho f(x_n, y) + \langle Ax_n - Ax_{n-1} + \rho N(Tx_{n-1}, Sx_{n-1}), \eta(y, x_n) \rangle + \rho[\langle b(x_{n-1}, y) \rangle - b(x_{n-1}, x_n)] \geq 0, \quad (8.4.3)
\]

and

\[
\rho f(x_{n+1}, y) + \langle Ax_{n+1} - Ax_n + \rho N(Tx_n, Sx_n), \eta(y, x_{n+1}) \rangle + \rho[\langle b(x_n, y) \rangle - b(x_n, x_{n+1})] \geq 0. \quad (8.4.4)
\]

Taking \( y = x_{n+1} \) in (8.4.3) and \( y = x_n \) in (8.4.4), respectively, we have

\[
\rho f(x_n, x_{n+1}) + \langle Ax_n - Ax_{n-1} + \rho N(Tx_{n-1}, Sx_{n-1}), \eta(x_{n+1}, x_n) \rangle + \rho[\langle b(x_{n-1}, x_{n+1}) \rangle - b(x_{n-1}, x_n)] \geq 0, \quad (8.4.5)
\]

\[
\rho f(x_{n+1}, x_n) + \langle Ax_{n+1} - Ax_n + \rho N(Tx_n, Sx_n), \eta(x_n, x_{n+1}) \rangle + \rho[\langle b(x_n, x_{n+1}) \rangle - b(x_n, x_{n+1})] \geq 0. \quad (8.4.6)
\]

Adding (8.4.5) and (8.4.6), we have

\[
-\rho[f(x_{n+1}, x_n) + F(x_n, x_{n+1}) + \langle Ax_{n+1} - Ax_n, \eta(x_{n+1}, x_n) \rangle \leq \langle Ax_n - Ax_{n-1} - \rho[N(Tx_n, Sx_n) - N(Tx_{n-1}, Sx_{n-1})], \eta(x_{n+1}, x_n) \rangle + \rho[\langle b(x_{n-1}, x_{n+1}) \rangle - b(x_n, x_{n+1}) + (b(x_n, x_n) - b(x_{n-1}, x_n)].
\]
Since \( f \) is \( \alpha \)-strongly monotone, \( A \) is \( \tau \)-strongly \( \eta \)-monotone and \( b \) is linear in the first argument, preceding inequality becomes

\[
\rho \alpha \|x_{n+1} - x_n\|^2 + \tau \|x_{n+1} - x_n\|^2 \leq \left[ \|Ax_n - Ax_{n-1} - \eta(x_n, x_{n-1})\| + \|\eta(x_n, x_{n-1}) - \rho[N(Tx_n, Sx_n) - N(Tx_{n-1}, Sx_{n-1})]\|\|\eta(x_{n+1}, x_n)\| + \rho[(b(x_n - x_{n-1}, x_n) - b(x_n - x_{n-1}, x_{n+1})].
\]

Using properties (ii) and (iii) of \( b \) and \( \delta \)-Lipschitz continuity of \( \eta \), we have

\[
(\tau + \rho \alpha) \|x_{n+1} - x_n\|^2 \leq \left[ \delta \|Ax_n - Ax_{n-1} - \eta(x_n, x_{n-1})\| + \delta \|\eta(x_n, x_{n-1}) - \rho[N(Tx_n, Sx_n) - N(Tx_{n-1}, Sx_{n-1})]\|\|\eta(x_{n+1}, x_n)\| \right. \\
\left. + \rho \gamma \|x_n - x_{n-1}\| \|x_{n+1} - x_n\| \right)
\]

(8.4.7)

Since \( N \) is \( \epsilon \)-strongly mixed \( \eta \)-monotone with respect to \( T \) and \( S \), and \((\beta_1, \beta_2)\)-Lipschitz continuous, and \( T \) and \( S \) are \( t \)-Lipschitz continuous and \( s \)-Lipschitz continuous, respectively, we estimate

\[
\|\eta(x_n, x_{n-1}) - \rho[N(Tx_n, Sx_n) - N(Tx_{n-1}, Sx_{n-1})]\|^2 \\
= \|\eta(x_n, x_{n-1})\|^2 - 2\rho\|N(Tx_n, Sx_n) - N(Tx_{n-1}, Sx_{n-1}), \eta(x_n, x_{n-1})\| \\
+ \rho^2\|N(Tx_n, Sx_n) - N(Tx_{n-1}, Sx_{n-1})\|^2 \\
\leq \delta^2\|x_n - x_{n-1}\|^2 - 2\rho \epsilon \|x_n - x_{n-1}\|^2 \\
+ \rho^2\|N(Tx_n, Sx_n) - N(Tx_{n-1}, Sx_{n-1})\|^2.
\]

(8.4.8)

\[
\|N(Tx_n, Sx_n) - N(Tx_{n-1}, Sx_{n-1})\| \\
\leq \beta_1\|Tx_n - Tx_{n-1}\| + \beta_2\|Sx_n - Sx_{n-1}\| \\
\leq (\beta_1 t + \beta_2 s)\|x_n - x_{n-1}\|. 
\]

(8.4.9)

From (8.4.8) and (8.4.9), we have

\[
\|\eta(x_n, x_{n-1}) - \rho[N(Tx_n, Sx_n) - N(Tx_{n-1}, Sx_{n-1})]\| \\
\leq (\delta^2 - 2\rho \epsilon + \rho^2(\beta_1 t + \beta_2 s)^2)\|x_n - x_{n-1}\|. 
\]

(8.4.10)
Since $A$ is $\tau$-strongly $\eta$-monotone and $\sigma$-Lipschitz continuous, we estimate

$$
\|Ax_n - Ax_{n-1} - \eta(x_n, x_{n-1})\| \leq (\delta^2 - 2\tau + \sigma^2)^{\frac{1}{2}}\|x_n - x_{n-1}\|. \quad (8.4.11)
$$

From (8.4.7), (8.4.10) and (8.4.11), we have

$$(\tau + \rho\alpha)\|x_{n+1} - x_n\|^2 \leq \{\delta(\delta^2 - 2\tau + \sigma^2)^{\frac{1}{2}} + (\delta^2 - 2\rho\epsilon + \rho^2(\beta_1 t + \beta_3 s)^2)^{\frac{1}{2}} + \rho\gamma\}\|x_n - x_{n-1}\|\|x_{n+1} - x_n\|
$$
or

$$
\|x_{n+1} - x_n\| \leq \theta\|x_n - x_{n-1}\|, \quad (8.4.12)
$$

where

$$
\theta := \left(\frac{\delta}{\tau + \rho\alpha}\right) \left[(\delta^2 - 2\tau + \sigma^2)^{\frac{1}{2}} + (\delta^2 - 2\rho\epsilon + \rho^2(\beta_1 t + \beta_3 s)^2)^{\frac{1}{2}} + \rho\gamma\right].
$$

By assumption (8.4.2), $\theta < 1$ and hence it follows from (8.4.12) that $\{x_n\}$ is a Cauchy sequence in $K \subset H$. Let $x_n \to \bar{x} \in H$ as $n \to \infty$. $\bar{x} \in K$ as $K$ is closed. Thus by the continuity of $f, A, T, S, N, \eta$ it follows from (8.4.1) that

$$
\rho f(\bar{x}, y) + \langle A\bar{x} - Ax + \rho N(T\bar{x}, S\bar{x}), (\eta(y, \bar{x})) + \rho[b(\bar{x}, y) - b(\bar{x}, \bar{x})] \geq 0, \quad \forall y \in K.
$$

Since $\rho > 0$, we have

$$
f(\bar{x}, y) + \langle N(T\bar{x}, S\bar{x}), \eta(y, \bar{x})\rangle + b(\bar{x}, y) - b(\bar{x}, \bar{x}) \geq 0, \quad \forall y \in K,
$$

that is, $\bar{x}$ is the unique solution of GMEP (8.2.1). This completes the proof.

We have the following consequences of Theorem 8.4.1.

**Corollary 8.4.1.** Let $K$ be a nonempty, closed and convex subset in $H$. Let $B : K \to H$ be $\epsilon$-strongly monotone and $\beta$-Lipschitz continuous; let $f : K \times K \to \mathbb{R}$ be convex and lower semicontinuous in the second argument, $f(x, x) = 0 \ \forall x \in K$, and $\alpha$-strongly monotone; let $A : K \to H$ be $\tau$-strongly $\eta$-monotone and $\sigma$-Lipschitz continuous; let $f$ and $A$ be jointly hemicontinuous. If there exists a nonempty compact subset $D$ of $H$ and $z_0 \in D \cap K$ such that for any $z \in K \setminus D$, we have

$$
-\rho f(z_0, z) + \langle Az_0 - Ax + \rho Bx, z_0 - z \rangle + \rho[b(x, z) - b(x, z)] < 0,
$$

100
for given $x \in K$, and if $\rho > 0$ satisfy

\[
\left| \frac{\epsilon + \alpha(\tau - k)}{\beta^2 - \alpha^2} \right| < \frac{\sqrt{[\epsilon + \alpha(\tau - k)]^2 - (\beta^2 - \alpha^2)[1 - (\tau - k)^2]}}{\beta^2 - \alpha^2}
\]

\[
\epsilon + \alpha(\tau - k) > \sqrt{(\beta^2 - \alpha^2)[1 - (\tau - k)^2]};
\]

\[
\beta > \alpha; \ \tau > k, \ k < 1,
\]

where $k := \sqrt{1 - 2\tau + \sigma^2}$. Then the sequence \( \{x_n\} \) generated by Algorithm 8.4.2 converges strongly to $x \in K$, where $x$ is the unique solution of problem (7.2.2).

**Corollary 8.4.2.** Let $K, f, A$ be same as Corollary 8.4.1, and let $b : H \times H \to \mathbb{R}$ be satisfy properties (i)-(iv). If there exists a nonempty compact subset $D$ of $H$ and $z_0 \in D \cap K$ such that for any $z \in K \setminus D$, we have

\[-\rho f(z_0, z) + \langle Az_0 - Ax, z_0 - z \rangle + \rho[b(x, z_0) - b(x, z)] < 0,
\]

for given $x \in K$ and if $\rho > 0$ satisfy $k + \rho \gamma < \tau + \rho \alpha$, where $k = \sqrt{1 - 2\tau + \sigma^2}$. Then the sequence \( \{x_n\} \) generated by Algorithm 8.4.3 converges strongly to $x \in K$, where $x$ is the unique solution of problem (8.2.3).

**Corollary 8.4.3.** Let $K, f, A$ be same as Corollary 8.4.1. If there exists a nonempty compact subset $D$ of $H$ and $z_0 \in D \cap K$ such that for any $z \in K \setminus D$, we have

\[-\rho f(z_0, z) + \langle Az_0 - Ax, z_0 - z \rangle < 0,
\]

for given $x \in K$ and if $\rho > 0$ satisfy $k < \tau + \rho \alpha$, where $k = \sqrt{1 - 2\tau + \sigma^2}$. Then the sequence \( \{x_n\} \) generated by Algorithm 8.4.4 converges strongly to $x \in K$, where $x$ is the unique solution of Problem 1.3.1.

We remark that the technique presented in this chapter can be applied for the mixed equilibrium problems involving set-valued mappings. Such problem will be the generalization of problems considered by [40,74,117,142].