Chapter II

ON SOME GENERALIZED SPACES OF DIFFERENCE SEQUENCES

2.1. Definitions and notations

We give here some more definitions and notations in addition to those given in the preceding chapters.

Let \( z \) be any sequence and let \( Y \) be any subset of \( w \). Then we shall write

\[ z^{-1} \cdot Y := \{ x \in w : zx = (z_k x_k) \in Y \}. \]

For any subset \( X \) of \( w \), we give here an alternative definitions of \( X^\alpha \), \( X^\beta \) : the \( \alpha \)-, \( \beta \)-duals of \( X \), by:

\[ X^\alpha = \bigcap_{x \in X} (x^{-1} \cdot \ell_{1}) \quad \text{and} \quad X^\beta = \bigcap_{x \in X} (x^{-1} \cdot cs). \]

We define the linear operators \( \Delta, \Delta^2, \Delta^{-1}, \Delta^{-2} : w \to w \) by:

\[ \Delta x = (\Delta x_k)_{k=1}^\infty = (x_k - x_{k+1})_{k=1}^\infty, \]

\[ \Delta^2 x = (\Delta^2 x_k)_{k=1}^\infty = (\Delta x_k - \Delta x_{k+1})_{k=1}^\infty, \]

\[ \Delta^{-1} x = (\Delta^{-1} x_k)_{k=1}^{k-1} = (\sum_{j=1}^{k-1} x_j)_{k=1}^\infty, \]

and

\[ \Delta^{-2} x = (\Delta^{-2} x_k)_{k=1}^{k-1} = (\sum_{j=1}^{k-1} \sum_{i=1}^{j-1} x_i)_{k=1}^\infty. \]

We shall write, for any subset \( E \) of \( w \),

\[ E(\Delta) := \{ x \in w : \Delta x \in E \}; \]

\[ E(\Delta^{-1}) := \{ x \in w : \Delta^{-1} x \in E \}; \]

and

\[ E(\Delta^2) := \{ x \in w : \Delta^2 x \in E \}; \]

\[ E(\Delta^{-2}) := \{ x \in w : \Delta^{-2} x \in E \}. \]
For any sequence \( p = (p_n) \) of strictly positive real numbers and any subset \( E \) of \( w \) we write

\[
E(p; \Delta) := \{ x \in w : \Delta x \in E(p) \};
\]

\[
E(p; \Delta^2) := \{ x \in w : \Delta^2 x \in E(p) \}.
\]

Let \( U \) be the set of all sequences \( u \) such that \( u_k \neq 0 \) (\( k = 1, 2, \ldots \)).

Given \( u \in U \), we define the sets

\[
E(u; \Delta) = (u^{-1} \cdot E)(\Delta) := \{ x \in w : (u_k \Delta x_k)_{k=1}^{\infty} \in E \};
\]

\[
E(u; \Delta^2) = (u^{-1} \cdot E)(\Delta^2) := \{ x \in w : (u_k \Delta^2 x_k)_{k=1}^{\infty} \in E \};
\]

\[
E(p; u, \Delta) = (u^{-1} \cdot E(p))(\Delta) := \{ x \in w : (u_k \Delta x_k)_{k=1}^{\infty} \in E(p) \};
\]

and

\[
E(p; u, \Delta^2) = (u^{-1} \cdot E(p))(\Delta^2) := \{ x \in w : (u_k \Delta^2 x_k)_{k=1}^{\infty} \in E(p) \}.
\]

We consider the subsets \( E, F \) to be any one of the spaces : \( \ell_\infty, c, \) or \( c_0 \). Then, for \( u = e \), we have

\[
E(u; \Delta) = E(\Delta); \quad E(u; \Delta^2) = E(\Delta^2), \quad \text{for } E = \ell_\infty \text{ or } c_0;
\]

\[
E(p; u, \Delta) = E(p; \Delta); \quad E(p; u, \Delta^2) = E(p; \Delta^2), \quad \text{for } E = \ell_\infty \text{ or } c_0;
\]

and for \( E = c, c(u; \Delta) = c(\Delta), c(u; \Delta^2) = c(\Delta^2), c(p; u, \Delta) = c(p; \Delta) \)

and \( c(p; u, \Delta^2) = c(p; \Delta^2) \), provided \( c(p; u) \) is defined as :

\[
c(p; u) := \{ x \in w : | u_n x_n - l e |^p_n \rightarrow 0, \quad \text{as } n \rightarrow \infty, \quad \text{for some } l \in C \}.
\]

Let us define the operator \( S : E(p; u, \Delta^2) \rightarrow E(p; u, \Delta^2) \) by \( x \rightarrow Sx = (0, 0, x_3, x_4, \ldots) \). It is clear that \( S \) is bounded linear operator on \( E(p; u, \Delta^2) \) with \( \| S \| = 1 \). Further
SE(p; u, Δ^2) := \{ x = (x_k) : x \in E(p; u, Δ^2), x_1 = x_2 = 0 \}

:= E(p; u, Δ^2) \cap \{ x = (x_k) : x_1 = x_2 = 0 \}.

Throughout we write

\[ \Delta_r x = (k^r \Delta x_k)_{k=1}^\infty. \]

**Remark 1.** \([SE(p; u, Δ^2)]^\dagger = [E(p; u, Δ^2)]^\dagger \) for \( \dagger = \alpha, \beta \).

### 2.2. Introduction

The difference sequence spaces \( E(\Delta) \) were introduced by Kizmaz [30], who showed that \( E \subseteq E(\Delta) \), since there exist a sequence \( x_k = k \) \((k = 1, 2, \ldots)\) for which \( \Delta x_k = 1 \), so that, although \( x \) is not convergent but it is \( \Delta \)-convergent. He also studied their topological properties, \( \alpha-, \beta-\) and \( \gamma\)-duals of these spaces and determined some matrix classes.

In 1987, Ahmad and Mursaleen [7] extended these spaces to \( E(p; \Delta) \) and studied related problems. In the same year, Sarigöl [77] generalized \( E(\Delta) \) in another direction, for \( E = \ell_\infty \) :

\[ (2.2.1) \quad E(\Delta_r) := \{ x \in w : \Delta_r x \in E, \text{ for } r < 1 \}, \]

which is the same as \( E(u; \Delta) \), for \( u_k = k^r, r < 1 \). While, in 1996, Mursaleen, Gaur and Saifi ([58]; see also [24]) defined and studied, for \( E = \ell_\infty \) :

\[ (2.2.2) \quad E(p; \Delta_r) := \{ x \in w : \Delta_r x \in E(p), \text{ for } r > 0 \}. \]

Thus \( (2.2.2) \) is more general than \( (2.2.1) \) if \( p_n = 1 \), for all \( n \), as it is true for \( r > 0 \) rather than \( r < 1 \).

Choudhary and Mishra [16] in 1993 (earlier than 1996) gave the following definition for \( E = c_0 \) :

\[ (2.2.3) \quad E(\Delta_r) := \{ x \in w : \Delta_r x \in E, r \geq 1 \} \]

:= \{ x \in w : k^r | \Delta x_k | \to 0, \text{ as } k \to \infty \},
so that $E(\Delta_r) \subset E(\Delta) = c_0(\Delta)$ and they observed that $E(\Delta_r) = c_0(\Delta_r)$ is a linear space and a $BK$ space normed by

$$\| x \| = | x_1 | + \sup_k \{ k^r | \Delta x_k | \}.$$ 

In 1996, generalizing the definitions of Kizmaz and Sarigol. Gnanaseelan and Srivastava [25] defined the spaces $E(u; \Delta)$ for a restricted $u$, that is, $u = (u_k)$ is a sequence of non-zero complex numbers such that

(i) $\frac{|u_k|}{|u_{k+1}|} = 1 + O(1/k)$, for each $k$;

(ii) $k^{-1} | u_k | \sum_{j=1}^k | u_i |^{-1} = O(1)$; and

(iii) $\{ k | u_k^{-1} | \}$ is a sequence of positive numbers increasing monotonically to infinity.

The above-said spaces are linear and that

(a) $E(u; \Delta) \subset E(\Delta)$, if $| u_k | \geq 1$ and $(| u_k |)$ increases to $\infty$;

(b) $E(\Delta) \subset E(u; \Delta)$, if $0 < | u_k | < 1$ and $(| u_k |)$ decreases to $0$.

These are Banach spaces with the norm :

$$\| x \|_u = | u_1 x_1 | + \sup_k | u_k \Delta x_k |.$$ 

In the same year (1996), Malkowsky [47] defined the space $E(u; \Delta)$ for an arbitrary, fixed sequence $u \in U$, putting no other restrictions on $u$ as in the case of Gnanaseelan and Srivastava [25]. As special cases he obtained the spaces defined by Choudhary and Mishra, and Kizmaz for $u = (k^r)_{k=1}^{\infty} (r \geq 1)$, and $u = e$, respectively. He pointed out that, for determining the $\alpha$- and $\beta$-duals, it seems that the converse parts of the proofs (cf. [16], pp. 144 and 145) do not hold, since for $x$ defined by :

$$x_k = \frac{1}{N} \sum_{j=1}^{k-1} \frac{1}{j^2} (k = 1, 2, \ldots) \text{ for } N > 1 \text{ fixed},$$

obviously $x \notin E(\Delta_r)$. He also observed and proved that $E(u; \Delta)$ are $BK$ spaces with respect to the norm $\| \cdot \|$ defined by :

$$\| x \| = \sup_{k>1} | u_{k-1}(x_{k-1} - x_k) |, \quad u_0 = 1, \quad x_0 = 0.$$
In the same year (1996), extending the definition of Choudhary and Mishra, Gaur and Mursaleen [24] defined the spaces $E(p; \Delta_r) = E(p; (k^r)_{i=1}^{\infty} \cdot \Delta_r), r \geq 1$, for $E = c_0$, and determined its $\alpha$- and $\beta$-duals and some matrix classes.

Recently, Malkowsky, Mursaleen and Qamaruddin [48], extended $E(u; \Delta)$ to $E(p; u, \Delta)$, generalizing the above-mentioned spaces and also getting the spaces $c_0(p; u), c(p; u)$ and $\ell_\infty(p; u)$ as special cases.

In 1993, Mikail [49] defined the spaces $E(\Delta^2)$. Recently, Mursaleen [57] defined the difference sequence spaces $E(u; \Delta^2)$, which exhibit some properties that the previously studied one do not. He also proved the following results.

Mursaleen [57] has determined the $\alpha$- and $\beta$-duals in:

**Theorem 2.2.1.** Let $u \in U$. Then $[SE(u; \Delta^2)]^\alpha = M_\alpha$, where

$$M_\alpha = (\Delta^{-2}(1/|u|))^{-1} \cdot \ell_1 := \{a \in w : \sum_{k=1}^{\infty} \left| a_k \right| \sum_{j=1}^{k-1} \sum_{i=1}^{j-1} \frac{1}{|u_i|} < \infty\}.$$

**Theorem 2.2.2.** Let $u \in U$. We write

$$M_\beta := \{a \in (\Delta^{-2}(1/|u|))^{-1} \cdot cs : R \in u \cdot \ell_1\}$$

$$:= \{a \in w : \sum_{k=1}^{\infty} a_k \sum_{j=1}^{k-1} \sum_{i=1}^{j-1} \frac{1}{|u_i|} \text{ converges}$$

and $\sum_{k=1}^{\infty} \frac{|R_k|}{|u_k|} < \infty\}.$

where, for all $a \in cs$

$$\gamma_k = \sum_{\nu = k+1}^{\infty} a_\nu \text{ and } R_k = \sum_{m = k+1}^{\infty} \gamma_m.$$

Then

(a) $[SE(u; \Delta^2)]^\beta = M_\beta$, for $E = \ell_\infty$ or $c$.

(b) If $1/u \in \ell_1$, then $[S_{c_0}(u; \Delta^2)]^\beta = M_3.$
(c) If \( 1/u \in \ell_{\infty} \setminus \ell_1 \), then \( [S_{c_0}(u; \Delta^2)]^\beta \neq M_\beta \).

He has also proved the following mapping theorems:

**Theorem 2.2.3.** \( A \in (E, F(u; \Delta^2)) \) if and only if

(i) \( \sum_k |a_{nk}| < \infty \), for each \( n \);

(ii) \( C \in (E, F) \),

where

\[
C = (c_{nk}) = ((\Delta a_{nk} - \Delta a_{n+1,k})u_k).
\]

**Theorem 2.2.4.** \( A \in (F(u; \Delta^2), E) \) if and only if

(i) \((a_{n1}) \) and \((a_{n2}) \in F(u; \Delta^2)\);

(ii) \( \sum_k a_{nk} \sum_{j=1}^{k-1} \sum_{i=1}^{j-1} \frac{1}{|u_j|} < \infty \);

(iii) \( Ru^{-1} \in (F, E) \),

where

\[
R = (R_{nk}) = \left( \sum_{m=1}^{\infty} \gamma_{nm} \right) \quad \text{and} \quad \gamma_{nm} = \sum_{i=m+1}^{\infty} a_{ni}.
\]

**Theorem 2.2.5.** \( A \in (F(u; \Delta^2), E(u; \Delta^2)) \), where \( F = c_0 \) or \( \ell_{\infty} \) and \( E = c_0 \) or \( \ell_{\infty} \), if and only if

(i) \( \sum_k a_{nk} \sum_{j=1}^{k-1} \sum_{i=1}^{j-1} |u_i|^{-1} < \infty \),

(ii) \( \sum_k |R_{nk}|/u_k|^{-1} < \infty \),

(iii) \( Ru^{-1} \in (F, E) \),

(iv) \((c_{n1}) \) and \((c_{n2}) \in F(u; \Delta^2)\),

where

\[
R = (R_{nk}), C = (c_{nk}) \text{ with}
\]
\[ R_{nk} = \sum_{m = k+1}^{\infty} \sum_{i = m+1}^{\infty} c_{ni} \text{ and } c_{nk} = (\Delta^2 a_{nk})u_k = (\Delta a_{nk} - \Delta a_{n+1,k})u_k. \]

He [57] proved the following as a corollary:

**Theorem 2.2.6.** \( A \in (c(u; \Delta^2), c(u; \Delta^2)) \) if and only if

1. \( \sum_k a_{nk} \sum_{j=1}^{k-1} \sum_{i=1}^{j-1} |u_i|^{-1} < \infty; \)

2. \( \sum_k |R_{nk}/u_k| < \infty; \)

3. \( Ru^{-1} \in (c,c), \)

4. \( (c_{n1}) \text{ and } (c_{n2}) \in c(u; \Delta^2). \)

In the present chapter we extend the spaces \( E(u; \Delta^2) \) to \( E(p; u, \Delta^2) \) and propose to determine its \( \alpha \)- and \( \beta \)-duals and to study some matrix maps. Our results generalize the above-mentioned theorems and yield them as special case for \( p_n = a \) constant sequence.

### 2.3. Köthe-Toeplitz duals

In this section we determine \( \alpha \)- and \( \beta \)-duals of \( E(p; u, \Delta^2) \) for \( E \in \{c_0, c, \ell_\infty\}. \)

**Theorem 2.3.1.** Let \( u \in U \). Then \( [SE(p; u, \Delta^2)]^a = M_\alpha(p) \), where for a constant \( M > 1 \),

\[ M_\alpha(p) := (\Delta^{-2}(M^{-1/p}/|u|))^{-1} \cdot \ell_1 \]

\[ := \{a \in w : \sum_{k=1}^{\infty} |a_k| \sum_{j=1}^{k-1} \sum_{i=1}^{j-1} M^{-1/p_i/|u_i|} |< \infty \}. \]

**Proof.** Let \( a \in M_\alpha(p) \) and \( x \in SE(p; u, \Delta^2) \). Then \( (u_k \Delta^2 x_k) \in E(p) \), and for some constant \( M > 1 \),

\[ |u_k \Delta^2 x_k|^{p_k} < M^{-1}. \]

Therefore,

\[ |\Delta^2 x_k| < M^{-1/p_k}/|u_k|. \]

for each \( k \), and
\[ x_k = \sum_{j=1}^{k-1} \sum_{i=1}^{j-1} \Delta^2 x_i \] (since \( x_1 = x_2 = 0 \)).

Hence

\[ \sum_{k=1}^{\infty} |a_k x_k| \leq \sum_{k=1}^{\infty} |a_k| \sum_{j=1}^{k-1} \sum_{i=1}^{j-1} |\Delta^2 x_i| \]

\[ \leq \sum_{k=1}^{\infty} |a_k| \sum_{j=1}^{k-1} \sum_{i=1}^{j-1} M^{-1/p_k} |u_i| < \infty. \]

It follows that \( a \in [SE(p; u, \Delta^2)]^o \), and so \( M_o(p) \subset [SE(p; u, \Delta^2)]^o \).

Conversely, let \( a \in [SE(p; u, \Delta^2)]^o \). Then, for \( x \in SE(p; u, \Delta^2) \),

\[ \sum_{k=1}^{\infty} |a_k x_k| < \infty. \] Let \( a \notin M_o(p) \). Then

\[ \sum_{k=1}^{\infty} |a_k| \sum_{j=1}^{k-1} \sum_{i=1}^{j-1} \frac{M^{-1/p_k}}{|u_i|} = \infty. \]

Since

\[ x = \left( \sum_{j=1}^{k-1} \sum_{i=1}^{j-1} \frac{M^{-1/p_k}}{|u_i|} \right)_{k=1}^{\infty} \in SE(p; u, \Delta^2), \]

so

\[ \sum_k |a_k x_k| = \infty. \]

Hence \( a \in M_o(p) \), and \([SE(p; u, \Delta^2)]^o \subset M_o(p)\). By Remark 1, \([E(p; u, \Delta^2)]^o = M_o(p)\).

This completes the proof of the theorem.

**Theorem 2.3.2.** Let \( u \in U \). We write,

\[ M_\beta(p) := \{ a \in (\Delta^{-2}(B^{1/p} \mid u \mid))^{-1} \cdot cs : R \in B^{-1/p} u \cdot \ell_1 \} \]

\[ := \{ a \in w : \sum_{k=1}^{\infty} a_k \sum_{j=1}^{k-1} \sum_{i=1}^{j-1} B^{1/p_t} \mid u_i \mid \text{ converges} \}

and \( \sum_{k=1}^{\infty} |R_k| \frac{B^{-1/p_k}}{||u||} < \infty \}, \]
for real $B > 0$, where, for all $a \in \mathbb{C}$,

$$
\gamma_k = \sum_{\nu = k+1}^{\infty} a_\nu \quad \text{and} \quad R_k = \sum_{m = k+1}^{\infty} \gamma_m.
$$

Then

(a) $[SE(p; u, \Delta^2)]^\beta = M_\beta(p)$, for $E = \ell_\infty$ or $c$.

(b) If $1/u \in \ell_1$, then $[SE(p; u, \Delta^2)]^\beta = M_\beta(p)$, for $E = c_0$.

(c) If $1/u \in \ell_\infty \setminus \ell_1$, then $[SE(p; u, \Delta^2)]^\beta \neq M_\beta(p)$, for $E = c_0$.

**Proof.** (a) Let $a \in M_\beta(p)$ and $x \in SE(p; u, \Delta^2)$. Then $(u_k \Delta^2 x_k) \in E(p)$ and we can choose an integer $B > 1$, such that $| u_k \Delta^2 x_k |_{p} < B^{-1}$.

Now, we have, by Abel's transformation

$$
(2.3.1) \quad \sum_{k=1}^{m} a_k x_k = \sum_{i=1}^{m-2} R_i \Delta^2 x_i - R_{m-1} \sum_{i=1}^{m-2} \Delta^2 x_i + \gamma_m \sum_{i=1}^{m-1} \Delta x_i.
$$

Since

$$
\sum_{i=1}^{\infty} | R_i | \Delta^2 x_i | \leq \sum_{i=1}^{\infty} | R_i | \frac{B^{-1/p_i}}{| u_i |} < \infty. \quad \text{(since } a \in M_\beta(p)),
$$

it follows that the first term on the right of (2.3.1) is absolutely convergent. Now, by Corollary 2 of Kizmaz [30], the convergence of $\sum_{k=1}^{\infty} a_k$ ($\sum_{j=1}^{k-1} \sum_{i=1}^{j-1} \frac{B^{-1/p_i}}{| u_i |}$) implies that the second and third terms on the right of (2.3.1) tend to zero, as $m \to \infty$, and we have

$$
(2.3.2) \quad \sum_{k} a_k x_k = \sum_{k} R_k \Delta^2 x_k.
$$

Hence $\sum_{k} a_k x_k$ is convergent for each $x \in SE(p; u, \Delta^2)$, and so $a \in [SE(p; u, \Delta^2)]^\beta$, which implies that $M_\beta(p) \subseteq [SE(p; u, \Delta^2)]^\beta$. 
Conversely, let \( a \in [SE(p; u, \Delta^2)]^\beta \). Then, for \( x \in SE(p; u, \Delta^2) \), \( \sum_k a_k x_k \) is convergent. Now, since \( x = (x_k) = \left( \sum_{j=1}^{k-1} \sum_{i=1}^{j-1} \frac{B^{1/p_i}}{|u_i|} \right)_{k=1}^\infty \in SE(p; u, \Delta^2) \), it follows that

\[
(2.3.3) \quad \sum_k a_k \sum_{j=1}^{k-1} \sum_{i=1}^{j-1} B^{1/p_i} / |u_i|
\]
is convergent. Further, by Corollary 2 of [30], we have the second and third terms on the right of (2.3.1) tend to zero. Therefore, also by (2.3.2), \( \sum_k R_k \Delta^2 x_k \) is convergent for all \( x \in SE(p; u, \Delta^2) \). We can write, for \( B > 1 \),

\[
\sum_k R_k \Delta^2 x_k = \sum_k R_k B^{-1/p_k} u_k \Delta^2 x_k B^{1/p_k},
\]
so that

\[
(2.3.4) \quad \sum_k |R_k| B^{-1/p_k} / |u_k| < \infty,
\]
since \( u_k \Delta^2 x_k \in E(p) \). Hence by (2.3.3) and (2.3.4), \( a \in M_{\beta}(p) \), which implies that \( [SE(p; u, \Delta^2)]^\beta \subseteq M_3(p) \). Hence \( [SE(p; u, \Delta^2)]^\beta = M_\beta(p) \).

Similarly, part (b) follows easily, provided \( 1/u \in \ell_1 \). For (c), let us define the sequence

\[
1/u = (1/u_k) = (-1)^{k+1}/k_{k=1}^\infty.
\]
Then \( 1/u \in \ell_\infty \ell_1 \) and it is easily seen that \( a \not\in M_\beta(p) \).

Again, by Remark 1, \( [E(p; u, \Delta^2)]^\beta \neq M_\beta(p) \).

**2.4. Matrix transformations**

In this section we characterize the matrix classes \((F, E(p; u, \Delta^2))\) and \((E(p; u, \Delta^2), F)\), where \( E, F \in \{c, c_0, \ell_\infty\} \) as in § 2.1.

**Theorem 2.4.1.** \( A \in (E, F(p; u, \Delta^2)) \) if and only if

(i) \( \sum_k |a_{nk}| < \infty \), for each \( n \);

(ii) \( C \in (E, F(p)) \),

where \( C = (c_{nk}) = ((\Delta a_{n,k} - \Delta a_{n-1,k})u_k) \).
**Proof.** It is easy consequence of the results due to Lascarides and Maddox [34] on the matrix classes \((E, F(p))\).

**Theorem 2.4.2.** \(A \in (F(p; u, \Delta^2), E)\) if and only if

(i) \((a_n)\) and \((a_{n2}) \in F(p; u, \Delta^2)\);

(ii) \(\sum_{k=1}^{\infty} a_{nk} \sum_{j=1}^{k-1} \sum_{i=1}^{j-1} B^{1/p_1} | u_i |^{-1} < \infty, (B > 1)\), for each \(n\);

(iii) \(\sum_{k=1}^{\infty} | R_{nk} | B^{-1/p_k} | u |^{-1} < \infty, (B > 1)\), for each \(n\);

(iv) \(Ru^{-1} \in (F(p), E)\),

where \(R = (R_{nk}) = (\sum_{m=1}^{\infty} \gamma_{nm})\) and \(\gamma_{nm} = \sum_{i=1}^{\infty} a_{ni}\).

**Proof.** *Necessity*: Let \(A \in (F(p; u, \Delta^2), E)\). Then, the series \(A_n(x) = \sum_{k=1}^{\infty} a_{nk} x_k\) converges for each \(n\) and \(Ax = (A_n(x)) \in E\), for each \(x \in F(p; u, \Delta^2)\). Hence \(A_n = (a_{nk})_{k=1}^{\infty} \in [F(p; u, \Delta^2)]^2\) for each \(n\). Therefore, by Theorem 2.3.2, conditions (ii) and (iii) are followed. Condition (i) follows easily, since the sequence \(x = e\) belongs to \(F(p; u, \Delta^2)\). Now, using (2.3.1), we have

\[
A_n(m, x) = \sum_{k=1}^{m-2} R_{nk} \Delta^2 x_k - R_{n,m-1} \sum_{k=1}^{m-2} \Delta^2 x_k + \sum_{k=1}^{m-1} \Delta x_k.
\]

Taking limit as \(m \to \infty\), and using conditions (ii) and (iii) together with Corollary 2 of Kizmaz [30], we have

\[
(2.4.1) \lim_m A_n(m, x) = A_n(x) = \sum_k R_{nk} \Delta^2 x_k = \sum_k R_{nk} B^{-1/p_k} u_k \Delta^2 x_k B^{1/p_k}.
\]

Define the sequence \(y = (y_k)\) by \(y_k = u_k \Delta^2 x_k B^{1/p_k}\). Then \(x \in F(p; u, \Delta^2)\) if and only if \(y \in F(p)\) and \(A_n(x) = R_n(y)\), where \(Ru^{-1} = (R_n(y)), R_n(y) = \sum_k (R_{nk}/u_k) y_k\). Hence, \(A \in (F(p; u, \Delta^2), E)\) implies \(Ru^{-1} \in (F(p), E)\), that is, condition (iv) follows.

* Sufficiency*: Let \(x \in F(p; u, \Delta^2)\). Define \(x = (x_k)\) by

\[
x_k = \begin{cases} 
  x_1, & k = 1 \\
  x_2, & k = 2 \\
  x_k, & k > 2;
\end{cases}
\]
where $x' = (x'_k) \in SF(p; u, \Delta^2)$. Again, by (2.3.1), we have

$$A_n(x) = a_{n1}x_1 + a_{n2}x_2 + \sum_k R_{nk} \Delta^2 x'_k$$

$$= a_{n1}x_1 + a_{n2}x_2 + \sum_k \frac{R_{nk}}{u_k} B^{-1/p_k} u_k \Delta^2 x'_k B^{1/p_k}.$$

Therefore, by conditions (i)-(iv), $Ax = (A_n(x))$ exists and $Ax = Ru^{-1} \cdot y \in E$. Hence $A \in (F(p; u, \Delta^2), E)$.

This completes the proof of the theorem.

Combining Theorems 2.4.1 and 2.4.2, we get :

**Theorem 2.4.3.** $A \in (E(p; u, \Delta^2), E(p; u, \Delta^2))$ if and only if

(i) $\sum_k a_{nk} \sum_{j=1}^{k-1} \sum_{i=1}^{j-1} B^{1/p_i} \left| u_i \right|^{-1} < \infty, (B > 1)$, for each $n$;

(ii) $\sum_k \frac{R_{nk}}{u_k} B^{-1/p_k} < \infty, (B > 1)$, for each $n$;

(iii) $Ru^{-1} \in (E(p), F(p))$;

(iv) $(c_{n1})$ and $(c_{n2}) \in E(p; u, \Delta^2)$,

where $R = (R_{nk}), C = (c_{nk})$, with

$$R_{nk} = \sum_{m=k+1}^{\infty} \sum_{i=m+1}^{\infty} c_{ni} \text{ and } c_{nk} = (\Delta^2 a_{nk}) u_k = (\Delta a_{nk} - \Delta a_{n-1,k}) u_k.$$ 

The proof easily follows by Theorems 2.3.2, 2.4.1 and 2.4.2.

We close this chapter with the following corollary :

**Corollary 2.4.1.** $A \in (c(p; u, \Delta^2), c(p; u, \Delta^2))$ if and only if

(i) $\sum_k a_{nk} \sum_{j=1}^{k-1} \sum_{i=1}^{j-1} B^{1/p_i} \left| u_i \right|^{-1} < \infty, (B > 1)$, for each $n$;

(ii) $\sum_k \frac{R_{nk}}{u_k} B^{-1/p_k} < \infty, (B > 1)$, for each $n$;

(iii) $Ru^{-1} \in (c(p), c(p))$;

(iv) $(c_{n1})$ and $(c_{n2}) \in c(p; u, \Delta^2)$,

where $R$ and $C$ are defined as in Theorem 2.4.3.