CHAPTER II

RATIO AND PRODUCT TYPE EXPONENTIAL ESTIMATORS

2.1 Introduction

In sample surveys, information on an auxiliary variate, which is highly correlated with the variable under study may be readily available and can be judiciously used for improving the precision of the estimators. The data on an auxiliary variate for all individual sampling units may not be available. But the aggregated data can be used at the estimation stage provided the data on an auxiliary variate for the sampled units can be obtained while observing the values of study variate. Let the aim is to estimate the average/total production in an industrial unit, the main variate of interest being the production. However, the information on the auxiliary variates may be available such as availability (and quality) of the raw material, economical production processes, quality control, trained workforce, management etc. may be used. Also to estimate the agriculturable land (study variate) for a region, the information on total population, total number
of cultivators, the number of agricultural labourers may be used as the auxiliary variates.

When the study variate and auxiliary variate(s) are highly correlated, the ratio and product methods of estimation provides an efficient estimator of the population mean/total, with the condition that the regression line passes through the origin. Several authors have worked over the ratio and product type of estimators in different sampling designs so as to improve the precision. I. Olkin (1958) proposed ratio estimator using multi-auxiliary information. Singh M.P. (1965) proposed estimation of ratio and product estimator of population parameters. Singh M.P. (1967) proposed multi-variate product method of estimation. Ray S. K., Singh R.K. (1981) proposed ratio estimator using two auxiliary variables. Bahl S. and Tuteja R.K. (1991) proposed exponential type ratio and product estimators using single auxiliary variable. In this chapter we formulate some sampling strategies using information on two or more auxiliary variables. A new ratio and product type exponential estimators is proposed with sampling scheme simple random sampling without replacement (SRSWOR). Their bias
(B) and mean square error (M) is compared with other estimator and is found to be efficient in many practical situations.

2.2 Ratio and Product Type Exponential Estimators

In this section, we propose ratio and product type exponential estimators using information on two-auxiliary variables.

Let for a finite (survey) population \( P \), a collection of known number \( N \) of identifiable units labeled as 1, 2, 3, ..., \( N \); \( P = \{1,2,3,...,i,...,N\} \), where 'i' stands for the physical unit labeled 'i'. Let 'y' be the study variable having value \( y_i \) on \( i = 1,2,...,N \). Associated with the population \( P \), we have a vector of real numbers \( y = (y_1, y_2,.., y_N) \) which constitutes the parameter space. The parameter, population mean

\[
\bar{Y} = \frac{\sum_{i=1}^{N} y_i}{N}
\]

is to be estimated by selecting a sample from population \( P \) and observing the value of study variable \( y \) only on the units in the sample.

Let a simple random sample of size \( n \) be drawn without replacement from the finite population \( P \), and observe the study variate
y, auxiliary variates \( x_1 \) and \( x_2 \). Let \( \bar{y}, \bar{x}_1 \) and \( \bar{x}_2 \) be the sample means and \( \bar{Y}, \bar{X}_1, \bar{X}_2 \) be the population means, with sample mean as an unbiased estimator of population mean respectively. The usual ratio and product estimator be as follows

\[
\bar{y}_R = \frac{\bar{y}}{\bar{x}} \quad \text{and} \quad \bar{y}_p = \frac{\bar{y}}{\bar{X}}
\]

Further, to the first degree of approximation i.e. upto terms of order \( 1/n \), the bias and means square error is given as

\[
B(\bar{y}_R) = (1 - f) \frac{\text{RS}_x^2 - S_{yx}}{nX} \quad \text{and} \quad \bar{y}_p = \frac{\bar{y}}{\bar{X}} \quad \frac{[C_1^2 - \rho_{01} C_0 C_1]}{n}
\]

\[
M[\bar{y}_R] = (1 - f) \frac{\bar{Y}^2 [C_0^2 + C_1^2 - 2\rho_{01} C_0 C_1]}{n}
\]

(2.2.1)

\[
B(\bar{y}_p) = (1 - f) \frac{S_{yx}}{\bar{X} \bar{Y}}
\]
\[ M[\bar{y}_p] = \left( \frac{1-f}{n} \right) \bar{Y}^2 \left[ C_0^2 + C_1^2 + 2\rho_{01}C_0C_1 \right] \]

\[ = \left( \frac{1-f}{n} \right) \left[ S_y^2 + R^2S_x^2 + 2RS_{yx} \right] \]  \hspace{1cm} (2.2.2)

\[ M[\bar{y}] = \left( \frac{1-f}{n} \right) \bar{Y}^2 C_0^2 = \left( \frac{1-f}{n} \right) S_y^2 \]

where \( f = (n/N) \), \( C_0^2 = \frac{S_y^2}{\bar{Y}^2} \), \( C_1^2 = \frac{S_x^2}{\bar{X}^2} \), \( R = (\bar{Y}/\bar{X}) \)

\[ S_y^2 = \frac{1}{N} \sum_{i=1}^{N} (y_i - \bar{Y})^2, \quad S_x^2 = \frac{1}{N} \sum_{i=1}^{N} (x_i - \bar{X})^2 \]

\[ \rho_{01} = \frac{1}{(N-1)S_yS_x} \sum_{i=1}^{N} (y_i - \bar{Y})(x_i - \bar{X}) \]

\[ S_{xy} = \frac{1}{(N-1)} \sum_{i=1}^{N} (y_i - \bar{Y})(x_i - \bar{X}) \]

\( C_0, C_1 \) are coefficients of variation of \( \bar{Y} \) and \( \bar{X} \), \( \rho_{01} \) be the correlation coefficient between \( \bar{Y} \) and \( \bar{X} \).

For estimating the population mean \( \bar{Y} \), the exponential type of estimator defined by S. Bahl and R.K. Tuteja(1991) as
\[ \bar{y}_{Re} = y \exp \left[ \frac{X_1 - x_1}{X_1 + x_1} \right] \]  
(2.2.3)

\[ \bar{y}_{Pe} = y \exp \left[ \frac{x_1 - X_1}{x_1 + X_1} \right] \]  
(2.2.4)

Let

\[ e_0 = \frac{\bar{y} - \bar{Y}}{\bar{Y}}, \quad e_1 = \frac{\bar{x}_1 - \bar{X}_1}{\bar{X}_1} \]

so that \( E(e_0) = E(e_1) = 0, \quad E(e_0^2) = \frac{(1 - f)}{n} C_0^2 \),

\[ E(e_1^2) = \frac{(1 - f)}{n} C_1^2, \quad E(e_0 e_1) = \frac{(1 - f)}{n} \rho C_0 C_1 \]

Further, it is assumed that the sample is large enough to make \( |e_0| \) and \( |e_1| \) so small that the terms involving \( e_0 \) and \( e_1 \) up to degree higher than two are negligible. Hence,

\[ B[ \bar{y}_{Re}] = \frac{(1 - f)}{n} \bar{Y} \left[ \frac{3 C_1^2}{8} - \frac{\rho_{01} C_0 C_1}{2} \right] \]  
(2.2.5)

\[ B[ \bar{y}_{Pe}] = \frac{(1 - f)}{n} \bar{Y} \left[ \frac{\rho_{01} C_0 C_1}{2} - \frac{C_1^2}{8} \right] \]  
(2.2.6)
For estimating of the population total $Y$, the usual ratio estimator is given by $\hat{Y}_R = (\hat{Y}/\hat{X}_1)X_1'$. Singh M.P. (1967) proposed the estimators using two auxiliary variables

$$\hat{Y}_{R_1}^* = \hat{Y}_R (\hat{X}_2/X_2)$$

$$\hat{Y}_{R_2}^* = \hat{Y}_R (X_2/\hat{X}_2)$$

Hence, to the first degree approximation, mean square error be given as

$$M[\hat{Y}_{R_1}^*] = M[\hat{Y}_R] + Y^2[C_2^2+2\rho_{02}C_0C_2 - 2\rho_{12}C_1C_2]$$

$$M[\hat{Y}_{R_2}^*] = M[\hat{Y}_R] + Y^2[C_2^2-2\rho_{02}C_0C_2 + 2\rho_{12}C_1C_2]$$

where $M[\hat{Y}_R] = Y^2[C_0^2+C_1^2 - 2\rho_{01}C_0C_1]$


$$\hat{Y}_{R_0} = (1+\theta)\hat{Y}_R - \theta\hat{Y}_{R_1}^*$$

where $\theta$ is a scalar constant which can be suitably chosen (depending on the value of $C_0$ or $C_1$). It may be observed that $\hat{Y}_{R_0}$ reduces to
\( \hat{Y}_r \) and \( \hat{Y}_{r_1}^* \) for \( \theta = 0 \) and \( \theta = -1 \) respectively. Hence, to the first degree approximation

\[
M[\hat{Y}_{r_0}] = M[\hat{Y}_r] + Y^2[\theta^2C_2^2 - 2\theta \rho_{02}C_0C_2 + 2\rho_{12}C_1C_2] \tag{2.2.14}
\]

for \( \theta = 2C_1 + 1 \), \( M[\hat{Y}_{r_0}] < M[\hat{Y}_{r_{S1}}] \)

and for \( \theta = 2C_1 - 1 \), \( M[\hat{Y}_{r_0}] < M[\hat{Y}_{r_{S2}}] \)

Singh M.P. (1965) proposed multivariate product estimator

\[
\hat{Y}_{MP1}^* = \hat{Y}_p \left( \hat{X}_2 / \hat{X}_1 \right) \tag{2.2.15}
\]

\[
\hat{Y}_{MP2}^* = \hat{Y}_p \left( \hat{X}_2 / \hat{X}_1 \right) \tag{2.2.16}
\]

where \( \hat{Y}_p = (\hat{Y} / \hat{X}_1) / \hat{X}_1 \)

Hence to the first degree of approximation, mean square error be given as

\[
M[\hat{Y}_{MP1}^*] = M[\hat{Y}_p] + \frac{(1-f)}{n} \bar{Y}^2 [C_2^2 + 2\rho_{02}C_0C_2 + 2\rho_{12}C_1C_2] \tag{2.2.17}
\]

\[
M[\hat{Y}_{MP2}^*] = M[\hat{Y}_p] + \frac{(1-f)}{n} \bar{Y}^2 [C_2^2 - 2\rho_{02}C_0C_2 - 2\rho_{12}C_1C_2] \tag{2.2.18}
\]

where \( M[\hat{Y}_p] = \frac{(1-f)}{n} \bar{Y}^2 [C_0^2 + C_1^2 + 2\rho_{01}C_0C_1] \)
Proposed ratio and product type estimators

We propose the ratio and product type estimators using two auxiliary variables for estimating the population mean $\bar{Y}$ as

\[
\bar{y}^{*}_{Re1} = \frac{\bar{y}_{Re} \bar{X}_2}{\bar{X}_2} \tag{2.2.19}
\]

\[
\bar{y}^{*}_{Re2} = \frac{\bar{y}_{Re} - \bar{x}_2}{\bar{X}_2} \tag{2.2.20}
\]

for

\[
\bar{y}_{Re} = \bar{y} \exp \left[ \frac{\bar{X}_1 - \bar{x}_1}{\bar{X}_1 + \bar{x}_1} \right] \tag{2.2.21}
\]

and

\[
\bar{y}^{*}_{Pe1} = \frac{\bar{y}_{Pe} \bar{X}_2}{\bar{X}_2} \tag{2.2.22}
\]

\[
\bar{y}^{*}_{Pe2} = \frac{\bar{y}_{Pe} - \bar{x}_2}{\bar{X}_2} \tag{2.2.23}
\]

for

\[
\bar{y}_{Pe} = \bar{y} \exp \left[ \frac{\bar{x}_1 - \bar{X}_1}{\bar{x}_1 + \bar{X}_1} \right] \tag{2.2.24}
\]

We propose an alternative ratio and product estimator as
\[ \bar{y}_{RZ} = (1+Z) \quad \bar{y}_{Re} - Z \quad \bar{y}_{Rel} \quad (2.2.25) \]

and

\[ \bar{y}_{PZ} = (1+Z) \quad \bar{y}_{Pe} - Z \quad \bar{y}_{Pel} \quad (2.2.26) \]

respectively. \( Z \) is a scalar constant which can be suitably chosen (depending on the value of \( C_0 \) or \( C_1 \) to make \( \bar{y}_{RZ} \) and \( \bar{y}_{PZ} \) more efficient than \( \bar{y}_{Re}, \bar{y}_{Rel}, \bar{y}_{Re2} \) and \( \bar{y}_{Pe}, \bar{y}_{Pel}, \bar{y}_{Pe2} \) respectively.

**Bias and Mean Square Error of the Proposed Estimator**

Let \( \bar{y} = \bar{Y}(1+e_0) \), \( \bar{x}_1 = \bar{X}_1(1+e_1) \), \( \bar{x}_2 = \bar{X}_2(1+e_2) \) with \( E(e_0) = E(e_1) = E(e_2) = 0 \), \( C_2 \) is the coefficient of variation of \( \bar{X}_2 \) and \( \rho_{02}, \rho_{12} \) be the correlation coefficients between the \( (\bar{Y}, \bar{X}_2) \) and \( (\bar{X}_1, \bar{X}_2) \) respectively.

\[ E(e_0^2) = \frac{(1-f)}{n} C_0^2, \quad E(e_1^2) = \frac{(1-f)}{n} C_1^2, \]

\[ E(e_2^2) = \frac{(1-f)}{n} C_2^2, \quad E(e_0 e_1) = \frac{(1-f)}{n} \rho_{01} C_0 C_1 \]

\[ E(e_0 e_2) = \frac{(1-f)}{n} \rho_{02} C_0 C_2, \quad E(e_1 e_2) = \frac{(1-f)}{n} \rho_{12} C_1 C_2 \]

\[ E(\bar{x}_2 - \bar{X}_2)^2 = \frac{(1-f)}{n} S_{x_2}^2, \quad S_{x_1}^2 = \frac{1}{(N-1)} \sum_{i=1}^{N} (x_{1i} - \bar{X}_1)^2 \]
\begin{align*}
S^2_{x_2} &= \frac{1}{(N-1)} \sum_{i=1}^{N} (x_{2i} - \bar{X}_2)^2 \\
S^2_{x_1} &= \frac{S^2_{x_2}}{\bar{X}_1^2} \quad , \quad C_1^2 = \frac{S^2_{x_1}}{\bar{X}_1^2} \\
C_2^2 &= \frac{S^2_{x_2}}{\bar{X}_2^2} \\
\rho_{01} &= \frac{1}{(N-1)S_y S_{x_1}} \sum_{i=1}^{N} (y_i - \bar{Y})(x_{1i} - \bar{X}_1) \\
\rho_{02} &= \frac{1}{(N-1)S_y S_{x_2}} \sum_{i=1}^{N} (y_i - \bar{Y})(x_{2i} - \bar{X}_2) \\
\rho_{12} &= \frac{1}{(N-1) S_{x_1} S_{x_2}} \sum_{i=1}^{N} (x_{1i} - \bar{X}_1)(x_{2i} - \bar{X}_2)
\end{align*}

It is assumed that the sample is large enough to make \( e_0, e_1 \) and \( e_2 \) so small that the terms involving \( e_0, e_1 \) and \( e_2 \) up to degree higher than two are negligible. Substituting the expression for \( \bar{y}, \bar{x}_1 \) and \( \bar{x}_2 \) in terms of \( e_0, e_1 \) and \( e_2 \) in

\[ y^*_{Re_1}, \quad y^*_{Re_2}, \quad y^*_{Pe_1}, \quad y^*_{Pe_2}, \quad y^*_{RZ} \] and \( y^*_{PZ} \) we get

\[
B[ \bar{y}^*_{Re_1}] = \frac{1-f}{n} \bar{Y} \left[ -\frac{C_1^2}{8} + \frac{C_2^2}{2} \right] + \frac{\rho_{01} C_0 C_1}{2} + \frac{\rho_{12} C_1 C_2}{2} - \rho_{02} C_0 C_2
\]

(2.2.27)
\[
B[ \bar{y}^*_{Re2}] = \frac{1-f}{n} \bar{V} \left[ \frac{3}{8} C_1 \frac{\rho_{01} C_0 C_1}{2} - \frac{\rho_{12} C_1 C_2}{2} + \rho_{02} C_0 C_2 \right]
\]

(2.2.28)

\[
B[ \bar{y}^*_{Pe1}] = \frac{1-f}{n} \bar{V} \left[ C_2 \frac{C_1^2}{8} + \frac{\rho_{01} C_0 C_1}{2} - \frac{\rho_{12} C_1 C_2}{2} - \rho_{02} C_0 C_2 \right]
\]

(2.2.29)

\[
B[ \bar{y}^*_{Pe2}] = \frac{1-f}{n} \bar{V} \left[ - \frac{C_1^2}{8} + \frac{\rho_{01} C_0 C_1}{2} + \frac{\rho_{12} C_1 C_2}{2} + \rho_{02} C_0 C_2 \right]
\]

(2.2.30)

\[
M[ \bar{y}^*_{Re1}] = \frac{1-f}{n} \bar{V}^2 \left[ C_0^2 + \frac{C_1^2}{4} + C_2^2 - \rho_{01} C_0 C_1 - 2\rho_{02} C_0 C_2 + \rho_{12} C_1 C_2 \right]
\]

(2.2.31)

\[
M[ \bar{y}^*_{Re2}] = \frac{1-f}{n} \bar{V}^2 \left[ C_0^2 + \frac{C_1^2}{4} + C_2^2 - \rho_{01} C_0 C_1 + 2\rho_{02} C_0 C_2 - \rho_{12} C_1 C_2 \right]
\]

(2.2.32)

\[
M[ \bar{y}^*_{Pe1}] = \frac{1-f}{n} \bar{V}^2 \left[ C_0^2 + \frac{C_1^2}{4} + C_2^2 + \rho_{01} C_0 C_1 + 2\rho_{02} C_0 C_2 - \rho_{12} C_1 C_2 \right]
\]

(2.2.33)

\[
M[ \bar{y}^*_{Pe2}] = \frac{1-f}{n} \bar{V}^2 \left[ C_0^2 + \frac{C_1^2}{4} + C_2^2 + \rho_{01} C_0 C_1 + 2\rho_{02} C_0 C_2 + \rho_{12} C_1 C_2 \right]
\]

(2.2.34)
\[
B[\bar{y}^*_{RZ}] = (\frac{1-f}{n}) \bar{Y} \left[ \frac{(1-\frac{C_1^2}{8} - \frac{ZC_2^2}{2})^{\frac{3}{2}}}{2} \right] + \rho_{01}C_0C_1 + Z\rho_{12}C_1C_2
\]

(2.2.35)

\[
B[\bar{y}^*_{PZ}] = (\frac{1-f}{n}) \bar{Y} \left[ \frac{(1-\frac{C_1^2}{8} - \frac{ZC_2^2}{2})^{\frac{3}{2}}}{2} \right] + \rho_{01}C_0C_1 + Z\rho_{12}C_1C_2
\]

(2.2.36)

\[
M[\bar{y}^*_{RZ}] = (\frac{1-f}{n}) \bar{Y}^2 \left[ C_0^2 + \frac{C_1^2}{4} + ZC_2^2 - \rho_{01}C_0C_1 + Z(2\rho_{02}C_0C_2 + \rho_{12}C_1C_2) \right]
\]

(2.2.37)

\[
M[\bar{y}^*_{PZ}] = (\frac{1-f}{n}) \bar{Y}^2 \left[ C_0^2 + \frac{C_1^2}{4} + ZC_2^2 + \rho_{01}C_0C_1 + Z(2\rho_{02}C_0C_2 + \rho_{12}C_1C_2) \right]
\]

(2.2.38)

Hence comparing with usual ratio and product estimator

\[
M[\bar{y}^*_{Rel}] - M[\bar{y}_R] = (\frac{1-f}{n}) \bar{Y}^2 \left[ C_2^2 - 0.75C_1^2 - 2\rho_{02}C_0C_2 + \rho_{12}C_1C_2 \right]
\]

(2.2.39)

\[
M[\bar{y}^*_{Rel_2}] - M[\bar{y}_R] = (\frac{1-f}{n}) \bar{Y}^2 \left[ C_2^2 - 0.75C_1^2 + \rho_{01}C_0C_1 + 2\rho_{02}C_0C_2 - \rho_{12}C_1C_2 \right]
\]

(2.2.40)

\[
M[\bar{y}^*_{Pel}] - M[\bar{y}_P] = (\frac{1-f}{n}) \bar{Y}^2 \left[ C_2^2 - 0.75C_1^2 - 2\rho_{02}C_0C_2 \right]
\]
\[ + \rho_{12}C_1C_2 - 3\rho_{01}C_0C_1 \]  

\[ (2.2.41) \]

\[ M[ \bar{y}^*_{pe2}] - M[ \bar{y}_p] = \frac{1-f}{n} \bar{Y}^2 \left[ C_2^2 - 0.75C_1^2 + 2\rho_{02}C_0C_2 + \rho_{12}C_1C_2 - \rho_{01}C_0C_1 \right] \]  

\[ (2.2.42) \]

Comparing the bias and mean square error of the proposed estimators with estimator proposed by S. Bahl (1991),

\[ B[\bar{y}^*_{R_e}] - B[\bar{y}_{Re}] = \frac{(1-f)}{n} \bar{Y} \left[ C_2^2 + (0.5)\rho_{12}C_1C_2 - \rho_{02}C_0C_2 \right] \]  

\[ (2.2.43) \]

\[ B[\bar{y}^*_{Re2}] - B[\bar{y}_{Re}] = \frac{(1-f)}{n} \bar{Y} \left[ - (0.5)\rho_{12}C_1C_2 + \rho_{02}C_0C_2 \right] \]  

\[ (2.2.44) \]

\[ B[\bar{y}^*_{pe1}] - B[\bar{y}_{pe}] = \frac{(1-f)}{n} \bar{Y} \left[ C_2^2 - (0.5)\rho_{12}C_1C_2 - \rho_{02}C_0C_2 \right] \]  

\[ (2.2.45) \]

\[ B[\bar{y}^*_{pe2}] - B[\bar{y}_{pe}] = \frac{(1-f)}{n} \bar{Y} \left[ (0.5)\rho_{12}C_1C_2 + \rho_{02}C_0C_2 \right] \]  

\[ (2.2.46) \]

\[ M[\bar{y}^*_{Re1}] - M[\bar{y}_{Re}] = \frac{(1-f)}{n} \bar{Y}^2 \left[ C_2^2 - 2\rho_{02}C_0C_2 + \rho_{12}C_1C_2 \right] \]  

\[ (2.2.47) \]

\[ M[\bar{y}^*_{Re2}] - M[\bar{y}_{Re}] = \frac{(1-f)}{n} \bar{Y}^2 \left[ C_2^2 + 2\rho_{02}C_0C_2 - \rho_{12}C_1C_2 \right] \]  

\[ (2.2.48) \]

\[ M[\bar{y}^*_{pe1}] - M[\bar{y}_{pe}] = \frac{(1-f)}{n} \bar{Y}^2 \left[ C_2^2 + 2\rho_{02}C_0C_2 - \rho_{12}C_1C_2 \right] \]  

\[ (2.2.49) \]

\[ M[\bar{y}^*_{pe2}] - M[\bar{y}_{pe}] = \frac{(1-f)}{n} \bar{Y}^2 \left[ C_2^2 + 2\rho_{02}C_0C_2 + \rho_{12}C_1C_2 \right] \]  

\[ (2.2.50) \]
Comparing with Ray and Singh (1981)

\[
M[ \bar{y}_{R1}^*] - M[ \bar{y}_{RS1}^*] = \frac{(1-f)}{n} \bar{Y}^2 \left[ \rho_{01}C_0C_1 - 4\rho_{02}C_0C_2 + 3\rho_{12}C_1C_2 - (0.75)C_1^2 \right]
\] (2.2.51)

\[
M[ \bar{y}_{R2}^*] - M[ \bar{y}_{RS2}^*] = \frac{(1-f)}{n} \bar{Y}^2 \left[ \rho_{01}C_0C_1 - 3\rho_{12}C_1C_2 - (0.75)C_1^2 \right]
\] (2.2.52)

And for product estimator, Singh M.P. (1965), comparing

\[
M[ \bar{y}_{P1}^*] - M[ \bar{y}_{MP1}^*] = \frac{(1-f)}{n} \bar{Y}^2 \left[ -\rho_{01}C_0C_1 - 3\rho_{12}C_1C_2 - (0.75)C_1^2 \right]
\] (2.2.53)

\[
M[ \bar{y}_{P2}^*] - M[ \bar{y}_{MP2}^*] = \frac{(1-f)}{n} \bar{Y}^2 \left[ -\rho_{01}C_0C_1 + 3\rho_{12}C_1C_2 - (0.75)C_1^2 + 4\rho_{02}C_0C_2 \right]
\] (2.2.54)

\[
M[ \bar{y}_{RZ}^*] - M[ \bar{y}_R^*] = \frac{(1-f)}{n} \bar{Y}^2 \left[ Z^2C_2^2 + \rho_{01}C_0C_1 - (0.75)C_1^2 + Z(2\rho_{02}C_0C_2 - \rho_{12}C_1C_2) \right]
\] (2.2.55)

\[
M[ \bar{y}_{PZ}^*] - M[ \bar{y}_P^*] = \frac{(1-f)}{n} \bar{Y}^2 \left[ Z^2C_2^2 - \rho_{01}C_0C_1 - (0.75)C_1^2 + Z(2\rho_{02}C_0C_2 + \rho_{12}C_1C_2) \right]
\] (2.2.56)

2.3 Multivariate Ratio and Product Type Exponential Estimators

Olkin I. (1958) proposed multivariate ratio estimation for finite populations
\[ \bar{y} = \sum_{i=1}^{p} w_i r_i \bar{X}_i \]  

(2.3.1)

\[ w = (w_1, w_2, \ldots, w_p), \quad \Sigma w_i = 1 \] is a weighting function and \( r_i = \frac{\bar{y}}{x_i} \), hence with optimum weight \( w \), is given by

\[ \hat{w} = \frac{eA^{-1}}{eA^{-1}e'} \]

\[ \bar{Y} (eA^{-1}b'). \]

then \( E(\bar{Y}) = \frac{\bar{Y}}{n (eA^{-1}e')} + \bar{Y} \)

(2.3.2)

\[ V(\bar{Y}) = \frac{\bar{Y}^2}{n eA^{-1}e'} \]

(2.3.3)

when weights are uniform, then \( \hat{w} = e/p \)

\[ E(\bar{y}) = \bar{Y} + \bar{Y} \left( \frac{eb'}{np} \right) \]

(2.3.4)

\[ V(\bar{y}) = \frac{\bar{Y}^2 k}{np} \]

(2.3.5)

As \( C_1 = C_2 \ldots C_p = C, \) \( \rho_{01} = \rho_{02} \ldots = \rho_{0p} = \rho_0, \) \( \rho_{ij} = \rho (i \neq j) \)

\[ \text{So } E(\bar{y}) = \bar{Y} + \left( \frac{(N-n) \bar{Y}}{N n} \right) \left( \frac{C^2 - \rho_0 C_0 C}{C^2} \right) \]

(2.3.6)

\[ V(\bar{y}) = \frac{(N-n) \bar{Y}^2}{N pn} \left[ C^2 (1-\rho) + p(C_0^2 - 2\rho_0 C_0 C + \rho C^2) \right] \]

(2.3.7)
In addition, $C_0 = C$, $\rho_0 = \rho$, then

$$E(\bar{Y}) = \bar{Y} + \frac{(N-n)\bar{Y}}{N \ n} C^2 (1-\rho) \quad (2.3.8)$$

$$V(\bar{Y}) = \frac{(N-n) \bar{Y}^2}{N \ p n} C^2 (1-\rho) (1+\rho) \quad (2.3.9)$$

Also Singh M.P. (1967) proposed multivariate product method of estimation, using $k$ supplementary variables $x_1, x_2, \ldots, x_k$. The multivariate product estimator

$$\hat{Y}_p = \sum_{i=1}^{k} \frac{w_i \ p_i}{X_i} \quad (2.3.10)$$

where $p_i = yx_i$ and $w_i$'s are weights such that $\sum_{i=1}^{k} w_i = 1$. Then the exact expression for the bias and mean square error of $\hat{Y}_p$ as

$$B[\hat{Y}_p] = Ywb'$$

$$M[\hat{Y}_p] = Y^2 w(A+B+C)w'$$

where $A = (a_{ij})$, $B = (b_{ij})$, $C = (c_{ij})$ are matrices of order $k \times k$ each, $b$ and $w$ are vectors $(b_1, b_2, \ldots, b_k)$ and $(w_1, w_2, \ldots, w_k)$ respectively.

$$a_{ij} = (V_0^2 + V_{oi}^{11} + V_{oj}^{11} + V_{ij}^{11})$$
\[ b_{ij} = (V_{oi}^{21} + 2V_{oij}^{111} + V_{oj}^{21}) \]

\[ c_{ij} = (V_{oij}^{211}) \]

\[ V_{oij}^{rst} = \frac{E[(y-Y)(x_i-Y)(x_j-Y)^t]}{Y^tX_iX_j} \]

The terms involving \( n^{-2} \) and \( n^{-3} \) in mean square error may be neglected for large values of \( n \), then

\[ M[\hat{Y}_{MP}] = Y^2wAw' \] (2.3.11)

Following the procedure used by Olkin (1958) for determination of optimum weights, we have

\[ w = \frac{eA^{-1}}{e' A^{-1} e}, e = (1, 1, \ldots, k \times 1), A^{-1} \text{ is matrix inverse of } A. \]

Assuming weights for all the supplementary variables are uniform which will happen only when the sums of each column of matrix \( A \) are equal to optimum \( \hat{w} \), given by \( e/k \) then corresponding bias and mean square error is

\[ B[\hat{Y}_r] = Y(eb'/k), \]

\[ M[\hat{Y}_r] = Y^2(d/k) \]

where \( d \) is the scalar such that \( eA = ed, (d \neq 0) \). For uniform weights and \( C_i = C, \rho_{oi} = \rho_0, \rho_{ij} = \rho (i \neq j) \), gives
\[ B[\hat{Y}_p] = Y \rho_0 C_0 C \]

\[ M[\hat{Y}_p] = [C^2(1-\rho) + k(C_0^2 + 2\rho_0 C_0 C + \rho C^2)] Y^2 / k \]

where \( C_{ij} = V_{ij}^{11} \approx \rho_{ij} C_i C_j \), where \( C_0, C_i, C_j \) being the coefficient of variation of the estimates \( y, x_i \) and \( x_j \) respectively. If \( \rho = \rho_0, C_0 = C \), then

\[ B[\hat{Y}_p] = Y \rho C^2 \quad (2.3.12) \]

\[ M[\hat{Y}_p] = Y^2 [k+1 + \rho(3k-1)] C^2 / k \quad (2.3.13) \]

For a finite population we propose the ratio and product type estimators using multi-auxiliary variables for estimating the population mean \( \bar{Y} \) as

\[ \bar{Y}_{ReM} = \frac{1}{k} \sum_{t=1}^{k} \frac{\bar{X}_t - \bar{x}_i}{\bar{X}_t + \bar{x}_i} \exp \left[ \frac{-1}{\bar{X}_t + \bar{x}_i} \right] \quad (2.3.14) \]

\[ \bar{Y}_{PeM} = \frac{1}{k} \sum_{t=1}^{k} \frac{\bar{x}_t - \bar{X}_t}{\bar{x}_t + \bar{X}_t} \exp \left[ \frac{-1}{\bar{x}_t + \bar{X}_t} \right] \quad (2.3.15) \]

where \( w_i \) be the weights such that \( \sum_{t=1}^{k} w_i = 1 \).
Let \( \bar{y} = \bar{Y}(1+e_0), \bar{x}_i = \bar{X}_i(1+e_i) \), \( E(e_0) = E(e_i) = 0 \), \( C_0, C_1 \) be the coefficients of variation of \((\bar{Y}, \bar{X}_i)\), \( \rho_{0i} \) is the correlation coefficient between study variable and auxiliary variable \( x_i \) for \( t = 1, 2, \ldots, k \) and

\[
1 - f \quad 1 - f
\]
\[
E(e_i^2) = \left( \frac{n}{\bar{x}_i^2} \right) C_i^2, \quad E(e_0 e_i) = \left( \frac{n}{\bar{x}_i^2} \right) \rho_{0i} C_0 C_i
\]

\[
1 - f \quad S_{x_i}^2
\]
\[
E(e_i e_i') = \left( \frac{n}{\bar{x}_i^2} \right) \rho_{ii'} C_i C_i', \quad C_i^2 = \left( \frac{\bar{x}_i^2}{\bar{x}_i^2} \right)
\]

\[
\rho_{0i} = \frac{1}{(N-1)S_y S_{x_i}} \sum_{i=1}^{N} (x_{ii} - \bar{X}_i) (y_i - \bar{Y})
\]

\[
E(\bar{x} - \bar{X}_i)^2 = \left( \frac{1}{n} \right) S_{x_i}^2
\]

\[
S_{x_i}^2 = \frac{1}{(N-1)} \sum_{i=1}^{N} (x_{ii} - \bar{X}_i)^2
\]

\[
\rho_{ii'} = \frac{1}{(N-1)S_{x_i} S_{x_i'}} \sum_{i=1}^{N} (x_{ii} - \bar{X}_i) (x_{i'i} - \bar{X}_{i'})
\]

It is assumed that the sample is large enough to make \( e_0, e_i \) so small that the terms involving \( e_0, e_i \) up to degree higher than two are negligible.
Substituting the expression for $\bar{y}$, $x_t$ in terms of $e_0$, $e_t$ and $e$ is the identity matrix.

$$w' = [w_1 \ w_2 \ w_3 \ ... \ w_k]_{1 \times k}$$

$$e' = [1 \ 1 \ 1 \ ... \ 1]_{1 \times k}$$

**Bias and Mean Square Error of the Proposed Estimator**

The expression for bias and mean square error of the proposed estimators be given as

$$B[\bar{y}_{Rem}] = \left(\frac{1 - f}{n}\right) \bar{Y} \left[ - \sum_{t=1}^{3} w_t C_t^2 - \sum_{t=1}^{1} w_t \rho_{\omega t} C_0 C_t \right]$$ (2.3.16)

$$M[\bar{y}_{Rem}] = \left(\frac{1 - f}{n}\right) \bar{Y}^2 \left[ \sum_{t=1}^{4} w_t C_t^2 + \sum_{t=1}^{4} w_t w_{t'} + \sum_{t \neq t'}^{4} \rho_{\omega t} C_t C_{t'} \right]$$ (2.3.17)

$$= \left(\frac{1 - f}{n}\right) \bar{Y}^2 [w' A_{Re} w]$$

$$A_{Re} = \{a_{\omega t}\} = \{C_0^2 - \frac{\rho_{\omega t} C_0 C_t}{2} - \frac{\rho_{\omega t} C_0 C_{t'}}{2} + \frac{\rho_{\omega t} C_t C_{t'}}{4}\}$$ (2.3.18)

t, t' = 1, 2, ..., k.
[\begin{align*}
B[\bar{y}_{pem}] &= \left( \frac{1-f}{n} \right) \bar{Y} \left[ \sum_{t=1}^{k} \frac{\rho_{0t} C_0 C_t}{2} + \sum_{t=1}^{k} \frac{C_t^2}{8} \right] \\
M[\bar{y}_{pem}] &= \left( \frac{1-f}{n} \right) \bar{Y} \left[ C_0^2 + \sum_{t=1}^{k} \frac{w_t \rho_{0t} C_0 C_t}{2} + \sum_{t=1}^{k} \frac{w_t^2 C_t^2}{4} + \sum_{t \neq t'} \frac{\rho_{tt'} C_t C_{t'}}{4} \right] \\
&+ \left( \frac{1-f}{n} \right) \bar{Y} \left[ w' A_{pe} w \right] \\
A_{pe} &= \{ a_{tt'} \} = \left\{ \frac{\rho_{0t} C_0 C_t}{2} + \frac{\rho_{0t} C_0 C_{t'}}{2} + \frac{\rho_{tt'} C_t C_{t'}}{4} \right\}
\end{align*}]

Choice of a weight function

Following the procedure used by I. Olkin (1958) for
determination of optimum weights it can be easily established that

\[ w_t = \frac{\text{sum of elements in } i^{th} \text{ column of } A^{-1}}{\text{sum of all } k^2 \text{ elements in } A^{-1}} \]

The weight vector \( w' \), \( \sum_{t=1}^{k} w_t = 1 \) then the optimum weights for ratio

\[ \hat{w}_R \] and product \( \hat{w}_p \) estimators are
\[
\hat{\omega}_R = \frac{A_{Re}^{-1}e}{e'A_{Re}^{-1}e}, \quad \hat{\omega}_p = \frac{A_{Pe}^{-1}e}{e'A_{Pe}^{-1}e}
\]

Assuming that weights for all the supplementary variables are uniform which happens only when the sum of each column of matrix \(A_{Re}\) (or \(A_{Pe}\)) are equal, then the optimum weights \(\hat{\omega}_R\) (or \(\hat{\omega}_p\)) is given by \(e/k\), hence corresponding bias and mean square error be given as

\[
B[\bar{y}_{Re}] = \bar{Y} \left( \frac{1-f}{n} \right) \frac{b_R A_{Re}^{-1}e}{e'A_{Re}^{-1}e} \tag{2.3.22}
\]

\[
b_R = \left[ \begin{array}{c}
\frac{3C_1^2}{8} - \frac{\rho_{01}C_0C_1}{2}, \quad \frac{3C_2^2}{8} - \frac{\rho_{02}C_0C_2}{2}, \quad \ldots, \quad \frac{3C_k^2}{8} - \frac{\rho_{0k}C_0C_k}{2}
\end{array} \right]_{1 \times k}
\]

\[
M[\bar{y}_{Re}] = \left( \frac{1-f}{n} \right) \frac{\bar{Y}^2}{e'A_{Re}^{-1}e} \tag{2.3.23}
\]

\[
B[\bar{y}_{Pe}] = \bar{Y} \left( \frac{1-f}{n} \right) \frac{b_p A_{Pe}^{-1}e}{e'A_{Pe}^{-1}e} \tag{2.3.24}
\]

\[
b_p = \left[ \begin{array}{c}
\frac{\rho_{01}C_0C_1}{2} - \frac{C_1^2}{8}, \quad \frac{\rho_{02}C_0C_2}{2} - \frac{C_2^2}{8}, \quad \ldots, \quad \frac{\rho_{0k}C_0C_k}{2} - \frac{C_k^2}{8}
\end{array} \right]_{1 \times k}
\]

\[
M[\bar{y}_{Pe}] = \left( \frac{1-f}{n} \right) \frac{\bar{Y}^2}{e'A_{Pe}^{-1}e} \tag{2.3.25}
\]
From equation (2.3.22) and (2.3.24) bias is zero if

\[ b_{RA}e^{-1}e = 0, \text{this will hold if } b_R = 0 \text{ and correspondingly} \]

\[ b_{Pe}e^{-1}e = 0, \text{this will hold if } b_p = 0. \]

Hence for \( b_R = 0 \)

\[ \frac{3C_t^2}{8} - \frac{\rho_0C_0C_t}{2} = 0, \]

\[ \bar{Y} = \frac{4\rho_0 \bar{X} \bar{S}_y}{3S_{xt}} \]

and \( b_p = 0 \) if

\[ \frac{\rho_0C_0C_t}{2} - \frac{C_t^2}{8} = 0, \]

\[ \bar{Y} = \frac{4\rho_0 \bar{X} \bar{S}_y}{S_{xt}} \]

Considering the particular case, i.e. for uniform weights and \( C_0 = C_1 = C, \rho_0t = \rho_0, \quad \rho_{tt'} = \rho, \quad (t \neq t') \)

\[ A_{Re} = [C^2 - 0.5\rho_0C^2 - 0.5\rho_0C^2 + 0.25\rho C^2] = C^2 [1 - \rho_0 + 0.25\rho] \]

\[ A_{Pe} = [C^2 + 0.5\rho_0C^2 + 0.5\rho_0C^2 + 0.25\rho C^2] = C^2 [1 + \rho_0 + 0.25\rho] \]

\[ B[\bar{Y}_{ReM}] = \frac{(1-f)C^2}{\bar{Y} - [0.75 - \rho_0]} \quad (2.3.26) \]
Considering the particular case where the coefficients of variation of \(x_1, x_2, \ldots, x_k\) are equal to \(C\), there is the same correlation \(\rho\) between \(y\) and \(x_t\) i.e. \(C_0 = C_1 = \ldots C_k = C\), \(\rho_{01} = \rho_{02} = \ldots = \rho_{0k} = \rho_{t'} = \rho\), \(t, t' = 1, 2, \ldots k\), then
\[
A_{Re} = C^2 [1 - 0.75\rho]
\]
\[
A_{Pc} = C^2 [1 + 1.25\rho]
\]
Weights will be uniform iff column sums of \(A_{Re}\) (or \(A_{Pc}\)) are equal such that \(\hat{w}_R = e / k\), \(\hat{w}_P = e / k\)

Hence for optimum conditions
\[
B_{opt} [\bar{y}_{ReM}] = \frac{1-f}{n} \bar{Y} \left[ \frac{1}{b_R \hat{w}_R} \right]
\]
\[
= \frac{1-f}{n} \frac{C^2}{\bar{Y} \left( \frac{1}{2} \right)} [0.75 - \rho]
\]
\[ V_{opt} [\bar{y}_{ReM}] = \left( \frac{1-f}{n} \right) \bar{Y}^2 \frac{C^2}{k} \{(k+0.25) - \rho(k-0.25)\} \] (2.3.31)

\[ B_{opt} [\bar{y}_{PeM}] = \bar{Y}\left( \frac{1-f}{n} \right) b_p \hat{\omega}_p \]

\[ \frac{1-f}{n} \frac{C^2}{2} \]

\[ V_{opt} [\bar{y}_{PeM}] = \left( \frac{1-f}{n} \right) \bar{Y}^2 \frac{C^2}{k} \{(1+\rho)(k+0.25)\} \] (2.3.33)

For ratio estimator

\[ \frac{1-f}{n} \bar{Y}C^2 = U \]

<table>
<thead>
<tr>
<th>( \rho )</th>
<th>( B_{opt} [\bar{y}_{ReM}] )</th>
<th>( M_{opt} [\bar{y}_{ReM}] )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( k = 1 )</td>
<td>( k = 2 )</td>
</tr>
<tr>
<td>1</td>
<td>-U/8</td>
<td>U ( \bar{Y}/2 )</td>
</tr>
<tr>
<td>.75</td>
<td>0</td>
<td>U ( \bar{Y}(11/16) )</td>
</tr>
<tr>
<td>0.5</td>
<td>U/8</td>
<td>U ( \bar{Y}(7/8) )</td>
</tr>
<tr>
<td>0.25</td>
<td>U/4</td>
<td>U ( \bar{Y}(17/16) )</td>
</tr>
<tr>
<td>0</td>
<td>3U/8</td>
<td>U ( \bar{Y}(5/4) )</td>
</tr>
</tbody>
</table>
For product estimator

\[ 1 - f \]

Let \( \frac{1}{n} \bar{Y}C^2 = T \)

<table>
<thead>
<tr>
<th>( \rho )</th>
<th>( B_{\text{opt}[\bar{Y}_{\text{PeM}}]} )</th>
<th>( M_{\text{opt}[\bar{Y}_{\text{PeM}}]} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>-5T/8</td>
<td>0</td>
</tr>
<tr>
<td>-0.75</td>
<td>-T/2</td>
<td>T \bar{Y}(5/16)</td>
</tr>
<tr>
<td>-0.5</td>
<td>-3T/8</td>
<td>T \bar{Y}(5/8)</td>
</tr>
<tr>
<td>-0.25</td>
<td>-T/4</td>
<td>T \bar{Y}(15/16)</td>
</tr>
<tr>
<td>0</td>
<td>-T/8</td>
<td>T \bar{Y}(5/4)</td>
</tr>
</tbody>
</table>

In case for two auxiliary variables, \( k = 2 \)

\[
B[\bar{Y}_{\text{Re2}}] = \left( \frac{1 - f}{n} \right) \frac{3}{8} \bar{Y} \left[ (w_1^2C_1^2 + w_2^2C_2^2) - \frac{(w_1\rho_{01}C_0 + w_2\rho_{02}C_0C_2)}{2} \right]
\]

\[ (2.3.34) \]

\[
M[\bar{Y}_{\text{Re2}}] = \left( \frac{1 - f}{n} \right) \frac{1}{4} \bar{Y}^2 \left[ C_0^2 + \frac{(w_1^2C_1^2 + w_2^2C_2^2)}{4} - (w_1\rho_{01}C_0 + w_2\rho_{02}C_0C_2) \right]
\]
Relative performance of the proposed estimators

Hence comparing with usual ratio estimator

\[ 1-f \]
\[ M[\bar{y}_{R1}^*]-M[\bar{y}_R]=(\ldots) \bar{Y}^2[C_2^2-0.75C_1^2+p_{01}C_0C_1-2p_{02}C_0C_2+p_{12}C_1C_2] \]
\[ n \]
\[ (2.2.38) \]

\[ 1-f \]
\[ M[\bar{y}_{R2}^*]-M[\bar{y}_R]=(\ldots) \bar{Y}^2[C_2^2-0.75C_1^2+p_{01}C_0C_1+2p_{02}C_0C_2+p_{12}C_1C_2] \]
\[ n \]
\[ (2.2.39) \]

\[ 1-f \]
\[ M[\bar{y}_{RZ}^*]-M[\bar{y}_R]=(\ldots) \bar{Y}^2[Z^2C_2^2-0.75C_1^2+p_{01}C_0C_1+Z(2p_{02}C_0C_2-p_{12}C_1C_2)] \]
\[ n \]
\[ (2.2.40) \]

with product estimator
\[ M[\bar{y}_{RZ}^*] - M[\bar{y}_{Rel}] = (1-f) \frac{\bar{Y}^2}{n} [(Z^2 - 1) C_2^2 + 2(Z + 1) \rho_{02} C_0 C_2] - \rho_{12} C_1 C_2] < 0. \]
(Z+1) C_2^2 [(Z-1) + (2 \rho_{02} C_0/C_2) - (\rho_{12} C_1/C_2)] < 0

Let C_\alpha = (2 \rho_{02} C_0/C_2) - (\rho_{12} C_1/C_2)

\therefore (Z+1) C_2^2 [(Z-1) + C_\alpha] < 0

\therefore Z lies between -1 and 1-C_\alpha,

choosing Z < -1, we have C_\alpha < Z - 1, hence for

C_\alpha < Z - 1, Z < 1 - C_\alpha or 1+C_\alpha < Z < 1 - C_\alpha.

\[ M[\bar{y}_{RZ}] - M[\bar{y}_R] = \frac{(1-f)}{\bar{Y}^2} [Z^2 C_2^2 - (3/4) C_1^2 + \rho_{01} C_0 C_1 - Z \rho_{12} C_1 C_2 + 2Z \rho_{02} C_0 C_2] < 0 \]

\[ [Z^2 C_2^2 - \frac{3}{4} C_1^2 + \rho_{01} C_0 C_1 + Z C_2^2 C_\alpha] < 0 \]

either Z < 0 or C_2^2(Z+C_\alpha) < \frac{(0.75) C_1^2 - \rho_{01} C_0 C_1}{Z}

\[ \frac{C_2^2}{C_1^2} < \frac{C_\beta}{Z(Z+C_\alpha)} \]

where C_\beta = (0.75) - \rho_{01} \frac{C_0}{C_1}

If Z = -C_\alpha, then \[ \frac{[(0.75) - \rho_{01} C_0/C_1]}{C_\alpha} < 0 \]
Concluding Remarks

1. The proposed estimators \( \bar{y}^{*}_{\text{Rel}}, \bar{y}^{*}_{\text{Re}2}, \bar{y}^{*}_{\text{Pe}1} \) and \( \bar{y}^{*}_{\text{Pe}2} \) are better than the usual estimators. The proposed estimators \( \bar{y}^{*}_{\text{Rel}} \) and \( \bar{y}^{*}_{\text{PZ}} \) are shown to be better in many practical situations. The difference of mean square error of \( \bar{y}^{*}_{\text{Pe}1} \) and \( \bar{y}^{*}_{\text{MPI}} \) comes out to be negative.

2. Using optimum weights, bias for ratio estimator is zero if 
\[
(0.75) C_{t}^{2} = \rho_{0t} C_{0} C_{t} \quad \text{and the bias for product estimator is zero if}
\]
\[
C_{t}^{2} = 4 \rho_{0t} C_{0} C_{t} .
\]

3. In case of \( C_{0} = C_{t} = C, \rho_{0t} = \rho_{0} = \rho_{tt} = \rho \), bias of ratio estimator is zero when correlation coefficient \( \rho = 0.75 \). For product estimator bias is zero when correlation coefficient \( \rho = 0.25 \). It is observed that as the number of auxiliary variables are increased the mean square error is decreased.

4. Mean square error of \( \bar{y}^{*}_{\text{RZ}} \) is less than mean square error of \( \bar{y}^{*}_{\text{Rel}} \) when \( Z \) lies in the interval \( (1 + C_{\alpha}, 1 - C_{\alpha}) \) where
\[
C_{\alpha} = \frac{2 \rho_{02} C_{0} - \rho_{12} C_{1}}{C_{2}} .
\]
5. Also mean square error of $\bar{y}_{RZ}$ is less than the mean square error of $\bar{y}_R$ when either $Z < 0$ or

$$
\frac{C_2^2(Z + C_a)}{C_1^2} < \frac{(0.75 - \frac{\rho_0 C_0}{C_1})}{Z}
$$
or

$$
Z < \frac{(0.75 - \frac{\rho_0 C_0}{C_1})C_1^2}{C_2^2(Z + C_a)}
$$

$$
Z(Z + C_a) < \frac{C_1^2}{C_2^2} (0.75 - \frac{\rho_0 C_0}{C_1})
$$

If $Z = -C_a$, then $0.75 - \frac{\rho_0 C_0}{C_1} > 0$
or

$$
\frac{\rho_0 C_0}{C_1} < (0.75).
$$

**Empirical Study**

To study the efficiency of the proposed estimators, consider population I of 1991 census data of Rohtak district with 487 villages. Here number of cultivators in the village is taken as $y_1$, number of agricultural labourers as $x_1$, population of village as $x_2$.

To observe the efficiency of ratio-type exponential estimators, the population mean
\[ \bar{Y} = 449.846, \quad \bar{X}_1 = 158.3265, \quad \bar{X}_2 = 2909.105 \]

\[ R = \frac{\bar{Y}}{\bar{X}_1} = \frac{449.846}{158.3265} = 2.84125 \]

\[ \rho_{YX_1} = 0.7166773, \quad C_0^2 = 0.787, \quad C_0 = 0.88713 \]

\[ \rho_{YX_2} = 0.881815, \quad C_1^2 = 0.9988, \quad C_1 = 0.9994 \]

\[ \rho_{X_1X_2} = 0.8099, \quad C_2^2 = 0.59226, \quad C_2 = 0.7696 \]

\[ \rho_{01C_0C_1} = 0.63540, \quad \rho_{02C_0C_2} = 0.6020462 \]

\[ \rho_{12C_1C_2} = 0.623 \]

Let \( \frac{(1-f)}{n} \bar{Y} = A \)

### Table 1

<table>
<thead>
<tr>
<th>Estimator</th>
<th>Bias(B)</th>
<th>MSE(M)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \bar{Y} )</td>
<td>0</td>
<td>A ( \bar{Y}(0.787) )</td>
</tr>
<tr>
<td>( \bar{Y}_R )</td>
<td>A(0.3834)</td>
<td>A ( \bar{Y}(0.515) )</td>
</tr>
<tr>
<td>( \bar{Y}_{Re} )</td>
<td>A(0.5685)</td>
<td>A ( \bar{Y}(0.4013) )</td>
</tr>
<tr>
<td>( \bar{Y}^*_{Re1} )</td>
<td>A(0.9710846)</td>
<td>A ( \bar{Y}(0.4124631) )</td>
</tr>
<tr>
<td>( \bar{Y}^*_{Re2} )</td>
<td>A(0.3474)</td>
<td>A ( \bar{Y}(1.574648) )</td>
</tr>
<tr>
<td>( \bar{Y}^*_{RZ} )</td>
<td>A(0.8701913)</td>
<td>A ( \bar{Y}(0.3491) )</td>
</tr>
<tr>
<td>( \bar{Y}_{Re2} )</td>
<td>A(0.1247344)</td>
<td>A ( \bar{Y}(-0.1019) )</td>
</tr>
</tbody>
</table>

\( w_1 = 1.5, \ w_2 = -0.5 \)
Also from equation (2.2.31) and (2.2.32) i.e. comparing with S.K. Ray and R.K. Singh (1991) estimators

\[
M[\tilde{Y}_{Re1}] - M[\tilde{Y}_{Rk1}] = \frac{(1-f)}{n} \bar{Y}^2 [-.65288507]
\]

\[
M[\tilde{Y}_{Re2}] - M[\tilde{Y}_{Rk2}] = \frac{(1-f)}{n} \bar{Y}^2 [-.4254845307]
\]

Population II, on the basis of investigation by Biometry Research unit of ISI (Indian Statistical Institute) Calcutta. To study the multivariate investigation of Blood Chemistry, data collected on 32 variables for three groups of individuals (findings given by Das 1966). ‘Eosinophil’ content be the study variable \( y \) based on data for group C collected on 69 individuals. The auxiliary variables are \( x_1 \) (height) and \( x_2 \) (weight) of the individuals

\[C_0 = 0.60 \quad \rho_{01} = -0.1752\]

\[C_1 = 0.033 \quad \rho_{02} = -0.2505\]

\[C_2 = 0.28 \quad \rho_{12} = 0.0099\]

\[\rho_{01}C_0C_1 = -.00346896, \quad \rho_{02}C_0C_2 = -.042084\]

\[\rho_{12}C_1C_2 = .000091476\]

The efficiency of product-type exponential estimators
<table>
<thead>
<tr>
<th>Estimator</th>
<th>Bias(B)</th>
<th>MSE(M)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \bar{y} )</td>
<td>0</td>
<td>( \bar{Y}(0.36) )</td>
</tr>
<tr>
<td>( \bar{y}_P )</td>
<td>(-0.00346896)</td>
<td>( \bar{Y}(0.35415108) )</td>
</tr>
<tr>
<td>( \bar{y}_{Pe} )</td>
<td>(-0.060242)</td>
<td>( \bar{Y}(0.3535) )</td>
</tr>
<tr>
<td>( \bar{y}_{*\text{pe}1} )</td>
<td>(0.118567657)</td>
<td>( \bar{Y}(0.350943814) )</td>
</tr>
<tr>
<td>( \bar{y}_{*\text{pe}2} )</td>
<td>(-0.04390886)</td>
<td>( \bar{Y}(3.351126766) )</td>
</tr>
</tbody>
</table>
| \( \bar{y}_{*\text{RZ}} \)  
\( Z = 0.1 \) | \(0.013914405\)  | \( \bar{Y}(0.3491797) \) |
| \( \bar{y}_{\text{Pe}2} \)  
\( w_1 = -0.4, w_2 = 1.4 \) | \(-0.02363\) | \( \bar{Y}(0.3409) \) |

For the proposed population I, the estimator \( \bar{y}_{*\text{Re}2} \) has less bias as compared to usual ratio estimator \( \bar{y}_R \) but mean square error is manifold times. The bias of \( \bar{y}_{*\text{RZ}} \) is much greater than \( \bar{y}_R, \bar{y}_{\text{Re}} \) and \( \bar{y}_{*\text{Re}2}, \) also the mean square error of estimator \( \bar{y}_{*\text{RZ}} \) is very much less than \( \bar{y}_R, \bar{y}_{\text{Re}} \) and \( \bar{y}_{*\text{Re}2}. \)

The case of ratio estimator using only two auxiliary variables \( \bar{y}_{\text{Re}2} \) with weights \( w_1 = 1.5, w_2 = -0.5 \) bias and mean square both are very small as compared to other proposed estimators.
In table 2, for population II, the bias of usual product estimator $\bar{y}_p$ is less as compared to other proposed estimators but $\text{MSE}(\bar{y}_p) > \text{MSE}(\bar{y}_{pe}) > \text{MSE}(\bar{y}_{pe2}) > \text{M}(\bar{y}_{pe1}) > \text{M}(\bar{y}_{pz}) > \text{M}(\bar{y}_{pe2})$.

For estimator $\bar{y}_p$ and $\bar{y}_{pe2}$,

$$|\text{Bias } \bar{y}_p| < |\text{Bias } \bar{y}_{pe2}|$$

but mean square error

$$\text{MSE}(\bar{y}_p) > \text{MSE}(\bar{y}_{pe2})$$

for weights $w_1 = -0.4$ and $w_2 = 1.4$. 