5.1. **INTRODUCTION**

In Chapter 4 we assumed that the cost incurred in enumerating the $j$th character in the $i$th stratum is proportional to $n_{ij}$, where $n_{ij}$ is the number on which the $j$th character is measured in the $i$th stratum. The total cost of the survey, there, was a linear function of $n_{ij}$. In multivariate sample surveys, however, there is usually an over-head cost which is associated with each individual of the sample in a stratum. The over-head cost together with the enumerating cost is now seen to be a convex function of $n_{ij}$.

After the same transformation as made in the previous chapter the problem finally reduces to minimize a convex function under linear constraints. A procedure is developed for finding the optimal solution. A numerical example is also presented. I presented the contents of this chapter in the seminar on "Probability and Statistical Inference" held at the Department of Mathematics and Statistics, Poona University, Poona (Ahm {1978a} ).
5.2. FORMULATION OF THE PROBLEM

Let $c_i$ be the overhead cost of approaching an individual in the $i$th stratum for measurements and $c_{ij}$ be the cost associated with measurement of the $j$th character of an individual in the $i$th stratum. Let $n_{ij}$ be the number of individuals in the $i$th stratum on which the $j$th character is measured. The sample sizes in the various strata are not fixed in advance, but they are determined as

$$n_i = \max_j n_{ij}, \quad i = 1, \ldots, k.$$  

This means that in each stratum there is at least one character which is measured on all the sampled individuals in that stratum. Then the total cost $C$ is given by

$$C = \sum_{i=1}^{k} c_i n_i + \sum_{i=1}^{k} \sum_{j=1}^{p} c_{ij} n_{ij}. \quad \ldots(5.2.1)$$

Note that $n_i$'s are the convex functions of $n_{ij}$ and thus $C$ is a convex function of $n_{ij}$. The constraints of the problem are same as in (4.5.5), chapter 4 i.e.

$$\sum_{i=1}^{k} \frac{n_{ij}^2}{(v_{ij})^2 + \frac{n_{ij}^2}{\sigma_{ij}^2}} \leq v_j, \quad j=1, \ldots, p. \quad \ldots(5.2.2)$$
and the restrictions on $n_{ij}$'s are

$$n_{ij} \geq 0, \quad i = 1, \ldots, k, \quad j = 1, \ldots, p.$$  ...(5.2.3)

The problem of finding an optimal stratified random sample now reduces to determine $n_{ij} > 0$ which minimize

(5.2.1) subject to the constraints (5.2.2).

Under the transformations

$$x_{ij} = \frac{1}{(v_j)_{ii} + \frac{n_{ij}}{n_s_{ij}}}^{\frac{1}{2}},$$

i.e.

$$n_{ij} = \left( \frac{1}{x_{ij}} - (v_j)_{ii} \right)^{2}_{ij}$$

and

$$n_{1} = \max_{j} \left( \frac{1}{x_{ij}} - (v_j)_{ii} \right)^{2}_{ij}$$

(5.2.1) changes to:

$$c(x) = \sum_{i=1}^{k} \sigma_{i} \max_{j} \left( \frac{1}{x_{ij}} - (v_j)_{ii} \right)^{2}_{ij} + \sum_{i=1}^{k} \sum_{j=1}^{p} \sigma_{ij} (\frac{1}{x_{ij}} - (v_j)_{ii})^{2}_{ij}$$

...(5.2.5)
where $x = (x_{11}, \ldots, x_{1p}, \ldots, x_{kl}, \ldots, x_{kp})$.

It can be easily verified that the function $C(x)$ is convex for $x > 0$. As stated in § 2.4, $\frac{1}{x_{ij}}$ are convex for $x_{ij} > 0$ and thus $\max_j \left( \frac{1}{x_{ij}} - (v_j)_{ii} \right)_{i=1}^k \leq 2$ are also convex (Zangwill [1972]). Therefore $C(x)$ being a positive combination of convex functions is also convex.

Under the transformations (5.2.4) the constraints of the problem now become

\[ \sum_{i=1}^k p_i^2 x_{ij} \leq v_j, \quad j = 1, \ldots, p \quad \ldots (5.2.6) \]

and

\[ x_{ij} \leq \frac{1}{(v_j)_{ii}}, \quad i=1, \ldots, k, \quad j=1, \ldots, p \quad \ldots (5.2.7) \]

5.3. SOLUTION TO THE PROBLEM

Define the sets of indices $I_j$, $j = 1, \ldots, p$ as follows:

\[ I_j = \{ l / \max_{l} a_{ij} = a_{ij} \}, \quad l=1, \ldots, p \quad \ldots (5.3.1) \]

This means that the maximum of $a_{ij}$ for each $i \in I_j$ is attained
The objective function in (5.3.2) can be written as

\[ c_2(x) = \sum_{j=1}^{p} \sum_{i \in I_j} c_i \left( \frac{1}{x_{ij}} - (v_j)_{il} \right) x_{ij}^2 + \sum_{j=1}^{k} \sum_{i=1}^{p} \frac{g_{jl}}{x_{ij}} + \ldots + \]

\[ \sum_{i=1}^{k} \frac{g_{lp}}{x_{ij}}. \]

The term \( \sum_{i=1}^{k} \frac{g_{jl}}{x_{ij}} \) in the objective function involves only those variables which occur in the \( j \)th constraint. The sets \( I_j \) however depend on \( x_{ij}, i=1,...,k, j=1,...,p. \) Were the sets \( I_j \)'s known, the solution to the problem (5.3.2) subject to (5.2.6) and (5.2.7) would have been obtain by solving the following problems separately for \( p \) characters, \( j=1,...,p \):

\[
\text{minimise } \sum_{i=1}^{k} \frac{g_{jl}}{x_{ij}} + \sum_{i \in I_j} c_i \left( \frac{1}{x_{ij}} - (v_j)_{il} \right) x_{ij}^2,
\]

such that \( \sum_{i=1}^{k} p_i^2 x_{ij} \leq w_j \)

and \( x_{ij} \leq \frac{1}{(v_j)_{il}}, i=1,...,k. \)
In the objective function of the above problem, for fixed $I_j$, the term $- \sum_{i \in I_j} c_i (v_j)_{ii} \cdot x_{ij}^2$ is constant with respect to $x_{ij}$. Therefore, instead of solving the above $p$ problems, we may solve the following $p$ problems for $j = 1, \ldots, p$ respectively,

\[
\text{Minimize } \sum_{i=1}^{k} \frac{\bar{c}_{ij}}{x_{ij}} + \sum_{i=1}^{k} \frac{\bar{c}_{ij}}{x_{ij}} + c_{ij} x_{ij}^2
\]

or minimize

\[
\sum_{i=1}^{k} \frac{\bar{c}_{ij}}{x_{ij}}
\]

such that

\[
\sum_{i=1}^{k} \frac{v_j^2}{x_{ij}} \leq w_j,
\]

and

\[
x_{ij} \leq \frac{1}{(v_j)_{ii}}, i=1, \ldots, k,
\]

where

\[
\bar{c}_{ij} = \begin{cases} 
\bar{c}_{ij} & \text{if } i \in I_j \\
\bar{c}_{ij} + c_{ij} \sigma_{ij}^2 & \text{if } i \notin I_j
\end{cases}
\]

The restrictions (5.3.5) prevent the negative values of $n_{ij}$. 
obtained from (5.3.8) be \( c^{(0)} \). The other \( k(p-1) \) configurations of \( I_j \)'s that we want to consider are those which are close to \( I_h^{(0)} \), in the sense that \( I_j \) consists of those indices \( i \) for which \( \sigma_{ij}^2 \) are large.

The procedure consists of \((k-1)\) phases:

Phase 1 proceeds as follows:

In each row of the matrix \( (\sigma_{ij}^2) \) we compute the differences in the maximum and 2nd maximum (i.e. next to the maximum) values of \( \sigma_{ij}^2 \). Let the minimum difference be attained in the \( s_{1}^{(1)} \)th row. Define \( I_h^{(1)} \) such that \( I_h^{(1)} = I_h^{(0)} \), \( k=1, \ldots, p \), except that the row \( s_{1}^{(1)} \) now belongs to \( I_h^{(1)} \), where \( t \) corresponds to that column in the \( s_{1}^{(1)} \)th row where next to the maximum over \( j \) of \( \sigma_{ij}^2 \) is attained. Let the corresponding value of the total cost be \( c^{(1)} \). (We will call the transfer of one index \( i \) from some \( I_h^{(0)} \) to some \( I_h^{(1)} \) as branching).

If \( c^{(1)} < c^{(0)} \), we branch from \( I_h^{(1)} \) otherwise from \( I_h^{(0)} \).

Suppose that \( c^{(1)} < c^{(0)} \). Let the next minimum difference be attained in \( s_{1}^{(2)} \)th row. Then \( I_h^{(2)} = I_h^{(1)} \) except that \( s_{1}^{(2)} \)th
row now belongs to $I^{(2)}_t$ where $t$ corresponds to the column in the $S_1^{(2)}$th row where next to maximum over $j$ of $\sigma_{ji}^{-2}$ is attained. Compute the corresponding value of $a^{(2)}$.

If $c^{(2)} < c^{(1)}$ we branch from $I^{(2)}_h$ and the process repeats. At $i$th step the branching is done from the sets $I^{(q)}_h$ where $q$ is such that

$$c^{(q)} = \min (c^{(1)}, c^{(2)}, ..., c^{(1)}).$$

Phase 2 starts at the $(k+1)$st step. Let $c^{(r)} = \min (c^{(1)}, c^{(2)}, ..., c^{(k)}$ and we compute for each row the differences in the maximum and 3rd maximum value of $c_{ij}^{-2}$. Let the minimum difference be attained in the $S_2^{(1)}$th row. Then $I^{(k+1)}_h = I^{(r)}_h$ except that the $S_2^{(1)}$th row now belongs to $I^{(k+1)}_t$ where $t$ corresponds to the column in the $S_2^{(2)}$th row where the 3rd maximum over $j$ of $c_{ji}^{-2}$ is attained.

Let the corresponding value of $c$ be $c^{(k+1)}$. This is step 1 of Phase 2. The other steps of this phase are parallel to the various steps of phase 1. The remaining phases are same as the second phase.

If the solution so obtained for the problem (5.3.3) and
(5.3.4) also satisfies (5.3.5) the optimal solution is attained. If some of the upper bounds in (5.3.5) are violated for a fixed \( j \) then we fix the corresponding \( x_{ij} \) equal to its upper limit \( \frac{1}{(v_j)_{ii}} \) given by (5.3.5). This amounts not to measure the \( j \)th character at all in the \( i \)th stratum.

5.4. A NUMERICAL EXAMPLE

Consider the following problem which will illustrate the computational details.

Let the population be divided into 4 strata i.e. \( k = 4 \) and 3 characteristics are defined on each unit of the population i.e. \( p = 3 \).

Suppose further that

\[
P = \begin{pmatrix}
0.19 & 0.39 & 0.27 & 0.15 \\
0.01 & 0.25 & 0.15 & 0.25 \\
0.12 & 0.20 & 0.28 & 0.30 \\
0.10 & 0.20 & 0.30 & 0.40
\end{pmatrix},
\]

\[
((e_{ij})) = \begin{pmatrix}
2 & 10 & 360 \\
2 & 16 & 280 \\
3 & 12 & 390 \\
3 & 18.9 & 410
\end{pmatrix}
\]
\[
((\sigma^{-2}_{ij})) = \begin{bmatrix}
15 & 89 & 12 \\
27 & 55 & 9 \\
71 & 46 & 35 \\
50 & 32 & 61
\end{bmatrix}
\]

\[A_1 = D(4.3 \ 6.5 \ 7.2 \ 11.1),\]
\[A_2 = D(10.2 \ 15.3 \ 8.8 \ 9.2),\]
\[A_3 = D(5.6 \ 8.7 \ 8.1 \ 10.3),\]

where \(D(\quad)\) represents a diagonal matrix. Also let
\[w_1 = 0.5961, \ w_2 = 0.8889, \ w_3 = 0.6281.\]
\[o_1 = 15, \ o_2 = 21, \ o_3 = 19, \ o_4 = 25.\]

The maximum of \(\sigma^{-2}_{ij}\)'s are
for \(i = 1\) at \(j = 2\),
for \(i = 2\) at \(j = 2\),
for \(i = 3\) at \(j = 1\),
for \(i = 4\) at \(j = 3\).

Thus we get
The corresponding value of the cost given by (5.2.1) is 
\( c(0) = 9238.1308 \).

The second maximum of \( \sigma_{ij}^2 \)'s are attained

- for \( i=1 \) at \( j=1 \),
- for \( i=2 \) at \( j=1 \),
- for \( i=3 \) at \( j=2 \),
- for \( i=4 \) at \( j=1 \).

**Phase 1:** The differences between the maximum and the second maximum values of \( \sigma_{ij}^2 \)'s are:

- for \( i=1 \), difference = 74,
- for \( i=2 \), difference = 28,
- for \( i=3 \), difference = 25,
- for \( i=4 \), difference = 11.

The minimum difference is 11 which is the difference between \( \sigma_{43}^2 \) and \( \sigma_{41}^2 \). Thus \( I_1^{(1)} = \{ 3, 4 \} \), \( I_2^{(1)} = \{ 1, 2 \} \) and \( I_3^{(1)} = \emptyset \). The corresponding cost \( c^{(1)} \) is 9284.5332 which
is greater than $c^{(o)}$. Therefore, the branching will be done from $I_j^{(o)}$, $j=1,2,3$.

The next minimum difference is 25 corresponding to $\sigma_{21}^{-2}$ and $\sigma_{32}^{-2}$. Thus $I_1^{(2)} = \phi$, $I_2^{(2)} = \{1,2,3\}$ and $I_3^{(2)} = \{4\}$, and the corresponding cost is 8522.6816 which is less than $c^{(o)}$, therefore, the branching will be done from $I_j^{(2)}$, $j=1,2,3$.

The next minimum difference is 28 corresponding to $\sigma_{22}^{-2}$ and $\sigma_{21}^{-2}$. Thus $I_1^{(3)} = \{2\}$, $I_2^{(3)} = \{1,3\}$ and $I_3^{(3)} = \{4\}$ and the corresponding cost $c^{(3)}$ is 9227.8906 which is greater than $c^{(2)}$. Therefore the branching will be done from $I_j^{(2)}$, $j=1,2,3$.

The next minimum difference is 74 corresponding to $\sigma_{12}^{-2}$ and $\sigma_{11}^{-2}$. Thus $I_1^{(4)} = \{1\}$, $I_2^{(4)} = \{2,3\}$ and $I_3^{(4)} = \{4\}$ and the corresponding cost $c^{(4)}$ is 8904.9082 which is greater than $c^{(2)}$. Therefore the branching will be done from $I_j^{(2)}$, $j=1,2,3$.

**Phase II:** The 3rd maximum of $\sigma_{ij}^{-2}$'s are attained

for $i=1$ at $j=3$,
for $i=2$ at $j=3$,
for $i=3$ at $j=3$,
for $i=4$ at $j=2$. 
The difference between the maximum and 3rd maximum values of $\sigma_{ij}^{-2}$ are:

for $i=1$, difference = 77,

for $i=2$, difference = 46,

for $i=3$, difference = 36,

for $i=4$, difference = 29.

The minimum difference is 29 corresponding to $\sigma_{43}^{-2}$ and $\sigma_{42}^{-2}$. Thus $I_1^{(5)} = \phi$, $I_2^{(5)} = \{1,2,3,4\}$ and $I_3^{(5)} = \phi$, and the corresponding cost $c^{(5)} = 8512.7285$ which is less than $c^{(2)}$. Therefore the branching will now be done from $I_j^{(5)}$, $j=1,2,3$.

Similarly we evaluate the solution for $I_j^{(6)}$, $I_j^{(7)}$ and $I_j^{(8)}$. The costs corresponding to them are:

$c^{(6)} = 8626.6523$, $c^{(7)} = 8665.711$ and $c^{(8)} = 8637.2261$ respectively.

The minimum cost is attained for $I_j^{(5)}$, $j=1,2,3$. So our approximate solution obtained by inserting $I_j^{(5)}$ in
(5.3.7) and rounding the corresponding $n_{ij}$'s to the nearest integer is

$$((n_{ij})) = \begin{pmatrix} 5 & 8 & 3 \\ 19 & 19 & 9 \\ 11 & 10 & 7 \\ 5 & 3 & 2 \end{pmatrix}.$$ 

It is also seen by the total enumeration of all the 81 possibilities for $I_j$'s that the above solution is also an optimal solution of the problem. The cost corresponding to the rounded $n_{ij}$'s is 8659.7 units.