CHAPTER-VI

USE OF MULTIVARIATE AUXILIARY INFORMATION THROUGH TWO-STAGE STRATIFIED SRSWOR SAMPLING IN ESTIMATING POPULATION PARAMETERS

6.1 Introduction

As seen in the previous chapters the double sampling scheme is applied to get better results. When the sampling frame within the strata and strata weights is not known, hence the cluster sampling is applied. Usually the sampled clusters are to be completely enumerated. Due to economic considerations, it may not be economically viable, so a sample of units in each sampled clusters is surveyed.

In sample surveys, the most common design is stratified multi-stage sampling. When the sampling frame within strata and strata weights is known then better results may be achieved through two stage stratified sampling scheme.
Let each unit in the population can be divided into a number of smaller units, or subunits. A sample of n units called cluster has been selected and measure a sample of the subunits in any chosen unit (cluster). This technique is called subsampling or cluster sampling or two-stage sampling. The sample is taken in two stages i.e. first is to select a sample of units called primary units (first stage units) and the second is to select a sample of subunits (second stage units) from each chosen primary unit. The situation arise when the list for all the units in the population may not exists but the list for groups of units (may be called clusters) exists.

Let for a finite (survey) population \( P \) is a collection of a known number \( N \) of identifiable units labeled 1, 2, ..., i, ..., N ; \( P = \{1,2,\ldots,i,\ldots,N\} \), where i stands for the physical unit labeled i. Let 'y' be a study variable having value \( y_i \) on i ( = 1,2,\ldots, N). Associated with \( P \), we have a vector of real numbers \( y = (y_1, y_2,\ldots, y_N) \) which constitutes the parameter for the model of a survey population, \( y \in \mathbb{R}^N \), the parameter space. The parameter population mean \( \bar{Y} = \frac{\sum_{i=1}^{N} y_i}{N} \) is to be estimated by selecting a sample from \( P \) through the scheme two-
stage stratified sampling and observing the value of $y$ only on the units in the sample.

For a finite population, if the supplementary information of one or more of the characteristics related to the values of survey variable (study variable) is known for each sampling unit then the population can be stratified before selection. Stratification is applied to the first stage units $fsus$ and within each stratum a sample of $fsus$ is selected SRSWOR, and each of the selected $fsus$ is further sub-sampled.

Let the population be divided into $L$ strata's with $N_t$ number of first-stage units ($fsus$) in the $t$-th stratum, so that

$$
\sum_{t=1}^{L} N_t = N
$$

Let $M_{i i}$ be the number of second-stage units ($ssus$) in the $i$-th $fsu$ of the $t$-th stratum, with $M_{i 0}$ be the total number of ssus in the $t$-th stratum,

$$
M_{i 0} = \sum_{i=1}^{N_t} M_{i i} = N_t \overline{M}_t
$$

Let $n_t$ be the number of $fsus$ selected by using SRSWOR from the $t$-th stratum, so that
and $m_{ti}$ be the number of ssus to be sampled from the i-th selected fsu for the t-th stratum using SRSWOR. The auxiliary information may be observed either at the first stage or at the second stage.

a) When auxiliary information is collected at the first stage.

Let a sample $s(1)_{i}$ of $n_{i}$ fsus is drawn SRSWOR and the auxiliary information for $p$ auxiliary variables is collected $\{x_{kii}\}$ for $k = 1,2,...,p$ $t = 1,2,...,L, i = 1,2,...,n_{i}$. Out of $n_{i}$ fsus a sub-sample $s(2)$ of $m_{ti}$ ssus is selected SRSWOR and study variable $\{y_{tij}\}$ is observed $j = 1,2,...,m_{ti}$.

b) When auxiliary information is collected at the second stage.

Let a sample $s(1)$ of $n_{i}$ fsus is drawn SRSWOR and out of $n_{i}$ fsus a sub-sample $s(2i)$ of $m_{ti}$ ssus is drawn SRSWOR from each selected fsus and observe the study variable, auxiliary variables $\{y_{tij}, x_{kij}\}$, $t = 1,2,...,L, k = 1,2,...,p, i = 1,2,...,n_{i}, j = 1,2,...,m_{ti}$.

The population and sample parameters are

a) Population total

$$Y = \sum_{t=1}^{L} W_{t} \bar{Y}_{t}, \quad Y_{t} = \sum_{i=1}^{N_{t}} \sum_{j=1}^{M_{ti}} y_{tij} = \sum_{i=1}^{N_{t}} M_{ti} \bar{Y}_{ti}.$$
\[
\bar{Y}_{ij} = \frac{1}{M_{ti}} \sum_{j=1}^{N_i} y_{tij} \\
X_k = \sum_{t=1}^{L} W_t x_{kt}, \quad \bar{X}_{kti} = \frac{1}{M_{ti}} \sum_{j=1}^{N_i} x_{ktij} \quad (\text{at fsu}) \\
N_t \quad M_{ti} \quad N_i \\
X_{kti} = \sum_{i=1}^{N_t} \sum_{j=1}^{M_{ti}} x_{ktij} = \sum_{i=1}^{N_t} \sum_{j=1}^{M_{ti}} \bar{X}_{kti} \quad (\text{at ssu}) \\
1 \quad M_{ti} \\
\bar{X}_{kti} = \frac{1}{M_{ti}} \sum_{j=1}^{N_i} x_{ktij} \\
\text{and sample total} \\
L \\
y = \sum_{t=1}^{L} W_t y_t, \quad y_t = \sum_{i=1}^{n_t} \sum_{j=1}^{m_{ti}} y_{tij} = \sum_{i=1}^{n_t} m_{ti} \quad y_{tii} \\
_1 \quad m_{ti} \\
y_{tii} = \frac{1}{m_{ti}} \sum_{j=1}^{n_t} y_{tij} \\
L \\
x_k = \sum_{t=1}^{L} W_t x_{kt}, \quad x_{kt} = \sum_{i=1}^{n_t} x_{kti} \quad (\text{at fsu}) \\
n_t \quad m_{ti} \quad n_t \\
x_{kti} = \sum_{i=1}^{n_t} \sum_{j=1}^{m_{ti}} x_{ktij} = \sum_{i=1}^{n_t} \sum_{j=1}^{m_{ti}} x_{kti} \quad (\text{at ssu}) \\
_1 \quad m_{ti} \\
x_{kti} = \frac{1}{m_{ti}} \sum_{j=1}^{n_t} x_{ktij}
b) Population mean (per fsu)

\[
\bar{Y} = \sum_{t=1}^{L} W_t \bar{Y}_t, \quad \bar{Y}_t = Y_t/N_t,
\]

\[
\bar{X}_{kl} = x_{kl}/N_{kl}, \quad \bar{X}_k = \sum_{t=1}^{L} W_t x_{kt}
\]

(\text{per ssu}) \ Y = \sum_{t=1}^{L} W_t Y_t,

\[
X_{kl} = x_{kl}/M_{t0} \quad X_k = \sum_{t=1}^{L} W_t x_{kt}
\]

\[Y_t = \text{the population mean per second stage unit in the } t\text{-th stratum} = \]

\[Y_t/M_{t0}\]

\[
f_t = \frac{n_t}{N_t}, \quad f_{2t} = \frac{m_{ti}}{M_{ti}}, \quad W_t = \frac{N_t}{N}
\]

\[M_{t0} = \sum_{i=1}^{L} M_{ii} = N_t \bar{M}_t, \quad M_0 = \sum_{t=1}^{L} M_{t0}, \]

Let

\[
W_t = \frac{N_t \bar{M}_t}{\sum_{t=1}^{L} M_{t0}} = \frac{M_{t0}}{M_0}
\]
is the relative size of the stratum in terms of second stage units.

Sample mean (per fsu)

\[ y = \sum_{t=1}^{L} W_t y_t, \quad y_t = y_t/n_t, \]

\[ x = \sum_{t=1}^{L} W_t x_{kt}, \quad x_{kt} = x_{kt}/n_t \]

and the sample mean (per ssu) based on cluster totals is given by

\[ y = \sum_{t=1}^{L} W_t y_t, \]

\[ x = \sum_{t=1}^{L} W_t x_{kt}, \quad x_{kt} = x_{kt}/M_{t0} \]

\( y_t \) = the sample mean per second stage unit in the t-th stratum = \( y_t/M_{t0} \)

The population parameters may be written as

\[ \Omega(y) = \sum_{t=1}^{L} W_t \phi_t(y) \]

\[ \phi_t(y) = (1/N_t) \sum_{i=1}^{N_t} Z_{ti} \bar{Y}_{ti}. \]
\[
\phi_t(y) = \begin{cases} 
Y_t & \text{if } Z_{ti} = N_t M_{ti} \\
\bar{Y}_t & \text{if } Z_{ti} = M_{ti} \\
Y_t & \text{if } Z_{ti} = N_t M_{ti}/M_{t0}
\end{cases}
\]  
(6.1.3)

With the observation \(\{y_{tij}\}\), an unbiased estimator for \(\phi_t(y)\) may be defined as

\[
\hat{\phi}(y) = y_{iz} = \left(1/n_t\right) \sum_{i=1}^{n_t} Z_{ti} - y_{ti}.
\]  
(6.1.4)

where

\[
\bar{y}_{tij} = \left(1/m_{ti}\right) \sum_{j=1}^{m_{ti}} y_{tij}
\]

is an unbiased estimator of \(\bar{Y}_{tij}\). Since sample is drawn from the t-th stratum in two stages. We have.

\[
E[\hat{\phi}_t(y)] = E_1 E_2[\hat{\phi}_t(y)]
\]

\[
V[\hat{\phi}_t(y)] = V_1 E_2[\hat{\phi}_t(y)] + E_1 V_2[\hat{\phi}_t(y)]
\]  
(6.1.5)

where \(E_1, V_1\) is the expected value and variance over first stage and \(E_2, V_2\) is the conditional expected value and conditional variance over second stage with the condition that a particular sample is selected at the first stage.
Let \( u_{ti} = Z_{ti} \bar{y}_{ti} \), \( \bar{u}_t = (1/n_t) \sum_{i=1}^{n_t} u_{ti} \).

\[
X_{ktZ} = \frac{1}{N_t} \sum_{i=1}^{N_t} Z_{ti} \bar{X}_{kTi}.
\]

\[
\bar{U}_t = \frac{1}{N_t} \sum_{i=1}^{N_t} Z_{ti} \bar{Y}_{ti}.
\]

Then the variance of \( y_{tz} \) is given as follows.

\[
V[ y_{tz} ] = \frac{1-f_t}{n_t} S_{ut}^2 + \frac{1}{n_t N_t} \sum_{i=1}^{n_t} Z_{ti}^2 \frac{1-f_{2t}}{m_{ti}} S_{iti}^2
\]  \hspace{1cm} (6.1.6)

where

\[
S_{ut}^2 = \left[ 1/(N_t-1) \right] \sum_{i=1}^{N_t} (u_{ti} - \bar{U}_t)^2
\]

\[
S_{iti}^2 = \left[ 1/(m_{ti}-1) \right] \sum_{j=1}^{m_{ti}} (y_{tij} - \bar{Y}_{ti})^2
\]

Hence the separate estimator is

\[
\Omega(y) = \sum_{t=1}^{L} W_t \hat{\phi}_t(y)
\]  \hspace{1cm} (6.1.7)

where
Let $a_{kt}$ be the weights such that $\sum_{k=1}^{p} a_{kt} = 1$. Also

$$E[\hat{\Omega}(y)] = \Omega(y), \quad E[\hat{\phi}_t(y)] = \phi_t(y)$$

with variance

$$V[\hat{\Omega}(y)] = \sum_{t=1}^{L} W_t^2 V[\hat{\phi}_t(y)]$$

(6.1.9)

The information on the auxiliary variables for the sampled units may be used in constructing the estimators for the population parameters. The auxiliary information may be observed at the first stage or at the second stage.

In this chapter, we propose different methods of estimation utilizing the multivariate auxiliary information in the following sub cases:

a) When the auxiliary information is available for fsus only and $M_{t0}$ is known.

b) When the auxiliary information is available for ssus only and $M_{t0}$ is known.
c) Estimation of $Y_1$ when $M_{i0}$ is not known.

6.2. When information is collected at first stage

Let the information on auxiliary variables be available at the first stage only. Let a sample $s(1)$ of $n_1$ fsus is selected using SRSWOR and the auxiliary information for $p$ variables is observed $\{x_{kt}\}$, where $k = 1, 2, \ldots, p$, $t = 1, 2, \ldots, L$, $i = 1, 2, \ldots, n_1$. Out of $n_1$ fsus a sub-sample $s(2)$ of $m_{ii}$ ssus is selected and the character of main study $\{y_{ij}\}$ is observed $j = 1, 2, \ldots, m_{ii}$. The multivariate auxiliary information is utilized to define estimator as for the parameters.

$$\hat{\phi}_i(y) = \sum_{k=1}^{p} a_{kt} d_{kt}$$

$$\hat{\phi}_i(y) = \begin{cases} \hat{\phi}_{iDM}(y) & \text{if } d_{kt} = \bar{y}_{tz} - \lambda_{kt} (\bar{x}_{kt} - \bar{X}_{kt}) \quad k = 1, 2, \ldots p \\ \hat{\phi}_{iRM}(y) & \text{if } d_{kt} = \bar{y}_{tz} \bar{X}_{kt} / \bar{x}_{kt} \quad k = 1, 2, \ldots p \\ \hat{\phi}_{iPM}(y) & \text{if } d_{kt} = \bar{y}_{tz} \bar{x}_{kt} / \bar{X}_{kt} \\ \hat{\phi}_{iRPM}(y) & \text{if } d_{kt} = \bar{y}_{tz} \bar{X}_{kt} / \bar{x}_{kt} \quad k = q+1, q+2, \ldots, p \end{cases}$$

$$= \bar{y}_{tz} \bar{x}_{kt} / \bar{X}_{kt} \quad k = q+1, q+2, \ldots, p$$
\( \lambda_{kl} 's \) be the suitably chosen constants, \( \sum_{k=1}^{p} a_{kl} = 1 \) is a weighing function, 

\( a_i = (a_{1i}, a_{2i}, \ldots, a_{pi}) \). Let let \( x_{1i}, x_{2i}, \ldots, x_{qi} \) be those values of \( x_{kl} \) in \( \phi_{RPM} \) which are positively correlated with \( y \) and \( x_{(q+1)i}, x_{(q+2)i} \ldots x_{pi} \) be those values of \( x_{kl} \) which are negatively correlated with \( y \).

The biases and mean square errors of the proposed estimators are calculated as follows.

As \( d_{kt} \) is an unbiased estimator of \( \phi_t (y) \) i.e.

\[
E[d_{kt}] = E_1 E_2 [d_{kt}] = \phi_t (y)
\]

then difference estimator \( \phi_{IDM}(y) \) is an unbiased estimator of \( \phi_{IDM}(y) \) and its variance is giving by

\[
V[\phi_{IDM}(y)] = \left( \frac{1-f_1}{n_t} \right) a'_t B_t a_t + \frac{1-f_2}{n_t N_t} \sum_{i=1}^{N_t} Z_{ti}^2 \left( \frac{1}{m_{ti}} \right) S_{ti}^2.
\]  

(6.2.3)

where \( B_t = b_{kt} S_{ut}^2 - \lambda_{kt} S_{ukt} - \lambda_{ht} S_{uh} + \lambda_{kt} \lambda_{ht} S_{kh} \)

and

\[
S_{ukt} = \sum_{i=1}^{N_t} (u_{iti} - \bar{U}_t) (x_{kii} - \bar{X}_{ki}) / (N_t - 1)
\]

\[
S_{ut} = \sum_{i=1}^{N_t} (u_{iti} - \bar{U}_t)^2 / (N_t - 1)
\]
\[ S_{kl} = \sum_{i=1}^{N_i} (x_{ki} - \bar{x}_{ki}) (x_{li} - \bar{x}_{li}) / (N_i - 1) \]

for \( k, l = 1, 2, 3, \ldots, p \).

Also it may be written as

\[ V[\phi_{IRM}(y)] = a'_i D_i a_i = \sum_{k, l=1}^{p} a_{ki} a_{li} \text{Cov}(d_{ki}, d_{li}) \]

where \( D_i = d_{kl} = \text{Cov}(d_{ki}, d_{li}) \)

and \( \text{Cov}[d_{ki}, d_{li}] = V[\bar{y}_{iz}] - \lambda_{ki} \text{Cov}[\bar{y}_{iz}, \bar{x}_{ki}] - \lambda_{li} \text{Cov}[\bar{y}_{iz}, \bar{x}_{li}] + \lambda_{ki} \lambda_{li} \text{Cov}[\bar{x}_{ki}, \bar{x}_{li}] \)

\[ 1 - f_i \]

\[ = \left( \frac{1}{n_i} \right) [S_{ul}^2 - \lambda_{kl} S_{ukl} - \lambda_{lt} S_{ult} + \lambda_{kl} \lambda_{lt} S_{kl}] \]

\[ + \frac{1}{n_i N_i} \sum_{i=1}^{N_i} \frac{1 - f_{2t}}{m_i} Z_{it}^2 \left( \frac{1}{\bar{x}_{kl}} \right) S_{iit}^2 \]

An approximation to the bias (B) and mean square error (M) of \( \hat{\phi}_{IRM}(y) \) for large samples be given as

\[ \text{B}[\hat{\phi}_{IRM}(y)] = \left( \frac{1}{n_i} \right) \sum_{k=1}^{p} \frac{a_{kt}}{\bar{x}_{kt}} (R_{kt} S_{kt}^2 - S_{ukt}) \]

\[ \text{M}[\hat{\phi}_{IRM}(y)] = V[\hat{\phi}_{IRM}(y)] \text{ with } \lambda_{kt} = R_{kt} = \frac{\phi_i(y)}{\bar{x}_{kt}} \]
Similarly the bias and mean square error for product and ratio-cum-product estimators be given as

\[
S_{kt} = \frac{N_i}{\sum_{i=1}^{\frac{(x_{kt} - \bar{X}_{kt})^2}} (N_i - 1)}
\]

\[
B[\hat{\phi}_{iPM}(y)] = \frac{1 - f_i}{n_t} \sum_{k=1}^{p} a_{kt} \bar{X}_{kt}
\]

\[
B[\hat{\phi}_{iRPM}(y)] = \frac{1 - f_i}{n_t} \sum_{k=1}^{q} \left( R_{kt} S_{kt}^2 - S_{ukt} \right) + \sum_{k=q+1}^{p} S_{ukt}
\]

\[
M[\hat{\phi}_{iPM}(y)] = V[\hat{\phi}_{iDM}(y)] \text{ with } \lambda_{kt} = -R_{kt}
\]

\[
M[\hat{\phi}_{iRPM}(y)] = V[\hat{\phi}_{iDM}(y)] \text{ with } \lambda_{kt} = R_{kt}, \lambda_{lt} = R_{lt}
\]

For \( k, l = 1, 2, \ldots, q \),

\[
\lambda_{kt} = -R_{kt}, \lambda_{lt} = -R_{lt}
\]

for \( k, l = q+1, q+2, \ldots, p \)

\[
\lambda_{kt} = R_{kt}, \lambda_{lt} = -R_{lt}
\]

for \( k = 1, 2, \ldots, q, l = q+1, \ldots, p \)

The variance of \( \Omega(y) \) is

\[
V[\Omega(y)] = \sum_{t=1}^{L} W_t^2 V[\hat{\phi}_{iDM}]
\]
and mean square error for ratio estimator be given as

\[
M[\Omega(y)] = \sum_{t=1}^{L} W_t^2 M[\hat{\phi}_{\text{RM}}]
\]  

(6.2.13)

Hence for product, ratio-cum-product estimators can be easily computed.

**Optimum Estimation:**

For the case of \( p = 1 \), when the information on only one auxiliary variable is available, say \( x_{kt} \), then

\[
V[\hat{\phi}_{\text{IDM}(y)}] = \left(\frac{1}{n_t}\right) \left[ S_{ut}^2 + 2\lambda_{kt} S_{ukt} + \lambda_{kt}^2 S_{kt}^2 \right]
\]

\[
+ \left[\frac{1}{n_t n_i} \sum_{i=1}^{n_i} \frac{1-f_{2i}}{m_{ti}} \right] \sum_{i=1}^{n_i} Z_{ti}^2 \left(\frac{S_{ti}^2}{m_{ti}}\right)
\]

(6.2.14)

the optimum \( \lambda_{kt(\text{opt})} = \beta_{\text{opt}} \) is the regression coefficient of \( u \) on \( x_{kt} \)
in the population and the sample estimate of regression coefficient is

\[
\hat{\lambda}_{kt(\text{opt})} = \hat{\beta}_{\text{opt}} = \frac{s_{ukt}}{S_{kt}^2}
\]

(6.2.15)

where \( s_{ukt} = [1/(n_t-1)] \sum_{i=1}^{n_t} (x_{kii} - \bar{x}_{kt}) (Z_{ti} - \bar{y}_{ti}) (y_{ti} - \bar{y}_{tz}) \)
\[ s_{kt}^2 = \frac{1}{(n_t-1)} \sum_{i=1}^{n_t} (x_{kti} - \bar{x}_{kt})^2 \] (6.2.16)

such that \( E[s_{ukt}] = S_{ukt}, \ E[s_{kt}^2] = S_{kt}^2 \).

Then the optimum variance

\[
V_{(opt)} [\phi_{tDM}(y)] = \frac{1-f_t}{n_t} \left( (1-\rho_{ukt}^2)S_{ut}^2 + \frac{1}{n_tN_t} \sum_{i=1}^{n_t} \sum_{m_{ti}} (\frac{1}{N_t}) Z_{ti}^2 \right) (6.2.17)
\]

and if optimum choices be made for each \( k \), then the optimum variance be given as

\[
V_0 [\phi_{tDM}(y)] = \frac{1-f_t}{n_t} (1-\rho_{ukt}^2 - \rho_{ult}^2 + \rho_{ukt} \rho_{ult} \rho_{kl} t) \quad \text{(6.2.18)}
\]

Let define a regression estimator (multi-variate) using estimated

values of \( \beta_{okt} \) as

\[
\phi^{(1)}_{trgm}(y) = y_{iz} - \sum_{k=1}^{p} a_{kt} \beta_{okt} (\bar{x}_{kt} - \bar{X}_{kt}) \quad \text{(6.2.19)}
\]

and for large samples

\[
M[\phi^{(1)}_{trgm}(y)] = V_0 [\phi_{tDM}(y)] \quad \text{(6.2.20)}
\]
Also optimum values of $T_{kt} = a_{kt} \lambda_{kt}$ may be calculated as, for $S_t = (S_{kt})$,

$$Q = (Q_1, Q_2, \ldots, Q_p), \quad Q_k = S_{ukt}.$$  Then

$$V[\phi_{IDM}(y)] = V[y_{iz}] - \frac{1 - f_t}{n_t} [2T'Q - T'S_tT] \quad (6.2.21)$$

Hence $T_{opt} = S_{1}^{-1}Q$,

$$V_0[\phi_{IDM}(y)] = \frac{1 - f_t}{n_t} S_{ut}^2 (1 - R_t^2) + \frac{1}{n_t N_t} \Sigma Z_{it}^2 \frac{1 - f_{2i}}{m_i} S_{ti}^2. \quad (6.2.22)$$

Let $R_t = Q'S_t^1Q$, $S_{ut}^2$ is the multiple correlation between $u$ and $x_1, x_2, \ldots x_p$, $\hat{T}_{okt} = \hat{S}_{1}^{-1}\hat{Q}$, $\hat{Q}_k = S_{ukt}$, $\hat{S}_{kt} = s_{kt}$

Hence regression estimator is

$$\phi_{regm}(y) = y_{iz} - \Sigma T_{okt}(x_{ki} - \overline{x}_{ki}) \quad (6.2.23)$$

and variance is given by (6.2.22).

### 6.3. When information is collected at second stage

Let the information on auxiliary variables be collected at the second stage only. Let a sample $s(1)$ of $n_1$ fsus is selected using SRSWOR and out of $n_1$ fsus a sub-sample $s(2i)$ of $m_{ii}$ ssus is selected, the character of main study and auxiliary information for p variables is
observed \{y_{tij}, x_{tij}\} k = 1, 2, \ldots p \ t = 1, 2, \ldots L \ i = 1, 2, \ldots n_t \ j = 1, 2, \ldots, m_{ui}$. The multivariate auxiliary information is utilized to define estimator as for the parameter

\[
\hat{\phi}_i(y) = \frac{1}{p} \sum_{k=1}^{p} a_{kt} d_{kt}
\]

(6.3.1)

\[
\begin{align*}
\hat{\phi}_{iDM}(y) & \text{ if } d_{kt} = \bar{y}_{tj} - \lambda_{kt} (\bar{x}_{kt} - \bar{X}_{kt}) \quad k = 1, 2, \ldots, p \\
\hat{\phi}_{iPM}(y) & \text{ if } d_{kt} = \frac{1}{\lambda_{kt}} \quad k = 1, 2, \ldots, p \\
\hat{\phi}_{iRM}(y) & \text{ if } d_{kt} = \frac{\bar{y}_{tj}}{\bar{X}_{kt}} \quad k = 1, 2, \ldots, q \\
\hat{\phi}_{iRM}(y) & \text{ if } d_{kt} = \frac{1}{\bar{X}_{kt}} \quad k = q+1, q+2, \ldots, p
\end{align*}
\]

(6.3.2)

\[\lambda_{kt} \text{'s be the suitably chosen constants, } \sum_{k=1}^{p} a_{kt} = 1 \text{ be a weighing function, } \bar{a}_i = (a_{1i}, a_{2i}, \ldots, a_{pi}) \text{, and } x_{1i}, x_{2i}, \ldots, x_{pi} \text{ be those values of } x_i \text{ in} \]
\( \phi_{\text{RPM}} \) which are positively correlated with \( y \), and \( x_{(q+1)k}, x_{(q+2)k}, \ldots, x_{pk} \) be those values of \( x_{kt} \) which are negatively correlated with \( y \). The bias and mean square error of the proposed estimators be calculated as follows.

The difference estimator \( \hat{\phi}_{\text{DM}}(y) \) is an unbiased estimators of \( \phi_{\text{DM}}(y) \) with variance

\[
V[\hat{\phi}_{\text{DM}}(y)] = \left( \frac{1}{n_t} \right) S_{ut}^2 + \left( \frac{1}{n_t N_t} \right) a_t B_t a_t \tag{6.3.3}
\]

where \( B_t = b_{kt} = \sum \left( \frac{1}{m_i} \right) Z_{ti}^2 [S_{ti}^2 - \lambda_{kt} S_{kti} - \lambda_{it} S_{it} + \lambda_{kt} \lambda_{it} S_{kti}] \)

\[
V[\hat{\phi}_{\text{DM}}(y)] = \sum a_{kt} a_{it} \text{Cov}(d_{kt}, d_{it}) \tag{6.3.4}
\]

\[
\text{Cov}(d_{kt}, d_{it}) = \text{Cov} \left[ \overline{Y}_{tZ} - \lambda_{kt} (\overline{X}_{kZ} - \overline{X}_{ktz}), \overline{Y}_{tZ} - \lambda_{it} (\overline{X}_{kZ} - \overline{X}_{1Z}) \right]
\]

\[
= \left( \frac{1 - f_i}{n_t} \right) S_{ut}^2 + \sum \left( \frac{1}{n_t N_t} \right) Z_{ti}^2 \left( \frac{1 - f_t}{m_i} \right) [S_{ti}^2 - \lambda_{kt} S_{kti} - \lambda_{it} S_{it} + \lambda_{kt} \lambda_{it} S_{kti}]
\]

where

\[
S_{kti} = \left[ 1/(M_{ti} - 1) \right] \sum (x_{klij} - \overline{X}_{kti}) (y_{tij} - \overline{Y}_{ti}).
\]
\[ S_{kli} = [1/(M_{li}-1)] \sum_{j=1}^{M_{li}} (x_{klij} - \bar{X}_{kli})(x_{luij} - \bar{X}_{lui}). \]

\[ S_{l_i}^2 = \frac{1}{M_{li}} \sum_{(M_{li}-1) i=1}^{M_{li}} (y_{lui} - \bar{Y}_{lui})^2 \]

\[ S_{kli}^2 = \frac{1}{M_{li}} \sum_{(M_{li}-1) i=1}^{M_{li}} (x_{klij} - \bar{X}_{kli})^2 \]

An approximation to the bias and mean square error of \( \hat{\phi}_{iRM}(y) \), \( \hat{\phi}_{iPM}(y) \) and \( \hat{\phi}_{iRPM}(y) \) for large sample is given by

\[ B[\hat{\phi}_{iRM}(y)] = \sum_{k=1}^{p} \frac{a_{kt} N_t}{\phi_i(x_{kt})} \left( 1-f_{2t} \right) \sum_{i=1}^{N_t} \sum_{m_{ti}} Z_{ti} (R_{kt} S_{kli}^2 - S_{kli}.). \]  

\[ M[\hat{\phi}_{iRM}(y)] = V[\hat{\phi}_{iDM}(y)] \text{ with } \lambda_{kt} = R_{kt} = \frac{\phi_i(x_{kt})}{\phi_i(x_{kt})} \]  

\[ B[\hat{\phi}_{iPM}(y)] = \sum_{k=1}^{p} \frac{a_{kt} N_t}{\phi_i(x_{kt})} \left( 1-f_{2t} \right) \sum_{i=1}^{N_t} \sum_{m_{ti}} Z_{ti} S_{kli}. \]  

\[ M[\hat{\phi}_{iPM}(y)] = V[\hat{\phi}_{iDM}(y)] \text{ with } \lambda_{kt} = -R_{kt} \]  

\[ B[\hat{\phi}_{iRPM}(y)] = \sum_{k=1}^{q} \frac{a_{kt} N_t}{\phi_i(x_{kt})} \left( 1-f_{2t} \right) \sum_{i=1}^{N_t} \sum_{m_{ti}} Z_{ti} [R_{kt} S_{kli}^2 - S_{kli}.]. \]
\[ p \sum a_{kt} n_t \sum_{i=1}^{m_i} (z_{tij} - s_{kji}) = (1-f_2) \]

\[ M[\hat{\phi}_{RPM}(y)] = V[\hat{\phi}_{DM}(y)] \]  \hspace{1cm} (6.3.10)

with

\[ \lambda_{kt} = R_{kt}, \quad \lambda_{1t} = R_{1t}, \quad k,1 = 1,2,...q \]

\[ \lambda_{kt} = -R_{kt}, \quad \lambda_{1t} = -R_{1t}, \quad k,1 = q+1, q+2,...p \]

\[ \lambda_{kt} = -R_{kt}, \quad \lambda_{1t} = -R_{1t}, \quad k = 1,2,...q \]

\[ l = q+1, q+2,...,p. \]

The variance of \( \Omega(y) \) is

\[ V[\Omega(y)] = \sum_{t=1}^{L} W_t^2 V[\phi_{DM}] \]  \hspace{1cm} (6.3)

and mean square error for ratio estimator be given as

\[ M[\Omega(y)] = \sum_{t=1}^{L} W_t^2 M[\phi_{RM}] \]  \hspace{1cm} (6.3.12)

Hence for product, ratio-cum-product estimators can be easily computed.

**Optimum Estimators:**

Let for the case of \( p = 1 \), when the information on only one auxiliary character is available say \( x_{kt} \), then
\[ V[\hat{\phi}_{tDM}(y)] = \left( \frac{1-f_t}{n_t} \right) S_{ut}^2 + \left( \frac{1-f_{2t}}{n_t N_t} \right). \] (6.3.13)

\[ B_t = (b_{kt}), \quad b_{kt} = \sum_{i=1}^{m_t} \left( \frac{Z_{ti}^2}{m_{ti}} \right) \left( S_{ti}^2 - 2\lambda_{kt}S_{k_{ti}} + \lambda_{kt}^2 S_{k_{ti}}^2 \right). \]

Then optimum \( \lambda_{kt} \) is

\[ \lambda_{kt(\text{opt})} = \frac{N_t \left( 1-f_{2t} \right)}{\sum_{i=1}^{m_t} m_{ti} \left( S_{k_{ti}}^2 \right)} \]

\[ = \frac{N_t \left( 1-f_{2t} \right)}{\sum_{i=1}^{m_t} m_{ti} \left( S_{k_{ti}}^2 \right)} \]

\[ = \frac{N_t}{\sum_{i=1}^{m_t} m_{ti}} = \beta_{okt}. \] (6.3.14)

where \( \beta_{k_{ti}} = \frac{S_{k_{ti}}}{S_{k_{ti}}^2} \) is the regression coefficient of \( y \) on \( x_{k_{ti}} \) in the population for the \( i \)-th fsu in \( t \)-th stratum.

\[ V_{\text{opt}}[\hat{\phi}_{tDM}(y)] = \left( \frac{1-f_t}{n_t} \right) S_{ut}^2 + \frac{N_t}{1/n_t N_t} \sum_{i=1}^{m_t} \left( \frac{Z_{ti}^2}{m_{ti}} \right) \left( S_{ti}^2 - (1-p_{kt}) \right). \] (6.3.15)
where

\[
\rho_{kti} = \frac{N_i \frac{1}{1-f_{2t}} \sum_{i=1}^{N_i} \frac{Z_{it}^2}{m_i} S_{kti} \left(1 - \frac{1}{n_i} \sum_{i=1}^{N_i} \frac{Z_{it}^2}{m_i} S_{kti}^2 \right)^{1/2}}{\left( \sum_{i=1}^{N_i} \frac{Z_{it}^2}{m_i} S_{kti}^2 \right) \left[ \sum_{i=1}^{N_i} \frac{Z_{it}^2}{m_i} S_{kti}^2 \right]} (6.3.16)
\]

is the correlation coefficient between y and x_{kti} in the t\textsuperscript{th} stratum for i-th fsu. In case the choices be made for each k, we get

\[
V_{opt} [\phi_i M(y)] = \frac{1}{n_i} S_{yt}^2 + \frac{1}{n_i} \sum_{i=1}^{n_i} \frac{Z_{it}^2}{m_i} S_{ti}^2 a_i B_i a_i \quad (6.3.17)
\]

\[
B_i = (b_{ki}), \quad b_{ki} = 1 - \rho_{kti} \rho_{ti} - \rho_{it} \rho_{kti} \rho_{ti} \rho_{ki}
\]

It may not be possible that exact value of \(\lambda_{kti}(opt)\) be available, so a sample value is used to estimate \(\beta_{okt}\)

\[
\beta_{okt} = \frac{n_i \frac{1}{1-f_{2t}} \sum_{i=1}^{N_i} \frac{Z_{it}^2}{m_i} S_{kti}}{\left( \frac{1}{n_i} \sum_{i=1}^{N_i} \frac{Z_{it}^2}{m_i} S_{ki}^2 \right) \left[ \frac{1}{n_i} \sum_{i=1}^{N_i} \frac{Z_{it}^2}{m_i} S_{kti} \right]^{1/2}} (6.3.18)
\]

where

\[
s_{kti}^2 = \frac{1}{(m_i - 1)} \sum_{j=1}^{m_i} (x_{ktij} - x_{kti})^2
\]
\[
    s_{kli} = \frac{1}{(m_{ti} - 1)} \sum_{j=1}^{m_{ti}} (x_{klij} - \bar{x}_{kli}) (y_{tij} - \bar{y}_{ti})
\]

and \(E_2(s_{kli}) = S_{kli}\), \(E_2(s_{kli}^2) = S_{kli}^2\)

so that \(s_{kli}\) and \(s_{kli}^2\) are unbiased estimators for \(S_{kli}\) and \(S_{kli}^2\) respectively. For these estimated values, we may define a new estimator by

\[
    \phi_{(1)}^{(1)}_{\text{regM}}(y) = y_{tz} - \sum_{k=1}^{p} a_{kt} \beta_{okt} (x_{kzt} - X_{kzt})
\]

and for large sample mean square error is

\[
    M[\phi_{(1)}^{(1)}_{\text{regM}}(y)] = M[\phi_{(1)}^{(1)}_{\text{regM}}(y)] = V_{\text{opt}}[\phi_{\text{DM}}(y)]
\] (6.3.19)

One may obtain simultaneous optimum values of \(T_{tk} = a_{kt} \lambda_{kt}\) for \(k = 1, 2, ..., p\). Let \(S_t = (S^*_{k1t}), Q = (Q_1, Q_2, ..., Q_p)\) and

\[
    S^*_{klt} = \sum_{i=1}^{N_t} \frac{1 - f_{i}}{m_{ti}} Z_{ti}^2 (---) S_{kli}, \quad Q_k = S^*_{kt}
\]

\[
    V[\phi_{\text{DM}}(y)] = \frac{1 - f_t}{n_t} S_{ut}^2 + \frac{1}{n_t N_t} [S^*_{t.}^2 - 2T'Q + T'S_t T]
\] (6.3.20)

which gives \(T_{kt(\text{opt})} = T_{kt} = S_t^{-1}Q\) and

\[
    V[\phi_{\text{DM}}(y)] = \frac{1 - f_t}{n_t} S_{ut}^2 + \frac{1}{n_t N_t} S_t^{-2} (1-R^2)
\] (6.3.21)
where \( R^2 = \frac{Q'S_i^{-1}Q}{S_i^2} \). \( R \) is the multiple correlation between \( y_{tz} \) and \( s_i \).

\[
(x_{kz} - \bar{x}_{kz}) \text{ whose variance for large sample is given by equation (6.3.21)}
\]

6.4. **Estimation of \( Y_t \) when \( M_{1o} \) is not known using multiauxiliary information.**

The population mean per ssu is estimated when \( M_{1o} \) is known, but the situation may arise when \( M_{1o} \) is not known. As \( \bar{Y}_t = \frac{\hat{Y}_t}{\hat{M}_t} \) may be estimated by

\[
\hat{Y}_t = \frac{\bar{Y}_t}{\hat{M}_t} \quad (6.4.1)
\]

When the auxiliary information is not used then

\[
\hat{Y}_t = \frac{n_t}{(1/n_t) \sum_{i=1}^{n_t} M_{ti} \bar{y}_{ti}}.
\]

\[
\bar{y}_{ti} = \frac{1}{m_{ti}} \sum_{j=1}^{m_{ti}} y_{tij}, \quad \hat{M}_t = \frac{n_t}{(1/n_t) \sum_{i=1}^{n_t} M_{ti}}
\]

Since \( \hat{Y}_t \) is ratio of two sample means, then for large sample

\[
B[\hat{Y}_t] = [\bar{Y}_t V(\hat{M}_t) - \text{Cov}(\hat{Y}_t, \hat{M}_t)] / \hat{M}_t^2 \quad (6.4.2)
\]
The availability of multivariate auxiliary information leads to the following sub cases.
a) when auxiliary information is available at fsu.

b) when auxiliary information is available at ssu.

6.4.1. Multivariate auxiliary information is available for fsus only

The estimator

\[ \hat{Y}_t = \frac{\bar{Y}_t}{\bar{M}_t} \]

where \( \bar{Y}_t = \sum_{k=1}^{p} a_{kt} d_{kt} \) (6.4.4)

\[ \begin{align*}
\hat{Y}_{tDM} & \text{ if } d_{kt} = \bar{y}_{tz} - \lambda_{kt} (\bar{x}_{kt} - \bar{X}_{kt}), k = 1,2,\ldots,p \\
\hat{Y}_{tRM} & \text{ if } d_{kt} = (\bar{y}_{tz} / \bar{x}_{kt}) \bar{X}_{kt}, k = 1,2,\ldots,p \\
\hat{Y}_{tPM} & \text{ if } d_{kt} = (\bar{y}_{tz} / \bar{x}_{kt}) \bar{x}_{kt}, k = 1,2,\ldots,p \\
\hat{Y}_{tRPM} & \text{ if } d_{kt} = (\bar{y}_{tz} / \bar{x}_{kt}) \bar{x}_{kt}, k = q+1, q+2,\ldots,p 
\end{align*} \] (6.4.5)

\( \lambda_{kt} \)'s are suitably chosen constants, \( \sum_{k=1}^{p} a_{kt} = 1 \) is a weighing function, \( \bar{a}_t = (a_{1t}, a_{2t}, \ldots, a_{pt}) \), and \( x_{1t}, x_{2t}, \ldots, x_{pt} \) be those values of \( x_t \) in \( \hat{\phi}_{tRPM} \) which are positively correlated with \( y \) and \( x_{(q+1)t}, x_{(q+2)t}, \ldots, x_{pt} \) be those values of \( x_{kt} \) which are negatively correlated with \( y \). The biases and
mean square errors are calculated as follows. The bias and mean square error can be computed as

\[
V(\hat{Y}_t) = \frac{1}{n_t} \sum_{i=1}^{n_t} M_{ti}^2 S_{ti}^2.
\]

(6.4.6)

where \( B_t = (b_{kt}) = S_{ut}^2 - \lambda_{kt} S_{ukt} - \lambda_{lt} S_{ult} + \lambda_{kt} \lambda_{lt} S_{klt} \)

\( S_{klt} = \left[ \frac{1}{(N_t - 1)} \right] \sum_{i=1}^{N_t} (x_{kti} - \bar{X}_{kt}) (x_{lti} - \bar{X}_{lt}) \)

\[
\text{Cov}(\hat{Y}_t, \hat{M}_t) = \frac{p}{n_t} \sum_{k=1}^{p} \text{Cov}(\hat{x}_{kt}, \hat{M}_t) = \frac{p}{n_t} \sum_{k=1}^{p} \text{Cov}(\hat{x}_{kt}, \hat{M}_t)
\]

Also \( V(\hat{M}_t) = \frac{S_{td}^2}{n_t} \)

Then
\[
B[ \mathbf{Y_{tDM}} ] = \frac{(1-f_t)}{n_t} \sum_{k=1}^{p} \mathbf{Y_tS}_{id}^2 - \mathbf{S}_{utd} + \lambda_{kt} \mathbf{S}_{ktd}
\]

\[
M[ \mathbf{\hat{Y}_{tDM}} ] = \frac{1-f_t}{n_t} \mathbf{a'}_i \mathbf{B'}_t \mathbf{a}_t + \left[ \frac{1}{(n_tN_t \bar{M}_t^2)} \right] \sum_{i=1}^{N_t} \frac{1-f_{2t}}{m_{i}} \mathbf{S}_{it}^2.
\]

\[
B^*_i = (b^*_k) = [b_{kt} - 2R_t(S_{utd} - \lambda_{kt} S_{ktd}) + R_t^2 S_{itd}^2]
\]

\[
M[ \mathbf{\hat{Y}_{tRM}} ] = M[ \mathbf{\hat{Y}_{tDM}} ] , \lambda_{kt} = R_{kt} , \lambda_{lt} = R_{lt} , R_{kt} = ( \bar{Y}_t' \bar{X}_{kt} )
\]

\[
M[ \mathbf{\hat{Y}_{tPM}} ] = M[ \mathbf{\hat{Y}_{tDM}} ] , \lambda_{kt} = R_{kt} , \lambda_{lt} = -R_{lt},
\]

\[
M[ \mathbf{\hat{Y}_{tRPM}} ] = M[ \mathbf{\hat{Y}_{tDM}} ] , \lambda_{kt} = R_{kt} ; \lambda_{lt} = R_{lt} , k = 1,2,..,q
\]

\[
\lambda_{kt} = -R_{kt} ; \lambda_{lt} = -R_{lt} , k, 1 = q+1,..,p
\]

\[
\lambda_{kt} = R_{kt} ; \lambda_{lt} = -R_{lt} , k, = 1,2,..,q
\]

\[
l = q+1, q+2,..,p
\]

6.4.2. **Multivariate auxiliary information is available for ssus and \( \lambda_{kt} \)'s (independent of i) are pre-assigned constants**

Let

\[
\mathbf{\hat{Y}_t} = \sum a_{kt} d_{kt}
\]
\[
\hat{Y}_{t} = \begin{cases} 
\hat{Y}_{tDM} \text{ if } d_{kt} = \bar{y}_{tz} - \lambda_{kt} (\bar{x}_{kt} - \bar{X}_{kt}), k = 1,2,\ldots,p \\
\hat{Y}_{tRM} \text{ if } d_{kt} = (\bar{y}_{tz} / \bar{x}_{kt}) \bar{X}_{kt}, k = 1,2,\ldots,p \\
\hat{Y}_{tPM} \text{ if } d_{kt} = (\bar{y}_{tz} / \bar{x}_{kt}) \bar{x}_{kt}, k = 1,2,\ldots,p \\
\hat{Y}_{tRPM} \text{ if } d_{kt} = (\bar{y}_{tz} / \bar{x}_{kt}) \bar{x}_{kt}, k = 1,2,\ldots,q \\
= (\bar{y}_{tz} / \bar{x}_{kt}) \bar{x}_{kt}, k = q+1, q+2,\ldots,p 
\end{cases}
\]

(6.4.13)

\(\lambda_{kt}\)'s be the suitably chosen constants, \(\Sigma a_{kt} = 1\) be a weighing function, \(k=1\)

\(\bar{a}_{i} = (a_{i1}, a_{i2}, \ldots, a_{ip})\), and \(x_{i1}, x_{i2}, \ldots, x_{qi}\) be those values of \(x_{i}\) in \(\hat{Y}_{tRPM}\) which are positively correlated with \(y\) and \(x_{(q+1)i}, x_{(q+2)i}, \ldots, x_{pi}\) be those values of \(x_{ki}\) which are negatively correlated with \(y\). The bias and mean square error can be computed as

\[
V[\hat{Y}_{t}] = \frac{1-f_{t}}{n_{t}} S_{ui}^{2} + \frac{1}{n_{t} N_{i}} \left( \frac{1}{n_{t}} \sum_{i=1}^{p} \frac{1}{M_{ii}} \left( 1-f_{2t} \right) / m_{ii} \right) a'_{i} B_{ii} a_{i}
\]

(6.4.14)

\(B_{ii} = (b_{kli})\)

\(b_{kli} = S_{ii}^{2} - \lambda_{ki} S_{ii-k} - \lambda_{lt} S_{ii-1} + \lambda_{kt} \lambda_{lt} S_{ikl}\)

\[
\text{Cov}(\hat{Y}_{t}, \hat{M}_{i}) = \sum_{k=1}^{p} \left[ \text{Cov}(\hat{y}_{tz}, \hat{M}_{i}) - \lambda_{kt} \text{Cov}(\hat{x}_{ktz}, \hat{M}_{i}) + \lambda_{ki} \text{Cov}(\bar{X}_{ktz}, \hat{M}_{i}) \right]
\]
\[
\text{Cov}(\bar{Y}_{iz}, \hat{M}_i) = [(1-f_i)/n_i] S_{utd}
\]
\[
\text{Cov}(\bar{X}_{kiz}, \hat{M}_i) = [(1-f_i)/n_i] S_{vtd}
\]
\[
\text{Cov}(\bar{X}_{kiz}, \bar{M}_i) = [(1-f_i)/n_i] S_{vtd}
\]
\[
S_{vtd} = [1/(N_t-1)] \sum_i (v_{kti} - \bar{V}_k) (M_{ti} - \bar{M}_t)
\]
with \( M_{ti} X_{ktt} = v_{kti} \)

\[
(1/n_i) \sum_i M_{ti} X_{ktt} = \bar{v}_{kt}
\]

\[
(1/N_t) \sum_i M_{ti} X_{ktt} = \bar{V}_k
\]

\[
M[\hat{Y}_{tDM}] = [(1-f_i)/n_i] \left( 1/\bar{M}_t^2 \right) \left[ S_{ut}^2 - 2R_t S_{utd} + R_t^2 S_{td}^2 \right]
\]
\[
+ (1/n_i N_t) \bar{M}_t^2 \sum_i M_{ti}^2 [(1-f_i)/m_{ti}] a_i B_{ti} a_i
\]

\[(6.4.15)\]

where \( B_{ti} = (b_{kli}) \) and

\[
b_{kli} = S_{ti}^2 - \lambda_{kt} S_{ti-k} + \lambda_{kt} \lambda_{li} S_{tikl} - \lambda_{lt} S_{ti-l}
\]

Hence the results for \( \bar{Y}_{tRM}, \bar{Y}_{tPM} \) and \( \bar{Y}_{tRPM} \) can be derived from \( \bar{Y}_{tDM} \).
6.5 **Relative performance of the proposed estimators**

The proposed estimators are compared with the usual estimator when no auxiliary information is used and when auxiliary information is used.

a) When multiauxiliary information is available for fsus only.

The variance for two-stage difference estimator is

\[
V[\phi_{DM}(y)] = \left(\frac{1-f_1}{n_t}\right) (S_{ut}^2 - \lambda_{kt} S_{ukt} - \lambda_{lt} S_{ult} + \lambda_{kt} \lambda_{lt} S_{kt})
\]

\[
+ \left(\frac{1}{n_t N_t}\right) \sum_{i=1}^{N_t} \left(\frac{1-f_2 t_i}{m_i}\right) Z_i^2 S_{i}^2
\]

b) When multi-auxiliary information is available at ssu level with \(\lambda_{kt}'s\) be pre-assigned constants, then

\[
V[\phi_{DM}(y)] = \left(\frac{1-f_1}{n_t}\right) S_{ut}^2 + \left(\frac{1}{n_t N_t}\right) \sum_{i=1}^{N_t} \left(\frac{1-f_2 t_i}{m_i}\right) Z_i^2 S_{i}^2 - \lambda_{kt} S_{kti}
\]
\[ -\lambda_{lt} S_{lii} + \lambda_{kt} \lambda_{lt} S_{kli} \]

\[
\nu[\Phi_{DM}(y)] = \left( \frac{1-f_i}{n_t} \right) S_u^2 + \frac{1-f_{2t}}{N_t} \sum_{i=1}^{m_{ti}} \left( \frac{1}{1/nN} \right) Z_{i}^2 \left[ S_{li}^2 - \lambda_i S_{li} \right]
- \lambda_{li} S_{li} + \lambda_{lk} S_{kli} \]