CHAPTER 7

RESOLVENT DYNAMICAL SYSTEM FOR A MIXED EQUILIBRIUM PROBLEM

7.1. INTRODUCTION

In 1999, Moudafi and Théra [112] considered a class of mixed equilibrium problems which includes variational inequalities as well as complementarity problems, convex optimizations, saddle point problems, problems of finding a zero of a maximal monotone operator, and Nash equilibria problems as special cases. They discussed proximal and dynamical approaches to solve mixed equilibrium problems and equilibrium problems in Hilbert space. Recently Moudafi [111] discussed sensitivity analysis and developed some iterative methods for mixed equilibrium problems. In recent past, much attention has been given to consider and analyze the projected dynamical systems associated with variational inequalities and nonlinear programming problems, in which the right hand side of the ordinary differential equation is a projection operator. Such types of the projected dynamical system were introduced and studied by Dupuis and Nagurney [56]. Projected dynamical systems are characterized by a discontinuous right-hand side. The discontinuity arises from the constraint governing the question. The innovative and novel feature of a projected dynamical system is that its variational inequality problems. It has been shown in [27, 66, 113, 135, 136, 143] that the dynamical systems are useful in developing efficient and powerful numerical technique for solving variational inequalities and related optimization problems. Xia and Wang [136], Zhang and Nagurney [143] and Nagurney and Zhang [113] have studied the globally asymptotic stability of these projected dynamical systems. Noor [118-120] has also suggested and analyzed similar resolvent dynamical systems for variational inequalities. It is worth mentioning that there is no such type of the dynamical systems for mixed equilibrium problems.

In this chapter, we show that such type of dynamical systems can be suggested for the mixed equilibrium problems. We consider a mixed equilibrium problem (for short, MEP) in $\mathbb{R}^n$ and give its related Wiener-Hopf equation (for short, WHE) and fixed point formulation. Using this fixed point formulation and Wiener-Hopf
equation, we suggest a resolvent dynamical system associated with mixed equi-
librium problem (for short, RDS-MEP). We use this dynamical system to prove the
uniqueness of a solution of mixed equilibrium problem. Further, we show that the
dynamical system has globally asymptotic stability property. Our results can be
viewed as significant and unified asymptotic stability property. Our results can be
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The remaining part of the chapter is organized as follows:

In Section 7.2, we consider MEP in $\mathbb{R}^n$ and give its related WHE and fixed-point
formulation. Further, using the fixed-point formulation and WHE, we consider a
RDS-MEP. Furthermore, we review some concepts and result which are needed in the
sequel. In Section 7.3, using Gronwall inequality, we establish existence theorem for
RDS-MEP. In Section 7.4, we show that RDS-MEP is solvable in sense of Lyapunov
and globally converges to the solution set of MEP. Further, we show that RDS-MEP
converges globally exponentially to the unique solution of MEP.

7.2. PRELIMINARIES

Let $\mathbb{R}^n$ be a Euclidean space, whose inner product and norm are denoted by
$\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ respectively. Let $K$ be a nonempty, closed and convex set in $\mathbb{R}^n$; let
$T, A : K \to K$ be nonlinear mappings and let $N : K \times K \to K$ be a nonlinear
mapping. If $f : K \times K \to \mathbb{R}$ is a given bifunction satisfying $f(x, x) = 0$ for all
$x \in K$. Consider the following mixed equilibrium problem (MEP): Find $x \in K$ such
that

$$f(x, y) + \langle N(Tx, Ax), y - x \rangle \geq 0, \quad \forall y \in K. \quad (7.2.1)$$

Some special cases:

(I) If $N(Tx, Ax) = Bx \quad \forall x \in K$, where $B : K \to K$, then MEP (7.2.1) reduces to
the mixed equilibrium problem of finding $x \in K$ such that

$$f(x, y) + \langle Bx, y - x \rangle \geq 0, \quad \forall y \in K, \quad (7.2.2)$$

which has been studied by Moudafi and Théra [112].

(II) If $N(Tx, Ax) = 0 \quad \forall x \in K$, then mixed equilibrium problem (7.2.1) reduces to
Problem 1.3.1 introduced by Blum and Oettli [27].
(III) If $F(x, y) = \sup_{w \in S^x} \langle w, y - x \rangle$ with $S : K \to 2^K$ a set-valued maximal monotone operator. Then MEP (7.2.1) is equivalent to find $x \in K$ such that

$$0 \in N(Tx, Ax) + Sx, \quad \forall y \in K,$$

which is known as variational inclusions. Related work on such variational inclusions, see Kazmi and Khan [92,94] and the references therein.

Let us recall the extension of the Yosida approximation notion introduced in [111,112]. Let $\mu > 0$, for a given bifunction $f$, the associated Yosida approximation, $f_\mu$, over $K$ and the corresponding regularized operator, $D^\mu_\mu$, are defined as follows:

$$f_\mu(x, y) = \langle \frac{1}{\mu}(x - J^\mu_\mu(x)), y - x \rangle \quad \text{and} \quad D^\mu_\mu := \frac{1}{\mu}(x - J^\mu_\mu(x)),$$

in which $J^\mu_\mu(x) \in K$ is the unique solution of

$$\mu f(J^\mu_\mu(x), y) + \langle J^\mu_\mu(x) - x, y - J^\mu_\mu(x) \rangle \geq 0, \quad \forall y \in K. \quad (7.2.3)$$

Remark 7.2.1 [111].

(i) The existence and uniqueness of the solution of problem (7.2.3) follows by invoking Theorem 1.2.5.

(ii) If $f(x, y) = \sup_{w \in S^x} \langle u, y - x \rangle$ and $K = \mathbb{R}^n$, $S$ being a maximal monotone operator, it directly yields

$$J^\mu_\mu(x) = (I + \mu S)^{-1}x \quad \text{and} \quad D^\mu_\mu(x) = S_\mu(x),$$

where $S_\mu := \frac{1}{\mu}(I - (I + \mu S)^{-1})$ is the Yosida approximation of $B$, and one recover classical concepts.

(iii) The operator $J^\mu_\mu$, known as extended resolvent operator, is 1-cocoercive and nonexpansive, i.e., $J^\mu_\mu$ satisfies, respectively,

$$\langle J^\mu_\mu(x) - J^\mu_\mu(y), x - y \rangle \geq \|J^\mu_\mu(x) - J^\mu_\mu(y)\|^2$$

and

$$\|J^\mu_\mu(x) - J^\mu_\mu(y)\| \leq \|x - y\|, \quad x, y \in \mathbb{R}^n.$$
Lemma 7.2.1. MEP (7.2.1) has a solution $x$ if and only if $x$ satisfies the equation

$$x = J^f_{\mu}(x - \mu N(Tx, Ax)),$$

for $\mu > 0$.

We now define the residue vector $R(x)$ by the relation

$$R(x) = x - J^f_{\mu}[x - \mu N(Tx, Ax)].$$

(7.2.4)

Invoking Lemma 7.2.1, one can observe that $x \in K$ is a solution of MEP (7.2.1) if and only if $x \in K$ is a zero of the equation

$$R(x) = 0.$$

Now related to MEP (7.2.1), we consider the following Wiener-Hopf equation (WHE): Find $z \in \mathbb{R}^n$ such that for $x \in K$,

$$N(Tx, Ax) + D^f_{\mu}(z) = 0$$

(7.2.5)

and

$$x = J^f_{\mu}(z), \quad \text{for } \mu > 0.$$  

(7.2.6)

Lemma 7.2.2. MEP (7.2.1) has a solution $x$ if and only if WHE (7.2.5)-(7.2.6) has a solution $z \in \mathbb{R}^n$ where

$$x = J^f_{\mu}(z)$$

(7.2.7)

and

$$z = x - \mu N(Tx, Ax), \quad \text{for } \mu > 0.$$  

(7.2.8)

Using (7.2.7)-(7.2.8), WHE (7.2.5)-(7.2.6) can be written as

$$x - \mu N(Tx, Ax) - J^f_{\mu}[x - \mu N(Tx, Ax)]$$

$$+ \mu N(T(J^f_{\mu}[x - \mu N(Tx, Ax)])], A(J^f_{\mu}[x - \mu N(Tx, Ax)])) = 0.$$  

(7.2.9)

Thus it is clear from Lemma 7.2.2, that $x \in K$ is a solution of MEP (7.2.1) if and only if $x \in K$ satisfies the equation (7.2.9).

Using this equivalence, we suggest a new dynamical system associated with MEP (7.2.1) as

80
\[
\frac{dx}{dt} = \lambda \{J^I_\mu [x - \mu N(Tx, Ax)] \\
- \mu N(T(J^I_\mu [x - \mu N(Tx, Ax)]), A(J^I_\mu [x - \mu N(Tx, Ax)])) + \mu N(Tx, Ax) - x\},
\]

\[
= \lambda \{-R(x) + \mu N(Tx, Ax) \\
- \mu N(T(J^I_\mu [x - \mu N(Tx, Ax)]), A(J^I_\mu [x - \mu N(Tx, Ax)]))\},
\]

where \(\lambda\) is a constant. The system of type (7.2.10) is called the resolvent dynamical system associated with mixed equilibrium problem (RDS-MEP) (7.2.10). Here the right-hand side is associated with resolvent and hence in discontinuous on the boundary of \(K\). It is clear from the definitions that the solution to (7.2.10) belongs to the constraints set \(K\). This implies that the results such as the existence, uniqueness and continuous dependence on the given data can be studied. It is worth mentioning that RDS-MEP (7.2.10) is different from the problems considered and studied in [118-120].

The following concepts and result are useful in the sequel.

Definition 7.2.1 [136]. The dynamical system is said to converge to the solution set \(K^*\) of MEP (7.2.1) if and only if, irrespective of the initial point, the trajectory of the dynamical system satisfies

\[
\lim_{t \to \infty} \text{dist}(x(t), K^*),
\]

(7.2.11)

where

\[
\text{dist}(x, K^*) = \inf_{y \in K^*} \|x - y\|.
\]

It is easy to see that, if the set \(K^*\) has a unique point \(x^*\), then (7.2.11) implies that \(\lim_{t \to \infty} x(t) = x^*\).

If the dynamical system is still stable at \(x^*\) in the Lyapunov sense, then the dynamical system is globally asymptotically stable at \(x^*\).

Definition 7.2.2 [136]. The dynamical system is said to be globally exponentially stable with degree \(\eta\) at \(x^*\) if and only if, irrespective of the initial point, the trajectory of the system \(x(t)\) satisfies

\[
\| x(t) - x^* \| \leq \mu_1 \| x(t_0) - x^* \| \exp(-\eta(t - t_0)), \quad \text{for all } t \geq t_0,
\]

(7.2.12)
where $\mu_1$ and $\eta$ are positive constants independent of the initial point.

It is clear that globally exponentially stability is necessarily globally asymptotical stability and the dynamical system converges arbitrarily fast.

**Lemma 7.2.3. (Gronwall [see 66])** Let $\bar{x}$ and $\bar{y}$ be real-valued nonnegative continuous function with domain $\{t : t \leq t_0\}$ and let $\alpha(t) = \alpha_0(|t - t_0|)$, where $\alpha_0$ is a monotone increasing function. If, for $t \geq t_0$,

$$\bar{x} \leq \alpha(t) + \int_{t_0}^{t} \bar{x}(s)\bar{y}(s)ds,$$

then

$$\bar{x}(s) \leq \alpha(t) + \exp \left( \int_{t_0}^{t} \bar{y}(s)ds \right).$$

In the sequel, we assume that the bifunction $f$ involved in MEP (7.2.1) satisfies conditions of Theorem 1.2.5. Further, from now onward we assume that $K^*$ is nonempty and bounded, unless otherwise specified. Furthermore, assume that for all $x \in K$, there exists a constant $\tau > 0$ such that

$$\|N(Tx, Ax)\| \leq \tau(\|Tx\| + \|Ax\|).$$

We study the some properties of RDS-MEP (7.2.10) and analyze the global stability of the system. First of all, we discuss the existence and uniqueness of RDS-MEP (7.2.10).

### 7.3. EXISTENCE AND UNIQUENESS OF SOLUTION

First, we define the following concepts:

**Definition 7.3.1.** Let $T, A : K \to K$, $f : K \times K \to \mathbb{R}$ and $N : K \times K \to K$ be nonlinear mappings. Then, for all $x, y, z, w \in K$,

(a) $T$ is $\delta$-Lipschitz continuous if there exists a constant $\delta > 0$ such that

$$\|Tx - Ty\| \leq \delta\|x - y\|;

(b) $N$ is $(\alpha, \beta)$-Lipschitz continuous if there exist constants $\alpha, \beta > 0$ such that

$$\|N(x, y) - N(z, w)\| \leq \alpha\|x - z\| + \beta\|y - w\|;$$

82
(c) \( N \) is mixed monotone with respect to \( T \) and \( A \), if
\[
\langle N(Tx, Ax) - N(Ty, Ay), x - y \rangle \geq 0;
\]

(d) \( f \) is said to be \( \theta \)-pseudomonotone, where \( \theta \) is a real-valued multivariate function, if
\[
f(x, y) + \theta \geq 0 \text{ implies } -f(y, x) + \theta \geq 0.
\]

**Theorem 7.3.1.** Let the mappings \( T, A \) and \( N \) be \( \delta \)-Lipschitz continuous, \( \gamma \)-Lipschitz continuous and \( (\alpha, \beta) \)-Lipschitz continuous, respectively. For each \( x_0 \in \mathbb{R}^n \), there exists a unique continuous solution \( x(t) \) of RDS-MEP (7.2.10) with \( x(t_0) = t_0 \) over \([t_0, \infty)\).

**Proof.** Let
\[
G(x) = \lambda \{ J_\mu^f[x - \mu N(Tx, Ax)] - \mu N(TJ_\mu^f[x - \mu N(Tx, Ax)]), AJ_\mu^f[x - \mu N(Tx, Ax)]
\]
\[
+ \mu N(Tx, Ax) - x\}
\]
where \( \lambda \) is a constant. For all \( x, y \in \mathbb{R}^n \), we have
\[
\|G(x) - G(y)\| \leq \lambda \{ \| J_\mu^f[x - \mu N(Tx, Ax)] - J_\mu^f[y - \mu N(Ty, Ay)]\|
\]
\[
+ \mu \| N(Tx, Ax) - N(Ty, Ay)\| + \| x - y\|
\]
\[
+ \mu \| N(TJ_\mu^f[x - \mu N(Tx, Ax)]), AJ_\mu^f[x - \mu N(Tx, Ax)]\|
\]
\[
- \| N(TJ_\mu^f[y - \mu N(Ty, Ay)]), AJ_\mu^f[y - \mu N(Ty, Ay)]\|\}
\]
\[
\leq \lambda \{ 2\| x - y\| + 2\mu \| N(Tx, Ax) - N(Ty, Ay)\|
\]
\[
+ \mu (\alpha \delta + \beta \gamma) \| x - y\| + \mu \| N(Tx, Ax) - N(Ty, Ay)\|\}
\]
\[
\leq \lambda \{ 2 + 3\mu (\alpha \delta + \beta \gamma) + \mu^2 (\alpha \delta + \beta \gamma)^2 \}
\]
This implies that the mapping \( G \) is a Lipschitz continuous in \( \mathbb{R}^n \). So, for each \( x_0 \in \mathbb{R}^n \), there exists a unique and continuous solution \( x(t) \) of RDS-MEP (7.2.10), defined in a interval \( t_0 \leq t < T \) with initial condition \( x(t_0) = x_0 \). Let \( [t_0, T) \) be its maximal interval of existence, we show that \( T = \infty \). We estimate
\[ \|G(x)\| = \lambda \|J^f_\mu[x - \mu N(Tx, Ax)] - \mu N(T(J^f_\mu[x - \mu N(Tx, Ax)], A(J^f_\mu[x - \mu N(Tx, Ax)])) + \mu N(Tx, Ax) - x\|.
\]
\[ \leq \lambda \|J^f_\mu[x - \mu N(Tx, Ax)] - x\| + \lambda \mu(\alpha \delta + \beta \gamma)\|J^f_\mu[x - \mu N(Tx, Ax)] - x\|
\]
\[ = \lambda(1 + \mu(\alpha \delta + \beta \gamma))\|J^f_\mu[x - \mu N(Tx, Ax)] - x\|
\]
\[ \leq \lambda(1 + \mu(\alpha \delta + \beta \gamma))(\|J^f_\mu[x - \mu N(Tx, Ax)] - J^f_\mu(x)\|
\]
\[ + \|J^f_\mu(x) - J^f_\mu(x^*)\| + \|J^f_\mu(x^*) - x\|)
\]
\[ \leq \lambda(1 + \mu(\alpha \delta + \beta \gamma))(\mu\|N(Tx, Ax)\| + \|x - x^*\| + \|J^f_\mu(x^*) - x\|)
\]
\[ \leq \lambda(1 + \mu(\alpha \delta + \beta \gamma))(\mu\|T\| + \|Ax\|) + \|x\| + \|x^*\| + \|J^f_\mu(x^*)\| + \|x\|,
\]
\[ \leq \lambda(1 + \mu(\alpha \delta + \beta \gamma))(\mu\|T\| + \|Ax\| + \|x\| + \|x^*\| + \|J^f_\mu(x^*)\| + \|x\|).
\]

for any \( x \in \mathbb{R}^n \), then
\[ \|x(t)\| \leq \|x_0\| + \int_{t_0}^t \|Gx(s)\|ds,
\]
\[ \leq (\|x_0\| + k_1(t - t_0)) + k_2 \int_{t_0}^t \|x(s)\|ds,
\]
where
\[ k_1 = \lambda(1 + \mu(\alpha \delta + \beta \gamma))(\|J^f_\mu(x^*)\| + \|x^*\|),
\]
and
\[ k_2 = \lambda(1 + \mu(\alpha \delta + \beta \gamma))(2 + \mu\|T\| + \|Ax\|).
\]

Therefore using Lemma 7.2.3, we have
\[ \|x(t)\| \leq (\|x_0\| + k_1(t - t_0)) \exp\{k_2(t - t_0)\}, t \in [t_0, T).
\]

Hence, the solution \( x(t) \) is bounded on \([t_0, T)\). So \( T = \infty \). This completes the proof.
7.4. STABILITY ANALYSIS

We now study the stability of RDS-MEP (7.2.10). The analysis is in the spirit of Xia and Wang [136].

**Theorem 7.4.1.** Let the mappings $T$, $A$ and $N$ be the same as Theorem 7.3.1. Let the function $f$ be $\theta$-pseudomonotone with respect to $\theta$, where $\theta$ is defined as

$$\theta(x, y) = \langle N(Tx, Ax), y - x \rangle, \quad \forall x, y \in K,$$

and let $N$ be mixed monotone with respect to $T$ and $A$. If $\mu < \frac{1}{(\alpha^2 + \beta^2)}$, then RDS-MEP (7.2.10) is stable in the sense of Lyapunov and globally converges to the solution set of MEP (7.2.1).

**Proof.** Since the mappings $T$, $A$ and $N$ are Lipschitz continuous, it follows from Theorem 7.3.1 that RDS-MEP (7.2.10) has a unique continuous solution $x(t)$ over $[t_0, T)$ for any fixed $x_0 \in K$. Let $x(t) = x(t, t_0; x_0)$ be the solution of the initial-value problem (7.2.10). For a given $x^* \in K$, consider the following Lyapunov function

$$L(x) = \|x - x^*\|^2, \quad x \in \mathbb{R}^n. \quad (7.4.1)$$

It is clear that $\lim_{n \to \infty} L(x_n) = +\infty$, whenever the sequence $\{x_n\} \subset K$ and $\lim_{n \to \infty} x_n = +\infty$. Consequently, we conclude that the level sets of $L$ are bounded.

Let $x^* \in K$ be a solution of MEP (7.2.1), then

$$f(x^*, y) + \langle N(Tx^*, Ax^*), y - x^* \rangle \geq 0, \quad \forall y \in K. \quad (7.4.2)$$

Since $f$ is $\theta$-pseudomonotone and $N$ is mixed monotone then (7.4.2) implies that

$$-f(y, x^*) + \langle N(Tx^*, Ax^*), y - x^* \rangle \geq 0$$

$$-f(y, x^*) \geq -\langle N(Tx^*, Ax^*), y - x^* \rangle$$

$$\geq -\langle N(Ty, Ay), y - x^* \rangle,$$

i.e.,

$$-f(y, x^*) + \langle N(Ty, Ay), y - x^* \rangle \geq 0, \quad \forall y \in K. \quad (7.4.3)$$

Taking $y = J^f_\mu[x - \mu N(Tx, Ax)]$ in (7.4.3), we have
\(-f(J^*_\mu[x - \mu N(Tx, Ax)], x^*)\)

\(+\langle N(T J^*_\mu[x - \mu N(Tx, Ax)]), A J^*_\mu[x - \mu N(Tx, Ax)] \rangle, J^*_\mu[x - \mu N(Tx, Ax)] - x^* \rangle \geq 0. \tag{7.4.4}\)

Setting \(y = x^*, \ x = x - \mu N(Tx, Ax), \) and \(J^*_\mu(x) = J^*_\mu[x - \mu N(Tx, Ax)]\) in (7.2.3), we have

\[\mu f(J^*_\mu[x - \mu N(Tx, Ax)], x^*) + \langle J^*_\mu[x - \mu N(Tx, Ax)] - x, +\mu N(Tx, Ax), x^* - J^*_\mu[x - \mu N(Tx, Ax)] \rangle \geq 0. \tag{7.4.5}\]

From (7.4.4), (7.4.5) and (7.2.4), we have

\[\langle -R(x) + \mu N(Tx, Ax)\]

\[-\mu N(T(J^*_\mu[x - \mu N(Tx, Ax)]), A(J^*_\mu[x - \mu N(Tx, Ax)]), x^* - x + R(x)) \geq 0,\]

which implies that

\[\langle x - x^*, R(x) - \mu N(Tx, Ax)\]

\[+\mu N(T(J^*_\mu[x - \mu N(Tx, Ax)]), A(J^*_\mu[x - \mu N(Tx, Ax)]))\]

\[\geq \|R(x)\|^2 - \mu \langle R(x), N(Tx, Ax)\]

\[-\mu N(T(J^*_\mu[x - \mu N(Tx, Ax)]), A(J^*_\mu[x - \mu N(Tx, Ax)]))\]

\[\geq \|R(x)\|^2 - \mu (\alpha \delta + \beta \gamma) \|R(x)\| \|x - J^*_\mu[x - \mu N(Tx, Ax)]\|\]

\[= (1 - \mu (\alpha \delta + \beta \gamma)) \|R(x)\|^2. \tag{7.4.6}\]

Thus from (7.2.10), (7.4.1) and (7.4.6), we have

\[
\frac{d}{dt} L(x) = \frac{dL}{dx} \frac{dx}{dt} = 2\lambda (x - x^*, -R(x) + \mu N(Tx, Ax)\]

\[-\mu N(T(J^*_\mu[x - \mu N(Tx, Ax)]), A(J^*_\mu[x - \mu N(Tx, Ax)]))\]

\[\leq -2\lambda (1 - \mu (\alpha \delta + \beta \gamma)) \|R(x)\|^2 \leq 0, \quad \text{for} \quad \mu < \frac{1}{(\alpha \delta + \beta \gamma)}.\]
This implies that $L(x)$ is a global Lyapunov function for RDS-MEP (7.2.10) is stable in the sense of Lyapunov. Since $\{x(t) : t \geq t_0\} \subset K_0$ where $K_0 = \{x \in K : L(x) \leq L(x_0)\}$ and the function $L(x)$ is continuously differentiable on the bounded and closed set $K$, it follows from LaSalle's invariance principle [66], that the trajectories $x(t)$ will converge to $\Omega$, the largest invariant subset of the following set:

$$E = \{x \in K : \frac{dx}{dt} = 0\}.$$ 

Note that, if $\frac{dL}{dt} = 0$, then

$$\|u - J'_\mu[x - \mu N(Tx, Ax)]\|^2 = 0,$$

and hence $x$ is an equilibrium point of RDS-MEP (7.2.10), that is, $\frac{dx}{dt} = 0$.

Conversely, if $\frac{dx}{dt} = 0$, then it follows that $\frac{dL}{dt} = 0$.

Thus, we conclude that $E = \{x \in K : \frac{dx}{dt} = 0\} = K_0 \cap K^*$, which is nonempty, convex and invariant set contained in the solution set $K^*$. So, $\lim_{t \to \infty} \text{dist}(x(t), E) = 0$.

Therefore RDS-MEP (7.2.10) converges globally to the solution set of MEP (7.2.1). In particular, if we set $E = \{x^*\}$, then

$$\lim_{t \to \infty} x(t) = x^*.$$

Hence RDS-MEP (7.2.10) is globally asymptotically stable. This completes the proof.

**Theorem 7.4.2.** Let the mappings $T, A$ and $N$ be the same as Theorem 7.3.1. If $\lambda < 0$, then RDS-MEP (7.2.10) converges globally exponentially to the unique solution of MEP (7.2.1).

**Proof.** It follows from Theorem 7.3.1 that there exists a unique continuously differentiable solution of RDS-MEP (7.2.10) over $[t_0, \infty)$. Then

$$\frac{dL}{dt} = 2\lambda(x(t)-x^*, J'_\mu[x(t)-\mu N(Tx(t), Ax(t))])$$

$$-\mu N(T(J'_\mu[x-\mu N(Tx, Ax)]), A(J'_\mu[x-\mu N(Tx, Ax)])) + \mu N(Tx(t), Ax(t)) - x(t))$$

87
\[ -2\lambda \| x(t) - x^* \|^2 + 2\lambda (x(t) - x^*, J_{\mu}^T[x(t) - \mu N(Tx(t), Ax(t))] \\
- \mu N(T(J_{\mu}^T[x - \mu N(Tx, Ax)]), A(J_{\mu}^T[x - \mu N(Tx, Ax)])) + \mu N(Tx(t), Ax(t)) - x^* \), \]

(7.4.7)

where \( x^* \in K \) is the solution of MEP (7.2.1). Thus

\[ x^* = J_{\mu}^T[x^* - \mu N(Tx^*, Ax^*)] - \mu N(T(J_{\mu}^T[x - \mu N(Tx, Ax)]), A(J_{\mu}^T[x - \mu N(Tx, Ax)])) \\
+ \mu N(Tx^*, Ax^*). \]

Now, we estimate

\[ \| J_{\mu}^T[x(t) - \mu N(Tx(t), Ax(t))] - \mu N(T(J_{\mu}^T[x(t) - \mu N(Tx(t), Ax(t))]) \\
A(J_{\mu}^T[x(t) - \mu N(Tx(t), Ax(t))]) + \mu N(Tx(t), Ax(t)) \| \\
\leq \| x(t) - x^* \| + 2\mu \| N(Tx(t), Ax(t)) - N(Tx^*, Ax^*) \| \\
+ \mu \| N(T(J_{\mu}^T[x(t) - \mu N(Tx(t), Ax(t))]), A(J_{\mu}^T[x(t) - \mu N(Tx(t), Ax(t))]) \\
- N(T(J_{\mu}^T[x^* - \mu N(Tx^*, Ax^*)]), A(J_{\mu}^T[x^* - \mu N(Tx^*, Ax^*)])) \| \\
\leq \| x(t) - x^* \| + 2\mu (\alpha \delta + \beta \gamma) \| x(t) - x^* \| \\
+ \mu (\alpha \delta + \beta \gamma) \{\| x(t) - x^* \| + \mu (\alpha \delta + \beta \gamma) \| x(t) - x^* \| \} \\
= \{1 + 3\mu (\alpha \delta + \beta \gamma) + \mu^2 (\alpha \delta + \beta \gamma)^2\} \| x(t) - x^* \|. \]

(7.4.8)

From (7.4.7) and (7.4.8), we have

\[ \frac{d}{dt} \| x(t) - x^* \|^2 \leq 2\lambda \theta \| x(t) - x^* \|^2, \]

where \( \theta = \mu (\alpha \delta + \beta \gamma) \{3 + \mu (\alpha \delta + \beta \gamma)\}. \)

Thus for \( \lambda = -\lambda_1 \), where \( \lambda_1 \) is a positive constant, we have

\[ \| x(t) - x^* \| \leq \| x(t_0) - x^* \| \exp\{-\theta \lambda_1 (t - t_0)\}, \]

which shows that the trajectory of the solution of RDS-MEP (7.2.10) converges globally exponentially to the unique solution of MEP (7.2.1). This completes the proof.