CHAPTER II

INTEGER SOLUTION TO BILEVEL LINEAR PROGRAMMING PROBLEM

2.1 Introduction

Many researchers have designed algorithms for the solution of the bilevel linear programming problem. However, there has been very little attention in the literature on both the solution and the application of bilevel problems involving discrete decisions. This is mainly because these problems pose major algorithmic challenges in the development of effective solution strategies. For attacking such problems cutting plane algorithms have found large attention in the past either as stand-alone algorithms or in connection with enumeration methods as branch-and-cut algorithms. The second approach explicitly uses the structure of the solution set mapping of the discrete lower level optimization problem.

Luh et al. (1982) discussed the case where all the decision variables are discrete for a three level hierarchical programming problem. For the solution of the all integer bilevel linear programming problem, a branch and bound type of enumerative solution algorithm has been developed by Moore and Bard (1990). They examined the case where each player tries to maximize the individual objective function over a jointly constrained polyhedron. The decision variables were variously partitioned into continuous and discrete sets. They developed a implicit enumeration scheme that finds a good feasible solutions within relatively few iterations and a set of heuristics were also developed that may be used

Part of the results of this chapter is contained in Rabbani and Adhami (2008) and Bari et al. (2009).
when the computations becomes too burdensome. Another branch and bound technique is developed by Wen and Yang (1990), where only the upper level problem has discrete decision variables and the lower level problem has continuous decision variables. Bard and Moore (1992) further presented an algorithm for solving the zero-one case in bilevel programming problem. They converted the leader’s objective function into a parameterized constraint, and then solved the resultant problem. Edmunds and Bard (1992) developed a branch and bound algorithm for the mixed integer nonlinear bilevel programming problem that finds a global optima for a class of bilevel programming problem where leader controls a set of continuous and discrete variables and tries to minimize a convex nonlinear objective function. Cutting plane and parametric solution approaches have also been developed by Dempe (1995) to solve problems where lower level has a separable upper variable in its objective function only. Saharidis and Ierapetritou (2008) gave an algorithm for the resolution of mixed integer bilevel programming problem based on Benders decomposition method.

Some other algorithms for mixed integer and all integer solution for bilevel programming problem can be found in the monograph by Bard (1998) and in the papers by Edmunds and Bard (1990), Chern et al. (1991), Jan and Chern (1994), Sahin and Ciric (1998), Zeynep and Floudas (2005).

In this chapter we focus on the integer linear bilevel programming problem, in which all the functions involved are linear. The aim of this chapter is to extend the Kth-best approach developed by Bialas and Karwan (1984) for finding the integer solution to a bilevel programming problem by introducing A-T cut (Bari and Alam, 2005) to the reduced
feasible region obtained after using an improved version of NAZ cut (Bari and Shoib, 2003). Here first of all we will give a brief description of Kth-best algorithm and then discuss the improvement in NAZ cut. An algorithm is then presented to solve the integer linear bilevel programming problem. An example is presented at the end of the chapter to highlight the computational details.

2.2 The Kth-Best Algorithm

Bialas and Karwan (1984) have proposed the Kth-best approach that compute global solutions of bilevel linear programming problems by enumerating the extreme points of the constraint region. The Kth-best approach has been proven to be a valuable analysis tool with a wide range of successful applications for bilevel linear programming problems.

Consider the following bilevel linear programming problem:

$$\max_{x_1, x_2} f_1(x_1, x_2) = c_1x_1 + c_2x_2, \text{ where } x_2 \text{ solves}$$  \hspace{1cm} (P1)

$$\max_{x_2} f_2(x_1, x_2) = d_1x_1 + d_2x_2$$  \hspace{1cm} (P2)

$$\text{s.t. } A_1x_1 + A_2x_2 \leq b,$$

$$x_1, x_2 \geq 0$$

where $x_1 \in \mathbb{R}^{n_1}$ and $x_2 \in \mathbb{R}^{n_2}$, $d_1$ and $c_1$ is an $n_1$-dimensional row vector, $c_2$ and $d_2$ are $n_2$- dimensional row vectors, $A_1$ and $A_2$ are $m \times n_1$ and $m \times n_2$ matrices respectively and $b$ is an $m$-dimensional column vector.

Let $S \subset \mathbb{R}^{n_1+n_2}$ denote the feasible choices of $(x_1, x_2)$. We assume that the polyhedron $S$ defined by the common constraints is nonempty and
bounded. For any fixed choice of $x_1$, lower level will choose a value of $x_2$ to maximize the objective function $f_2(x_1, x_2)$. Hence, for each feasible value of $x_1$, lower level will react with a corresponding value of $x_2$. This induces a functional relationship between the decisions of upper level, $x_1$, and the reactions of lower level. Hence, upper level is really restricted to choosing a point in the set, say $\psi_{f_2}(S)$ given by

$$\psi_{f_2}(S) = \left\{ (x_1^*, x_2^*) \in S \mid f_2(x_1^*, x_2^*) = \max_{x_2} f_2(x_1, x_2) \right\}.$$  

This is the set of rational reactions of $f_2$ over $S$. For convenience of notation and terminology we will refer to $S^1 = \psi_{f_2}(S)$ as the upper level feasible region, and $S^2 = S$ as the lower level feasible region. Also it is shown that if $x$ is an extreme point of $S^1$, then $x$ is an extreme point of $S^2$. Let $x_{[1]}^*, x_{[2]}^*, \ldots, x_{[N]}^*$ denote the $N$ ordered basic feasible solutions to the linear programming problem

$$\begin{align*}
\max & \quad cx \\
\text{s.t.} & \quad Ax = b \\
& \quad x \geq 0
\end{align*}$$

such that $cx_{[i]}^* \geq cx_{[i+1]}^*, (i = 1, \ldots, N - 1)$. Then solving (P1) is equivalent to finding the index $K^* = \min\left\{ i \in (1, \ldots, N) \mid x_{[i]}^* \in S^1 \right\}$ yielding the global optimal solution $x_{[K^*]}^*$. This requires finding the $(K^*)$th best extreme point solution to the problem given in (2.1.1). The $K$th-best algorithm performs this search and thus obtains a global optimal solution.
2.3 An Improved NAZ Cut and A-T Cut for Integer Linear Programming Problem

We have improved NAZ cut which has been proposed by Bari and Shoib (2003) that reduces the feasible region of an integer program considerably. NAZ cut is derived parallel to the objective function curve, on the integer point inside the feasible region, having minimum perpendicular distance from the objective function curve passing through the non-integer solution.

Consider the integer linear programming problem as follows:

\[
\begin{align*}
\max f(x_1, x_2) &= c_1 x_1 + c_2 x_2 \\
\text{s.t.} \quad &A_1 x_1 + A_2 x_2 \leq b \\
&x_1, x_2 \geq 0
\end{align*}
\]

where \( x_1 \in \mathbb{R}^{n_1} \) and \( x_2 \in \mathbb{R}^{n_2} \), where \( n_1 + n_2 = n \), \( c_1 \) is an \( n_1 \)-dimensional row vector, \( c_2 \) is an \( n_2 \)-dimensional row vectors, \( A_1 \) and \( A_2 \) are \( m \times n_1 \) and \( m \times n_2 \) matrices respectively and \( b \) is an \( m \)-dimensional column vector.

The linear programming relaxation can be obtained by omitting the integer restrictions.

First we solve the linear programming relaxation. Let the solution be \( x^* = (x_1^*, x_2^*) \).

If \( x^* \) is integer, then the problem is solved otherwise, let the \( k^{th} \) component of \( x^* \) be non integer with \( x_k = a_k^* \).
The nearest integer values to \( x \) are

\[
x_k^1 = \lfloor a_k^* \rfloor \quad \text{and} \quad x_k^2 = \lfloor a_k^* \rfloor + 1 = \{ a_k^* \}, \quad k = 1, 2.
\]

where \( \lfloor \cdot \rfloor \) is the largest integer less than or equal to \( t \) and \( \{ \cdot \} \) is the smallest integer greater than or equal to \( t \).

With such bifurcations we can find all the \( 2^n \) points in the surrounding of the non-integer solution \( x^* \). Denote the set of indices of these \( 2^n \) points by \( S^0 \). If all these points lie outside the feasible region we move to the next integer feasible points obtained from \( x_k = a_k^* - 1 \).

Let the objective value at \( x^* \) be \( Z^* \). Thus, the objective function level plane at \( x^* \) will be \( cx^* = Z^* \).

Now we find the difference \( d_i = Z^* - cx_i^0 \), \( i \in S^0 \), i.e., the difference between the objective function value at non integer solution and the objective function values at the surrounding integer points, instead of using the formula of perpendicular distance as discussed by Bari and Shoib (2003). Here \( x_i^0 \)'s, \( i \in S^0 \), are surrounding integer points around \( x^* \).

Now we search for the feasible point \( x_i^0 \), which has a minimum positive difference from the objective function value.

Let \( G \) be the set of indices \( i \in S^0 \) for which \( x_i^0 \)'s are feasible.

Let \( x^0 = \left\{ x_k^0 \mid d_k = \min_{i \in G} d_i \right\} \)
A plane passing through this integer point and parallel to the objective hyperplane will be \( cx^0 = Z^0 \).

Clearly \( Z^0 < Z^* \)

The NAZ cut is now introduced as

\[ cx^0 \geq Z^0 \]

which reduces the feasible region.

Here \( Z^0 \) acts as a lower bound for the integer solution to the problem.

Let \( x^0 \) be defined as:

\[ x^0 = (\alpha_1^{x^0}, \alpha_2^{x^0}, \ldots, \alpha_n^{x^0}); \quad k^0 \in S^0 \]

Now to find the integer optimum solution we add the A-T cut at \( x^0 \) as

\[ \sum_{j=1}^{n} x_j = \sum_{k^0 \in S^0} \alpha_j^{k^0}. \]

2.4 The Algorithm to Solve Integer Linear Bilevel Programming

Using the common notation in bilevel programming, the integer linear bilevel programming problem can be written as follows:

\[
\begin{align*}
\max_{x_1} f_1(x_1, x_2) &= c_1 x_1 + c_2 x_2, \quad \text{where } x_2 \text{ solves} \\
\max_{x_2} f_2(x_1, x_2) &= d_1 x_1 + d_2 x_2 \\
\text{s.t. } A_1 x_1 + A_2 x_2 &\leq b \\
x_1, x_2 &\geq 0, \quad x_1, x_2 \text{ are integers}
\end{align*}
\]
where \( x_1 \in \mathbb{R}^n \) and \( x_2 \in \mathbb{R}^m \), \( d_1 \) and \( c_1 \) is an \( n_1 \)-dimensional row vector, \( c_2 \) and \( d_2 \) are \( n_2 \)-dimensional row vectors, \( A_1 \) is an \( m \times n_2 \)-matrix and \( b \) is an \( m \)-dimensional column vector. We assume that the polyhedron \( S \) defined by the common constraints is nonempty and bounded.

First, we solve the linear programming relaxation for leader’s problem associated with (2.4.1) using simplex method i.e., we solve,

\[
\max f_1(x_1, x_2) = c_1^T x_1 + c_2^T x_2 \\
\text{s.t. } A_1 x_1 + A_2 x_2 \leq b, \\
x_1, x_2 \geq 0
\]

(2.4.2)

Let the solution be \( x^* \). If the solution is non-integer we add the NAZ cut \( c_1 x_1 + c_2 x_2 \geq c_1^0 x_1^0 + c_2^0 x_2^0 = z^0 \), which passes through \( x^0 \), where \( x^0 = (x_1^0, x_2^0) \) is the integer point inside the feasible region and \( z^0 \) is the value of leader’s problem at \( x^0 \).

Now to find the integer optimum solution we add the A-T cut

\[ \sum_{j=1}^{n} x_j = \sum_{j=1}^{n} \alpha_j^k \] at \( x^0 \).

Let \( x_{[1]}^*, x_{[2]}^*, \ldots, x_{[N]}^* \) denote the \( N \) ordered basic feasible solutions to the problem (2.4.2) such that \( c x_{[i]}^* \geq c x_{[i+1]}^*, (i = 1, \ldots, N-1) \).

Let \( S_1 \) be the projection of \( S \) onto the leader’s decision space. For each \( x_{[i]}^* \in S_1 \), a feasible solution to the bilevel linear integer problem (2.4.1) is obtained by solving the following integer linear programming problem:
\[
\max_{x_2} d_1 x_1 + d_2 x_2 \\
\text{s.t.} \quad A_2 x_2 \leq b - A_1 x_1^*, \tag{2.4.3}
\]

\[x_2 \geq 0 \text{ and integer.}\]

For the above problem also we can find the integer optimum by using NAZ cut and A-T cut. Let \( M(x_{[il]}^*) \) denote the set of optimal solution to (2.4.3). We assume that for any fixed choice of leader, follower has some room to respond, i.e., \( M(x_{[il]}^*) \neq \emptyset \). Hence, the feasible region of the leader, called the inducible region IR, is

\[ IR = \{ (x_1, x_2) : x_1 \in S_1, x_2 \in M(x_{[il]}^*) \} \]

With the above extensions in the Kth-best algorithm we can find the integer optimum solution for the bilevel programming problems.

The procedure can be summarized in the following steps:

**Step 1.** Set \( i = 1 \). Solve (2.4.2) with the simplex method. If the solution is non-integer then add NAZ cut and A-T cut to obtain integer optimum solution as \( x_{[il]}^* \). Let \( W = (x_{[il]}^*) \) and \( T = \emptyset \). Go to Step 2.

**Step 2.** Solve (2.4.3) for integer optimum solution using NAZ cut and A-T cut. Let this solution be denoted by \( \bar{x} \). If \( \bar{x} = x_{[il]}^* \), stop; \( x_{[il]}^* \) is the global optimum to (2.4.1). Otherwise, go to Step 3.

**Step 3.** Let \( W_{[il]} \) denote the set of adjacent extreme points \( x \) of \( x_{[il]}^* \) such that \( c x \leq c x_{[il]}^* \).
Let $T = T \cup (x^*_{[i]})$ and $W = (W \cup W_{[i]}) \cap T^c$. Go to step 4.

**Step 4.** Set $i = i + 1$ and choose $x^*_{[i]}$ so that $cx^*_{[i]} = \max_{x \in W} (cx)$. Go to step 2.

### 2.5 Numerical Illustration

Consider the following integer linear bilevel programming problem

$$\max_{x_1} f(x_1, x_2) = 18x_1 + 22x_2$$

where $x_2$ solves:

$$\max_{x_2} f_1(x_1, x_2) = 2x_1 + x_2$$

s.t. $17x_1 + 24x_2 \leq 102$

$84x_1 + 76x_2 \leq 399$

$x_1, x_2 \geq 0$, $x_1, x_2$ are integers.

The first step of the above procedure is to solve the linear programming problem

$$\max f(x_1, x_2) = 18x_1 + 22x_2$$

s.t. $17x_1 + 24x_2 \leq 102$

$84x_1 + 76x_2 \leq 399$

$x_1, x_2 \geq 0$

We get the non-integer solution as $x_1^* = 2.66$, $x_2^* = 2.36$ and $f(x_1^*, x_2^*) = 99.96$
We round off the non integer solution to the nearest four integer points as 
(2, 2), (2, 3), (3, 2) and (3, 3). The respective differences are

\[ 99.96 - 80 = 19.96; \quad 99.96 - 102 = -2.04; \quad 99.96 - 98 = 1.96; \quad 99.96 - 120 = -20.04 \]

We are left with only one feasible point (2, 2), which gives the minimum positive difference. Now the NAZ cut and A-T cut passing through the integer point (2, 2) can be derived respectively as

\[ 18x_1 + 22x_2 \geq 80 \]

and

\[ x_1 + x_2 = 4 \]

Now solving the problem (2.4.4) with these additional constraints we obtain the integer optimum solution as

\[ x_1^* = 0, \ x_2^* = 4 \quad \text{and} \quad f(x_1^*, x_2^*) = 88 \]

Let \( x_{ij}^* = (0,4) \), the first best solution. Set \( W = \{(0,4)\} \) and \( T = \emptyset \).

To determine if \( x_{ij}^* \) is an element of \( M(x_{ij}^*) \) we solve

\[
\begin{align*}
\max f_1(x_1, x_2) &= 2x_1 + x_2 \\
\text{s.t.} \quad 17x_1 + 24x_2 &\leq 102 \\
84x_1 + 76x_2 &\leq 399 \\
x_1, x_2 &\geq 0
\end{align*}
\]

(2.4.6)
$x_1 = 0$

$x_1, x_2$ integer.

After adding the required NAZ cut and A-T cut we get the integer optimal solution as

$\tilde{x} = (0,4)$. Hence, $\tilde{x} = x^*_1$

Therefore, $x^* = (0,4)$ is the global optimal solution to bilevel linear integer programming problem (2.4.4).