ABSTRACT

1. INTRODUCTION

Spectral theory is a thrust area of research in Functional analysis. The spectrum of an operator is a generalization of the notion of eigenvalues for matrices. Goldberg's [39] classify the spectrum of an operator over a Banach space is partitioned into three parts, which are the point spectrum, the continuous spectrum, and the residual spectrum. The calculation of these three parts of the spectrum of an operator is called the fine spectrum of the operator. Several authors have studied the spectrum and fine spectrum of linear operators defined by some particular limitation matrices over some sequence spaces. It was followed by the works due to J.B. Reade, J.T Okutoyi, M. González, K. Kyaduman, B.E. Rhoades, M. Yildrim, P. Srivastava, and R.B. Wenger are a few to be named.

The spectra of the difference operator have also been investigated on different classes of sequences. Altay and Basar ([11], [12]) studied the spectra of difference operator $\Delta$ and generalized difference operator on $c_0$ and $c$. Recently, the fine spectrum of $B(r, s, t)$ over the sequence spaces $\ell_p$ and $bv_p$ has been studied by Furkan et. al. [37]. Further P.D. Srivastava, S. Kumar, B. de Malafosse, S.R. El-Shabrawy, A.M. Akhmedov, B. L. Panigrahi, S. Dutta, V. Karakaya, M. D. Manafov, N. Şimşek and many others concentrated their study on different types of difference operators. Still there is lot to be exploring on the spectra of matrix operators on different sequence spaces.

Throughout $w, \ell_\infty, c, c_0, \ell_p, \ell_1, bv, bv_p, bv_0$ denote the classes of all, bounded, convergent, null, $p$-absolutely summable, absolutely summable, bounded variation and $p$-absolutely bounded variation and null as well as bounded variation classes of sequences respectively.
The following are the objectives and chapters of the thesis.

The Chapter-I of the thesis is introductory in nature. In this chapter most of the definitions and results have been collected from different journals, books, thesis etc. and these are used in the subsequent chapters of the thesis. Further, the preliminaries of the works carried are given to have a clear picture of the background and the development of the topics on which the works in the subsequent chapters have been carried out.

2. OBJECTIVE 1

(The Spectrum of the Operator \( D(r, 0, 0, s) \) Over \( c_0 \) and \( c \))

In Chapter-II, we introduce the operator \( D(r, 0, 0, s) \) and call it the Difference type operator whose matrix representation is as follows

\[
D(r,0,0,s) = \begin{bmatrix}
0 & 0 & 0 & 0 & \ldots \\
0 & r & 0 & 0 & \ldots \\
0 & 0 & r & 0 & \ldots \\
0 & 0 & 0 & r & \ldots \\
s & 0 & 0 & 0 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \\
\ldots & \ldots & \ldots & \ldots & \\
\ldots & \ldots & \ldots & \ldots & \\
\ldots & \ldots & \ldots & \ldots & \\
\ldots & \ldots & \ldots & \ldots & \\
\end{bmatrix}
\]

If we consider \( r = -1 \) and \( s = 1 \) this operator converts a sequence \( (x_k) \) into its sequence of differences \( (x_k - x_{k+1}) \).

In this chapter, we establish the following results.

THEOREM 2.1. \( \sigma_{Q}(r,0,0,s), c_0 \supseteq \lambda \in C : | \lambda - r | \leq | s | \).

THEOREM 2.2. \( \sigma_p(D(r, 0, 0, s), c_0) = \emptyset. \)
THEOREM 2.3. $\sigma_p (r, 0, 0, s)^*, c_0^* \vdash A \in C : |\lambda - r| < s$.

THEOREM 2.4. $\sigma (r, 0, 0, s)^*, c_0^* \vdash A \in C : |\lambda - r| \leq s$.

THEOREM 2.5. $\sigma, (r, 0, 0, s), c_0 \vdash A \in C : |\lambda - r| \leq s$.

THEOREM 2.6. $\sigma, (r, 0, 0, s), c \vdash A \in C : |\lambda - r| = s$.

THEOREM 2.7. $\sigma (r, 0, 0, s), c \vdash A \in C : |\lambda - r| = s$.

THEOREM 2.8. $\sigma_p (D, 0, 0, s), c) = \emptyset$.

THEOREM 2.9. $\sigma_p (r, 0, 0, s)^*, c^* \vdash A \in C : |\lambda - r| < s \cup \{r + s\}$.

THEOREM 2.10. $\sigma, (r, 0, 0, s), c \vdash \sigma_p (r, 0, 0, s)^*, c^*$.

THEOREM 2.11. $\sigma, (r, 0, 0, s), c \vdash A \in C : |\lambda - r| = s \setminus \{r + s\}$.

THEOREM 2.12. If $\alpha = r$, then $\alpha \in III_1 (r, 0, 0, s), c_0)$.

THEOREM 2.13. $\sigma_{ap} (D, 0, 0, s), c_0) = \{\lambda \in C : |\lambda - r| \leq s\} \setminus \{r\}$.

THEOREM 2.14. $\sigma_\delta (D, 0, 0, s), c_0) = \{\lambda \in C : |\lambda - r| \leq s\}$.

THEOREM 2.15. $\sigma_{co} (D, 0, 0, s), c_0) = \{\lambda \in C : |\lambda - r| < s\}$.

THEOREM 2.16. (i) $\sigma_{ap} (D, 0, 0, s)^*, \ell_1) = \{\lambda \in C : |\lambda - r| \leq s\}$,

(ii) $\sigma_\delta (D, 0, 0, s)^*, \ell_1) = \{\lambda \in C : |\lambda - r| \leq s\} \setminus \{r\}$.
THEOREM 2.17. If $\alpha = r$, then $\alpha \in \text{III}_1 \sigma (D(r, 0, 0, s), c)$.

THEOREM 2.18. $\sigma \circ \Theta (r, 0, 0, s), c \supseteq \bar{A} \subseteq C : |\lambda - r| \leq |s| \setminus \bar{r}$.

THEOREM 2.19. $\sigma \circ \Theta (r, 0, 0, s), c \supseteq \bar{A} \subseteq C : |\lambda - r| \leq |s| \setminus \bar{r}$.

THEOREM 2.20. $\sigma \circ \Theta (r, 0, 0, s), c \supseteq \bar{A} \subseteq C : |\lambda - r| \leq |s| \setminus \bar{r}$.

3. OBJECTIVE II

(The Spectrum of the Operator $D(r, 0, 0, s)$ over $\ell_p$ and $bv_p$ ($1 < p < \infty$))

In Chapter-III, we study the spectra and fine spectra of the same operator $D(r, 0, 0, s)$ which is introduced in chapter-II over the sequence spaces $\ell_p$ and $bv_p$.

In this chapter, we establish the following results.

THEOREM 3.1. $D(r, 0, 0, s) : \ell_p \rightarrow \ell_p$ is a bounded linear operator satisfying the inequalities $\ell^p_{\rho} + |s|^{\rho} \lesssim D(r, 0, 0, s) \parallel \rho \lesssim |r| + |s|$.

THEOREM 3.2. $\sigma \circ \Theta (r, 0, 0, s), \ell_p \supseteq \bar{A} \subseteq C : |\lambda - r| \leq |s| \setminus \bar{r}$.

THEOREM 3.3. $\sigma (D(r, 0, 0, s), \ell_p) = \emptyset$.

THEOREM 3.4. $\sigma_p (D(r, 0, 0, s), \ell_p^*) \supseteq \bar{A} \subseteq C : |\lambda - r| \leq |s| \setminus \bar{r}$.

THEOREM 3.5. $\sigma (D(r, 0, 0, s), \ell_p^*) \supseteq \bar{A} \subseteq C : |\lambda - r| \leq |s| \setminus \bar{r}$.
THEOREM 3.6. \( \sigma_c \Omega(r, 0, 0, s), \ell_p \notdarrow \mathcal{A} \in C : \lambda - r \leq s \). \( \_ \_ \_ \_ \_ \)

THEOREM 3.7. \( D(r, 0, 0, s) \in B(bv_p) \).

THEOREM 3.8. \( \sigma \Omega(r, 0, 0, s), bv_p \notdarrow \mathcal{A} \in C : \lambda - r \leq s \). \( \_ \_ \_ \_ \_ \)

THEOREM 3.9. (i) \( \sigma_p(D(r, 0, 0, s), bv_p) = \emptyset \),

(ii) \( \sigma_p \Omega(r, 0, 0, s)^*, bv_p^* \notdarrow \mathcal{A} \in C : \lambda - r \leq s \),

(iii) \( \sigma_r \Omega(r, 0, 0, s), bv_p \notdarrow \mathcal{A} \in C : \lambda - r \leq s \),

(iv) \( \sigma_c \Omega(r, 0, 0, s), bv_p \notdarrow \mathcal{A} \in C : \lambda - r \leq s \). \( \_ \_ \_ \_ \_ \)

THEOREM 3.10. If \( \alpha = r \), then \( a \in \text{III}_1 \sigma(D(r, 0, 0, s), \ell_p) \).

THEOREM 3.11. \( \sigma_{ap}(D(r, 0, 0, s), \ell_p) = \{ \lambda \in C : |\lambda - r| \leq |s| \} \backslash \{ r \} \).

THEOREM 3.12. \( \sigma_p \Omega(r, 0, 0, s), \ell_p \notdarrow \mathcal{A} \in C : \lambda - r \leq s \).

THEOREM 3.13. \( \sigma_c \Omega(r, 0, 0, s), \ell_p \notdarrow \mathcal{A} \in C : \lambda - r \leq s \). \( \_ \_ \_ \_ \_ \_ \_ \)

THEOREM 3.14. Let \( p^{-1} + q^{-1} = 1 \) then, the following results hold:

(i) \( \sigma_{ap} \Omega(r, 0, 0, s)^*, \ell_q \notdarrow \mathcal{A} \in C : \lambda - r \leq s \),

(ii) \( \sigma_p \Omega(r, 0, 0, s)^*, \ell_q \notdarrow \mathcal{A} \in C : \lambda - r \leq s \). \( \_ \_ \_ \_ \) \( \{ r \} \).

THEOREM 3.15. The following results hold:

(i) \( \sigma_{ap}(D(r, 0, 0, s), bv_p) = \{ \lambda \in C : |\lambda - r| \leq |s| \} \backslash \{ r \} \),

(ii) \( \sigma_p (D(r, 0, 0, s), bv_p) = \{ \lambda \in C : |\lambda - r| \leq |s| \}, \)
\[(iii) \sigma_{co}(D(r, 0, 0, s), bv_p) = \{ \lambda \in C : |\lambda - r| < |s| \},\]

\[(iv) \sigma_{ap}(D(r, 0, 0, s), bv^*_p) = \{ \lambda \in C : |\lambda - r| \leq |s| \},\]

\[(v) \sigma_s(D(r, 0, 0, s), bv^*_p) = \{ \lambda \in C : |\lambda - r| \leq |s| \} \setminus \{ r \}.

### 4. OBJECTIVE III

( THE SPECTRUM OF THE OPERATOR $D(r, 0, 0, s)$ OVER $bv_0$)

In Chapter-IV, we study the spectra and fine spectra of the same operator $D(r, 0, 0, s)$ which is introduced in the chapter-II over the sequence space $bv_0$.

In this chapter, we establish the following results.

**THEOREM 4.1.** $\sigma (r, 0, 0, s), bv_0 \supseteq A \subseteq C : |\lambda - r| \leq |s| \supseteq$

**THEOREM 4.2.** $\sigma_p(D(r, 0, 0, s), bv_0) = \emptyset$.

**THEOREM 4.3.** $\sigma (r, 0, 0, s)^*, bv_0^* \supseteq A \subseteq C : |\lambda - r| \leq |s| \supseteq$

**THEOREM 4.4.** $\sigma_p (r, 0, 0, s)^*, bv_0^* \supseteq A \subseteq C : |\lambda - r| \leq |s| \supseteq$

**THEOREM 4.5.** $\sigma (r, 0, 0, s), bv_0 \supseteq A \subseteq C : |\lambda - r| \leq |s| \supseteq$

**THEOREM 4.6.** $\sigma_s (r, 0, 0, s), bv_0 \supseteq A \subseteq C : |\lambda - r| \leq |s| \supseteq$
5. OBJECTIVE IV

(THE SPECTRUM OF THE OPERATOR \( D(r, 0, s, 0, t) \) OVER \( c_0 \) AND \( c \))

In Chapter V, we introduce the operator \( D(r, 0, s, 0, t) \) whose matrix representation is

\[
D(r, 0, s, 0, t) = \begin{bmatrix}
r & 0 & 0 & 0 & 0 & \ldots \\
0 & r & 0 & 0 & 0 & \ldots \\
s & 0 & r & 0 & 0 & \ldots \\
0 & s & 0 & r & 0 & \ldots \\
0 & t & s & 0 & r & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}
\]

We assume here and hereafter that \( s \) and \( t \) are complex parameters which do not simultaneously vanish.

In particular if we consider \( r = 1, s = -2 \) and \( t = 1 \) then \( D(1, 0, -2, 0, 1) = \Delta_2^2 \).

In this chapter, we establish the following results.

**THEOREM 5.1.** Let \( s \) be a complex number such that \( \sqrt{s^2} = -s \) and defined the set by

\[
S = \left\{ \lambda \in C : \frac{2(r - \lambda)}{-s + \sqrt{s^2 - 4t(r - \lambda)}} \leq 1 \right\}.
\]

Then \( \sigma(D(r, 0, s, 0, t), c_0) = S \).

**THEOREM 5.2.** \( \sigma_p(D(r, 0, s, 0, t), c_0) = \emptyset \).

**THEOREM 5.3.** \( \sigma_p(\Phi(r, 0, s, 0, t)^*, c_0^*) = S_1 \), where

\[
S_1 = \left\{ \lambda \in C : \frac{2(r - \lambda)}{-s + \sqrt{s^2 - 4t(r - \lambda)}} < 1 \right\}.
\]

**THEOREM 5.4.** \( \sigma(D(r, 0, s, 0, t), c_0) = S_1 \), where \( S_1 \) is defined as in Theorem 5.3.
THEOREM 5.5. $\sigma_c(D(r, 0, s, 0, t), c_0) = S_2$, where

$$S_2 = \left\{ \lambda \in C : \frac{2(r - \lambda)}{-s + \sqrt{s^2 - 4t(r - \lambda)}} = 1 \right\}.$$  

THEOREM 5.6. $\sigma(D(r, 0, s, 0, t), c) = S$, where $S$ is defined as in Theorem 5.1.

THEOREM 5.7. $\sigma_p(D(r, 0, s, 0, t), c) = \emptyset$.

THEOREM 5.8. $\sigma_p(D(r, 0, s, 0, t), c^*) = S_1 \cup \{ r + s + t \}$, where $S_1$ is defined as in Theorem 5.3.

THEOREM 5.9. $\sigma_r(D(r, 0, s, 0, t), c) = \sigma_p(D(r, 0, s, 0, t), c^*)$.

THEOREM 5.10. $\sigma_c(D(r, 0, s, 0, t), c) = S_2 \setminus \{ r + s + t \}$, where $S_2$ is defined as in Theorem 5.5.

6. OBJECTIVE V

THE SPECTRUM OF THE OPERATOR $D(r, 0, s, 0, t)$ OVER $\ell_p$ AND $bv_p$

In Chapter VI, we investigate the spectra and fine spectra of the same operator $D(r, 0, s, 0, t)$ which is introduced in Chapter V, over the sequence spaces $\ell_p$ and $bv_p$.

In this chapter, we establish the following results.

THEOREM 6.1. $D(r, 0, s, 0, t): \ell_p \to \ell_p$ is a bounded linear operator satisfying the inequalities

$$\| r^p + |s|^p + |t|^p \|_{\ell_p} \leq \| D(r, 0, s, 0, t) \|_{\ell_p} \leq |r| + |s| + |t|.$$
THEOREM 6.2. Let $s$ be a complex number such that $\sqrt{s^2} = -s$ and defined the set by

$$S = \left\{ \lambda \in C : \frac{2(r - \lambda)}{-s + \sqrt{s^2 - 4t(r - \lambda)}} \leq 1 \right\}.$$ Then $\sigma(D(r, 0, s, 0, t), \ell_p) = S$.

THEOREM 6.3. $\sigma_p(D(r, 0, s, 0, t), \ell_p) = \emptyset$.

THEOREM 6.4. $\sigma_p(D(r, 0, s, 0, t)^*, \ell_p^*) = S_1$, where

$$S_1 = \left\{ \lambda \in C : \frac{2(r - \lambda)}{-s + \sqrt{s^2 - 4t(r - \lambda)}} < 1 \right\}.$$

THEOREM 6.5. $\sigma_r(D(r, 0, s, 0, t), \ell_p) = S_1$, where $S_1$ is defined as in Theorem 6.4.

THEOREM 6.6. $\sigma_c(D(r, 0, s, 0, t), \ell_p) = S_2$, where

$$S_2 = \left\{ \lambda \in C : \frac{2(r - \lambda)}{-s + \sqrt{s^2 - 4t(r - \lambda)}} = 1 \right\}.$$

THEOREM 6.7. $D(r, 0, s, 0, t) \in B(bv_p)$.

THEOREM 6.8. $\sigma(D(r, 0, s, 0, t), bv_p) = S$, where $S$ is defined as in Theorem 6.2.

THEOREM 6.9. $\sigma_p(D(r, 0, s, 0, t)^*, bv_p^*) = S_1$, where $S_1$ is defined as in Theorem 6.4.

THEOREM 6.10. (i) $\sigma_p(D(r, 0, s, 0, t), bv_p) = \emptyset$,

(ii) $\sigma_s(D(r, 0, s, 0, t), bv_p) = S_1$,

(iii) $\sigma_c(D(r, 0, s, 0, t), bv_p) = S_2$,

where $S_1$ and $S_2$ be the sets defined as in the Theorem 6.4 and Theorem 6.6 respectively.

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