CHAPTER - IV

OPTIMUM INVESTMENT POLICY FOR SAVINGS

4.1 INTRODUCTION:

The money savings schemes for the welfare of the members of the frugal societies are organised by them in diverse departments. According to such schemes the fixed amount deposited monthly by the society and the same is deducted directly from the salaries of its members and such member is free to make withdrawal of his total amount at any time he desires. In some more way other benefit can also be enjoyed in accidental case because in such cases a good amount is refunded which is quite independent of his deposits. Thus the functions of the society is to invest its monthly savings in some business. The problem is to determine the maximum amount which may be invested in such a way to meet the demands on a sudden withdrawal or payment in accidental matter. Thus the problem is formulated in a two stage stochastic programming format Madansky(1962). The objective function is seen to be strictly concave and thus the solution can be obtained by equating differential coefficient to zero. When the distribution of the totals of deposits minus withdrawals approximates a normal distribution, it is seen that the maximum return is obtained by investing the average of deposits minus withdrawals. Maximum return can also be obtained in case of gamma distribution.
4.2 THE EXPECTED PROFIT

Let \( r \) represent the amount of deposits minus withdrawals per unit of time. The management wants to invest the maximum possible of this amount in a business.

Let \( d_1 \) denote the profit per unit of investment per unit of time. If the withdrawals at some point of time exceed the amount left with the society after investment then a per unit shortage cost of \( d_2 \) per unit of time is needed. In practice, we have \( d_2 > d_1 \).

Let \( y_1 \) be the amount invested over a unit of time, which yields a profit of \( d_1 y_1 \), and \( y_2 \), the amount procured due to shortage on a rate of \( d_2 \) per unit of time.

Let \( r \) be the non-negative continuous random variable with density function \( f(r) \).

Let us denote by

\[
P(r) = d_1 y_1 - d_2 y_2
\]

\[
\text{Max } P(r) = \begin{cases} 
  d_1 y_1, & \text{if } r > y_1 \\
  y_2 & \text{if } r \leq y_1 
\end{cases}
\]

Our aim is to maximize the function

\[
\phi(y_1) = E_r \max_{y_2} \left[ P(r) \right]
\]

where \( y_2 = y_1 - r \) and \( y_1, y_2 \geq 0 \).

Note that \( P \) is a continuous function of \( r \) because if \( r = y_1 \), the top and bottom expressions in \( P(r) \) are identical.

The expected profit is given by

\[
\phi(y_1) = \int_{r \leq y_1} \left[ d_1 y_1 - d_2 (y_1 - r) \right] f(r) \, dr + \int_{r > y_1} d_1 y_1 f(r) \, dr \quad \cdots (4.2.1)
\]

Note that the objective function in \( (4.2.1) \) is strictly concave.
Proposition: $\phi(y_1)$ is a strictly concave function of $y_1$.

Proof: consider 
\[ \hat{y}_1 = \lambda y_3 + (1-\lambda) y_4 \]
where $0<\lambda<1$ and thus $y_3 < \hat{y}_1 < y_4$ for $y_3 \neq y_4$

We have

\[ \phi(y_1) = d_1 \hat{y}_1 - d_2 \int_0^{y_1} (y_1-r)f(r)dr \]
\[ = d_1 \{\lambda y_3 + (1-\lambda) y_4\} - d_2 \int_0^{y_1} \{\lambda(y_3-r) + (1-\lambda)(y_4-r)\}f(r)dr \]
\[ = d_1 \lambda y_3 + d_1(1-\lambda) y_4 - d_2 \int_0^{y_1} \{\lambda(y_3-r) + (1-\lambda)(y_4-r)\}f(r)dr \]
\[ = \lambda \left\{ d_1 y_3 - d_2 \int_0^{y_1} (y_3-r)f(r)dr \right\} + (1-\lambda) \left\{ d_1 y_4 - d_2 \int_0^{y_1} (y_4-r)f(r)dr \right\} \]
\[ = \lambda \left[ d_1 y_3 - d_2 \left\{ \int_0^{y_3} (y_3-r)f(r)dr + \int_{y_3}^{y_1} (y_3-r)f(r)dr \right\} \right] \]
\[ + (1-\lambda) \left[ d_1 y_4 - d_2 \left\{ \int_0^{y_1} (y_4-r)f(r)dr - \int_{y_1}^{y_4} (y_4-r)f(r)dr \right\} \right] \]
\[ = \lambda \phi(y_3) - \lambda d_2 \int_{y_3}^{y_1} (y_3-r)f(r)dr + (1-\lambda) \phi(y_4) \]
\[ + (1-\lambda)d_2 \int_{y_1}^{y_4} (y_4-r)f(r)dr \]
\[ \phi(\hat{y}_1) = \lambda \phi(y_3) + (1-\lambda) \phi(y_4) + I_1 + I_2 \]
\[ > \lambda \phi(y_3) + (1-\lambda) \phi(y_4) \]
\[ \hat{y}_1 \]

where

\[ I_1 = -\lambda d_2 \int_{y_3}^{y_1} (y_3-r)f(r)dr \]
\[ I_2 = (1-\lambda)d_2 \int_{y_1}^{y_4} (y_4-r)f(r)dr \]
and \[ I_2 = (1-\lambda)d_2 \int_{y_1}^{y_4} (y_4-r)f(r)dr \]

In \( I_1 \), \( r > y_3 \), \( (y_3-r) < 0 \) \( \Rightarrow \) \( \lambda d_2(y_3-r) > 0 \) \( \Rightarrow I_1 > 0 \)

and in \( I_2 \), \( r < y_4 \), \( (y_4-r) > 0 \) \( \Rightarrow (1-\lambda)d_2(y_4-r) > 0 \) \( \Rightarrow I_2 > 0 \)

Thus \( \phi(y_1) \) is a strictly concave function of \( y_1 \).

4.3 DISTRIBUTION OF THE RANDOM VARIABLES

The random vector \( r \) is generally composed of a subvector of random variables representing demand and a subvector of random variables representing available amounts. It is now necessary to determine the most appropriate distribution to use to describe the behaviour of these random variables.

It is reasonable to able to derive the distributions of certain of the subvectors using statistical procedures. For example, in the case of the demand for direct labour that is associated with the production of a good or a service, a model for the forecasting of demand for that good or service can be used to derive manpower demand. Most time series forecasting techniques suppose the distribution of forecast errors to be normal, and it is therefore reasonable to assume the probability distribution of the random variables themselves to be normal.

However, in manpower planning problems, it is not always appropriate to determine the type and parameters of the required probability distributions from past data. When many intangible factors are involved, such an objective approach is unlikely to be fruitful and the distributions must be determined subjectively.

For example, in case of a government or public service organizations where the level of service provided is a question of
policy, the available amount or the demand for manpower will be a random variable whose eventual observed value will depend on the evolution of these policies. At the moment when a decision must be taken, the evolution of policy is essentially unknown but it is possible to capture likely scenarios through the use of a probability distribution. It is desirable to work with a univariate probability distribution which is such that its parameter can be estimated subjectively in manner likely to be familiar to managers. The Normal distribution, which is commonly used, would seem to be a good choice in this context (see Martel and Price(1981)).

4.4 THE NORMAL DISTRIBUTION

When a statistical technique is used to forecast the future outcome, it is usually reasonable to characterize the vector predicted for planning period by a normal distribution with a mean \( \mu \) and variance \( \sigma^2 \). Under this assumption, the expression in (4.2.1) can be simplified as follows:

\[
\phi(y_1) = \frac{1}{\sigma \sqrt{2\pi}} \int_{-\infty}^{\infty} \left\{ d_1y_1 - d_2(y_1 - r) \right\} e^{-\frac{1}{2} \left( \frac{r - \mu}{\sigma} \right)^2} dr \\
+ \frac{1}{\sigma \sqrt{2\pi}} \int_{y_1}^{\infty} d_1y_1 e^{-\frac{1}{2} \left( \frac{r - \mu}{\sigma} \right)^2} dr \\
= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-\infty}^{\infty} d_1y_1 - d_2(y_2 - \mu - oz) e^{-\frac{1}{2} z^2} dz \\
+ \int_{y_1}^{\infty} d_1y_1 e^{-\frac{1}{2} z^2} dz \right\}
\]

where \((r-\mu)/\sigma = z\), \(dr = \sigma dz\), \(y_1' = (y_1 - \mu)/\sigma\)
\[ d_1 y_1 - \frac{\sigma d_2}{\sqrt{2\pi}} e^{-\frac{1}{2} y_1^2} + \frac{d_2(\mu - y_1)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2} z^2} \, dz \quad \cdots (4.4.1) \]

Since \[ \frac{d_1 y_1}{\sqrt{2\pi}} \left\{ \int_{-\infty}^{\infty} e^{-\frac{1}{2} z^2} \, dz + \int_{-\infty}^{y_1} e^{-\frac{1}{2} z^2} \, dz = d_1 y_1 \right\} \]

and \[ \int_{-\infty}^{\infty} z e^{-\frac{1}{2} z^2} \, dz = e^{-\frac{1}{2} y_1^2} \]

We maximize the expected profit, which is attained when \( \frac{d\phi(y_1)}{dy_1} = 0 \).

This yields

\[ \frac{d_1}{d_2} - F(y_1') + (\sigma - 1)y_1' f(y_1') = 0 \]
or

\[ \frac{d_1}{d_2} = F(y_1') - (\sigma - 1)y_1' f(y_1') \quad \cdots (4.4.2) \]

where \( F(y_1') = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y_1} e^{-\frac{1}{2} z^2} \, dz \) and \( f(y_1') = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} y_1'^2} \)

**OPTIMAL INVESTMENT SCENARIOS:**

1. The first case, when \( y_1 = \mu, y_1' = 0 \), the problem (4.4.2) reduces

\[ \frac{d_1}{d_2} = \frac{1}{2} \]

i.e., if the shortage cost is two times the gain from investment, the maximum return is obtained.

2. The second case, when \( y_1' = +\infty \), the problem (4.4.2) becomes

\[ \frac{d_1}{d_2} = 1 \]

i.e., whatever the amount is available, may be invested.

3. The third case, when \( \sigma \) is greater, the problem (4.4.2) becomes

\[ \frac{d_1}{d_2} < 1 \]

i.e., if shortage cost, \( d_2 \), increases, investment rises.
4.5 THE GAMMA DISTRIBUTION

The probability density function of gamma variate, \( \gamma \), is

\[
f(y) = \begin{cases} 
\frac{\lambda^p}{\Gamma(p)} \, e^{-\lambda y} \, y^{p-1}, & 0 < y < \infty, \, \lambda, p > 0 \\
0, & \text{otherwise}
\end{cases}
\]

where \( \lambda \) and \( p \) are parameters to be estimated. Its mean and variance are \( \lambda \).

Then, the expression obtained from (4.2.1) is:

\[
\phi(y_1) = \int_0^{y_1} \left[ d_1 y_1 - d_2 (y_1 - r) \right] \frac{\lambda^p}{\Gamma(p)} \, e^{-\lambda r} \, r^{p-1} \, dr + \int_{y_1}^{\infty} d_1 y_1 \frac{\lambda^p}{\Gamma(p)} \, e^{-\lambda r} \, r^{p-1} \, dr
\]

\[
= d_1 y_1 - d_2 \int_0^{y_1} (y_1 - r) \frac{\lambda^p}{\Gamma(p)} \, e^{-\lambda r} \, r^{p-1} \, dr \quad \ldots \ldots (4.5.1)
\]

If we maximize the expected profit, i.e., \( \frac{d\phi(y_1)}{dy_1} = 0 \), then

\[
d_1 - d_2 \, \gamma(\lambda, p, y_1) = 0
\]

or

\[
\frac{d_1}{d_2} = \gamma(\lambda, p, y_1) = \int_0^{y_1} \frac{\lambda^p}{\Gamma(p)} \, e^{-\lambda r} \, r^{p-1} \, dr \quad \ldots \ldots (4.5.2)
\]

Using the tables, we have

1. If \( p=1, \lambda=1, \gamma_1=1 \), then \( d_1/d_2 = 3/5 \), i.e., if shortage cost is one and a half times the gain from investment, the maximum return is obtained.

2. If \( \lambda=1, \gamma_1 = \infty \), then \( d_1/d_2 = 1 \), which is similar to the investment behaviour as in previous section.

Note: The value of \( d_1/d_2 \) is between 0 and 1 in practical situation.
4.5 NUMERICAL EXAMPLE

The data of monthly totals (deposits—withdrawals) with the society over one year is given in table 1.

Table 1:

<table>
<thead>
<tr>
<th>S.N.</th>
<th>Deposits-Withdrawals (Rs. in 1000's)</th>
<th>Frequency (Observed)</th>
<th>Frequency (Expected)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(-175)--(-125)</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>(-125)--(-75)</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>(-75)--(-25)</td>
<td>7</td>
<td>15</td>
</tr>
<tr>
<td>4</td>
<td>(-25)--(25)</td>
<td>80</td>
<td>70</td>
</tr>
<tr>
<td>5</td>
<td>(25)--(75)</td>
<td>125</td>
<td>115</td>
</tr>
<tr>
<td>6</td>
<td>(75)--(125)</td>
<td>48</td>
<td>62</td>
</tr>
<tr>
<td>7</td>
<td>(125)--(175)</td>
<td>23</td>
<td>20</td>
</tr>
<tr>
<td>8</td>
<td>(175)--(225)</td>
<td>4</td>
<td>5</td>
</tr>
</tbody>
</table>

The mean of the above data is 49.828 and standard deviation is 53.002. The value of $\chi^2$ is computed from the above observed and corresponding theoretical frequencies of a normal distribution is 8.33, which is not significant at 1% level of significance. Thus the data approximate the normal distribution with mean 49.828 and standard deviation 53.002.