Chapter 0

NOTE ON CONVENTIONS

Here we state a few conventions regarding notations and definitions, which will be used throughout; others will be introduced as they become necessary.

0.1. The symbols $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$

$\mathbb{N} :=$ The set of natural numbers.

$\mathbb{R} :=$ The set of real numbers.

$\mathbb{C} :=$ The set of complex numbers.

0.2. Limit, supremum and infimum

$\lim_k :$ means $\lim_{k \to \infty}$.

$\sup_k :$ means $\sup_{k=1,2,...}$, unless otherwise stated.

$\inf_k :$ means $\inf_{k=1,2,...}$, unless otherwise stated.

0.3. Summation convention

$\sum_k :$ means summation over $k = 1$ to $k = \infty$, unless otherwise stated.

0.4. $O$ and $o$

If $\phi$ is a positive function of a variable which tends to a limit, we shall write

\[ f = O(\phi), \]

if $|f|/\phi < K$, where $K$ is an absolute constant, and

\[ f = o(\phi), \]

if $f/\phi \to 0$. (see Hardy [26]).
0.5. Sequences

\( x = (x_k) \), the sequence whose \( k \)th term is \( x_k \).

\( \theta = (0, 0, 0, \ldots) \), the zero sequence.

\( e_k = (0, 0, 0, \ldots, 0, 1, 0, 0, \ldots) \), the sequence whose \( k \)th component is 1 and others are zeros, for all \( k \in \mathbb{N} \).

\( e = (1, 1, 1, \ldots) \).

\( p = (p_k) \), the sequence of strictly positive reals.

0.6. Difference sequences

For any sequence \( x \), the difference sequences \( \Delta x \) and \( \Delta^2 x \) are defined by:

\[
\Delta x = \Delta^1 x_n = x_n - x_{n+1} \quad (\Delta^0 x = x),
\]

\[
\Delta^2 x = \Delta^2 x_n = \Delta x_n - \Delta x_{n+1}.
\]

0.7. Sequence spaces

\( w := \{ x = (x_k) : x_k \in \mathbb{R} \text{ (or } \mathbb{C}) \} \), the space of all sequences, real or complex.

\( \phi := \) The space of finite sequences, i.e. of all sequences terminating in zeros.

\( \ell := \{ x \in w : \sum_k |x_k| < \infty \} \), (see [42]).

\( \ell_\infty := \{ x \in w : \sup_k |x_k| < \infty \} \), the space of bounded sequences, (see [42], p. 29).

\( c_0 := \{ x \in w : \lim_k x_k = 0 \} \), the space of null sequences, (see [42], p. 29).

\( c := \{ x \in w : \lim_k x_k = l \}, \text{ for some } l \in \mathbb{C} \}, \) the space of convergent sequences, (see [42], p. 29).

\( \ell_\infty, c_0 \) and \( c \) are Banach spaces with the usual norm:

\[
\| x \| = \sup_k |x_k|, \quad (\text{see Maddox [42], p. 104}).
\]
\[ C := \{x \in w : |x_m - x_n| \to 0, \text{as } m, n \to \infty\}, \text{the space of Cauchy sequences, (see [42], p. 14).} \]

\[ cs := \{a = (a_k) : x = (x_n) = (\sum_{k=1}^{n} a_k) \in c\}, \text{the space of convergent series.} \]

\[ \ell_1 := \{a = (a_k) : \sum |a_k| < \infty\}, \text{the space of absolutely convergent series.} \]

We observe that \[ \ell_1 \subset cs \subset c_0 \subset c = C \subset \ell_\infty, \]
all inclusions being strict, (see [42], p. 15; see also [1]).

\[ BV := \{x \in w : \sum |\Delta x_k| = \sum |x_k - x_{k-1}| < \infty, x_0 = 0\}, \text{the space of sequences of bounded variation, (see [42], p. 32).} \]

This is a Banach space with the usual norm:

\[ \| x \| = \sum_{k} |x_k - x_{k-1}|. \]

If, for a given series \( \sum a_n, x_n = a_1 + a_2 + \cdots + a_n \). Then \( \ell_1 = BV \), and \( \ell \subset BV \subset c \). BV and cs overlap but neither contains the other, (see [42], p. 32).

\[ W_\infty := \{x \in w : \sup_n \frac{1}{n} \sum_{k=1}^{n} |x_k| < \infty\}, \text{the space of strongly Cesàro-bounded sequences, (see [38]).} \]

\[ W_0 := \{x \in w : \lim_n \frac{1}{n} \sum_{k=1}^{n} |x_k| = 0\}, \text{the space of strongly Cesàro-null sequences, (see [38]).} \]

\[ W := \{x \in w : \lim_n \frac{1}{n} \sum_{k=1}^{n} |x_k - l| = 0, \text{for some } l \in c\}, \text{the space of strongly Cesàro-summable sequences, (see [38]).} \]

\[ \ell_p := \{x \in w : \sum_{k} |x_k|^p < \infty\}, 0 < p < \infty, \text{ (see [42], p. 30).} \]

\[ W_p := \{x \in w : \lim_n \frac{1}{n} \sum_{k=1}^{n} |x_k - l|^p = 0, \text{for some } l \in c\}, 0 < p < \infty. \]

In the case \( 1 \leq p < \infty \), the spaces \( \ell_p \) and \( W_p \) are Banach spaces normed by

\[ \| x \| = \left( \sum_{k} |x_k|^p \right)^{1/p}. \]
and
\[ \| x \| = \sup \left( \frac{1}{n} \sum_{k=1}^{n} |x_k|^p \right)^{1/p} \]
respectively. If $0 < p < 1$, then $\ell_p$ and $W_p$ are complete $p$-normed spaces, $p$-normed by
\[ \| x \| = \sum_{k} |x_k|^p \]
and
\[ \| x \| = \frac{1}{n} \sum_{k=1}^{n} |x_k|^p \]
respectively. (see [17], p. 244). For the concept of $p$-norm, refer ([42], p. 103).

The following subspaces of $w$ were first introduced and discussed by Simons [82], and Maddox [38].

\[ \ell(p) := \{ x \in w : \sum_{k} |x_k|^{p_k} < \infty \}, \] [82];
\[ \ell_\infty(p) := \{ x \in w : \sup_{k} |x_k|^{p_k} < \infty \}, \] [82];
\[ c(p) := \{ x \in w : \lim_{k} |x_k - l|^{p_k} = 0, \text{ for some } l \in \mathbb{C} \}, \] [39];
\[ c_0(p) := \{ x \in w : \lim_{k} |x_k|^{p_k} = 0 \}, \] [39];
\[ W_\infty(p) := \{ x \in w : \sup_n \left( \frac{1}{n} \sum_{k=1}^{n} |x_k|^{p_k} \right) < \infty \}, \] [38];
\[ W(p) := \{ x \in w : \lim_{l \in \mathbb{C}} \frac{1}{n} \sum_{k=1}^{n} |x_k - l|^{p_k} = 0, \text{ for some } l \in \mathbb{C} \}, \] ([38], [41]);
\[ W_0(p) := \{ x \in w : \lim_{l \in \mathbb{C}} \frac{1}{n} \sum_{k=1}^{n} |x_k|^{p_k} = 0 \}, \] [38];
and
\[ M_0(p) := \bigcup_{N>1} \{ x \in w : \sum_{k} |x_k|^{N^{-1/p_k}} < \infty \}, \] [41].

Let $p = (p_k)$ be bounded. Then $c_0(p)$ is a linear metric space paranormed by:
\[ g_1(x) = \sup_{k} |x_k|^{p_k/M}, \]
where \( M = \max(1, \sup_k p_k) \). \( \ell_\infty(p) \) and \( c(p) \) are paranormed by \( g_1(x) \) defined above if and only if \( \inf_k p_k > 0 \). \( \ell(p) \) and \( W(p) \) are paranormed by:

\[
g_2(x) = \left( \sum_k |x_k|^{p_k} \right)^{1/M}
\]

and

\[
g_3(x) = \sup \left( \frac{1}{n} \sum_{k=1}^n |x_k|^{p_k} \right)^{1/M}
\]

respectively (definition of a paranorm is given in the subsequent section of this chapter).

All the spaces defined above are complete in their topologies. In general \( \ell_\infty(p), c_0(p), c(p), \ell(p) \) and \( W(p) \) are not normed spaces. (see [17]). If \( p_k = p \) for all \( k \), then \( \ell_\infty(p) = \ell_\infty, c_0(p) = c_0, c(p) = c, \ell(p) = \ell, M_0(p) = l \) and \( W(p) = W_p \). \( \ell_p \) and \( W_p \) are Banach spaces for \( 1 \leq p < \infty \) and complete \( p \)-normed spaces for \( 0 < p < 1 \).

0.8. Paranorm and total paranorm

A paranorm on a linear topological space \( X \) is a function \( g : X \to \mathbb{R} \) which satisfies the following axioms:

for any \( x,y,x_0 \in X \) and \( \lambda, \lambda_0 \in C \),

(i) \( g(\theta) = 0 \).

(ii) \( g(x) = g(-x) \),

(iii) \( g(x + y) \leq g(x) + g(y) \) (subadditivity),

and

(iv) the scalar multiplication is continuous, that is,

\[
\lambda \to \lambda_0, x \to x_0 \text{ imply } \lambda x \to \lambda_0 x_0;
\]

in others words,

\[
| \lambda - \lambda_0 | \to 0, g(x - x_0) \to 0 \text{ imply } g(\lambda x - \lambda_0 x_0) \to 0.
\]

A paranormed space is a linear space \( X \) with a paranorm \( g \) and it is written as \( (X, g) \), (see Maddox [42], p. 92).
Any function \( g \) which satisfies all the conditions (i)-(iv) together with the condition

\[(v) \ g(x) = 0 \text{ if and only if } x = 0,\]

is called a total paranorm on \( X \), and the pair \( (X, g) \) is called a total paranormed space, (see Maddox [42], p. 92).

If \( X \) is a linear space and \( g \) is any given function having the properties (i)-(v), then it follows that \( d \) defined by \( d(x, y) = g(x - y) \) is such that \( (X, d) \) is a linear metric space, (see [42]).

Let \( (X, g) \) be a paranormed space. A sequence \( (b_k) \) of elements of \( X \) is called a basis for \( X \) if and only if, for each \( x \in X \), there exists a unique sequence \( (\lambda_k) \) of scalars such that

\[x = \sum_{k=1}^{\infty} \lambda_k b_k,\]

that is,

\[g(x - \sum_{k=1}^{n} \lambda_k b_k) \to 0, \text{ as } n \to \infty.\]

This idea of a basis was introduced by J. Schauder [81] and is often called a Schauder basis, (see Maddox [42], p. 98).

0.9. r-convex space

For \( 0 < r \leq 1 \), a non-empty subset \( U \) of a linear space is said to be absolutely r-convex if \( x, y \in U \) and \( \lambda \mid \lambda \mid^r + \mu \mid \mu \mid^r \leq 1 \) together imply that \( \lambda x + \mu y \in U \), for \( \lambda, \mu \in \mathbb{C} \). It is clear that if \( U \) is absolutely r-convex, then it is absolutely t-convex for \( t < r \). A linear topological space \( X \) is said to be r-convex if every neighbourhood of \( 0 \in X \) contains an absolutely r-convex neighbourhood of \( 0 \in X \). The r-convexity for \( r > 1 \) is of little interest, since \( X \) is r-convex for \( r > 1 \) if and only if, \( X \) is the only neighbourhood of \( 0 \in X \) (see Maddox and Roles [44]).

0.10. Bounded and locally bounded sets

A subset \( E \) of \( X \) is said to be bounded if, for each neighbourhood \( U \) of \( 0 \in X \), there is some positive scalar \( \alpha \) such that \( E \subset \alpha U \). \( X \) is called locally bounded if there exists in it a bounded neighbourhood of \( 0 \), (see [85]).
0.11. Continuous and Köthe-Toeplitz duals

If $X$ is a space of sequences $x \in w$, then we denote the continuous dual of $X$ by $X^*$, that is, the set of all continuous linear functionals on $X$. (see [42], p. 113).

We denote the absolute Köthe-Topelitz dual (or $\alpha$-dual) and generalized Köthe-Toeplitz dual (or $\beta$-dual) by $X^\alpha$ and $X^\beta$ respectively, defined by:

$$X^\alpha := \{ a \in w : \sum_k |a_kx_k| < \infty, \text{ for all } x \in X\},$$

and

$$X^\beta := \{ a \in w : \sum_k a_kx_k < \infty, \text{ for all } x \in X\}. \text{ (see [41] and [46]).}$$

0.12. Summability methods and class of matrices

Let $X$ and $Y$ be two non-empty subsets of the space $w$. Let $A = (a_{nk}), (n, k = 1, 2, 3, \ldots)$ be an infinite matrix with elements of real or complex numbers. We write

$$A_n(x) = \sum_k a_{nk}x_k.$$ 

Then $Ax = (A_n(x))$ is called the $A$-transform (or $A$-mean) of $x$. Also

$$\limnAx = \limnA_n(x),$$

whenever it exists. Let this limit, if it exists, be a finite number $s$. Then, we say that the sequence $x$ (or the series $\sum_k a_k$, of which $x_k$ is its $k$th partial sum) is summable by the matrix method $A$, to the value $s$, and we write

$$x \in s(A).$$

(For a study of the Theory of Summability and its Applications, reference may be made to Borel [14], Cooke [18], Dienes [21], Ford [23], Hardy [26], Knopp [31], Petersen [65], Peyerimhoff [66], Powell and Shah [67], Szász [84]. Zeller and Beekman [87], Zygmund [88] and others; see also Ahmad [2]).

If $x \in X$ implies $Ax \in Y$, we say that $A$ defines a (matrix) transformation from $X$ into $Y$. denoted by $A : X \rightarrow Y$. By $(X, Y)$ we mean the class
of matrices $A$ such that $A : X \rightarrow Y$. By $(X, Y; P)$ we mean the subset of $(X, Y)$ for which limits or sums are preserved. We also write $(X, Y)_{\text{reg}}$ for $(X, Y; P)$.

Thus, a summability method $A$ is nothing but $A \in (X, Y)$. If $A$ is regular we write $A \in (X, Y)_{\text{reg}}$.

0.13. Bounded variation

By $\phi(x) \in BV(h, k)$ we mean that the function $\phi(x)$ is of bounded variation in the interval $(h, k)$, that is,

$$\int_h^k | \phi(x) | < \infty.$$ 

0.14. Some inequalities

The following inequalities will be useful.

0.14.1. The triangle inequality, (see [42] and [27]).

For any $a, b \in \mathbb{C}$, $|a + b| \leq |a| + |b|$.

0.14.2. Hölder's inequality, (see [42], p. 20; see also [27]).

Let $p > 1, 1/p + 1/q = 1, a_1, \ldots, a_n \geq 0$ and $b_1, \ldots, b_n \geq 0$. Then

$$\sum_{k=1}^n a_kb_k \leq \left( \sum_{k=1}^n a_k^p \right)^{1/p} \left( \sum_{k=1}^n b_k^q \right)^{1/q}.$$ 

Also,

$$\sum_{k=1}^n a_kb_k \leq \left( \sum_{k=1}^n a_k \right) \max_b b_k.$$ 

0.14.3. Minkowski's inequality, (see [42]; see also [27]).

Let $p \geq 1, a_1, \ldots, a_n \geq 0$ and $b_1, \ldots, b_n \geq 0$. Then

$$\left( \sum (a_k + b_k)^p \right)^{1/p} \leq \left( \sum a_k^p \right)^{1/p} + \left( \sum b_k^p \right)^{1/p},$$

where sums run from $k = 1$ to $k = n$.

0.14.4. Let $0 < p \leq 1, a_1, \ldots, a_n \geq 0$ and $b_1, \ldots, b_n \geq 0$. Then
\[
\sum_{k=1}^{n} (a_k + b_k)^p \leq \sum_{k=1}^{n} a_k^p + \sum_{k=1}^{n} b_k^p, \quad \text{(see [42] and [27]).}
\]

Inequalities 0.14.1, 0.14.3 and 0.14.4 yield the following frequently used results valid for complex \(a_k, b_k\) : (see Maddox [42], pp. 20-22 and Hardy, Littlewood and Polya [27]),

\[
\left( \sum |a_k + b_k|^p \right)^{1/p} \leq \left( \sum |a_k|^p \right)^{1/p} + \left( \sum |b_k|^p \right)^{1/p} (p \geq 1),
\]

\[
\sum |a_k + b_k|^p \leq \sum |a_k|^p + \sum |b_k|^p \quad (0 < p \leq 1).
\]

**0.14.5.** For any \(E > 0\) and any two complex numbers \(a, b\), (see Maddox [41] and Maddox and Willey [45]),

(i) \(|ab| \leq E (|a|^q E^{-q} + |b|^q)\), \((p > 1, 1/p + 1/q = 1)\),

(ii) \(|a|^p - |b|^p \leq |a + b|^p \leq |a|^p + |b|^p \quad (0 < p \leq 1)\),

and

(iii) \(|\lambda|^p \leq \max(1, |\lambda|) \quad (0 < p \leq 1)\),

where \(\lambda\) is complex scalar, and for \(p = (p_k)\), a strictly positive real sequence, such that \(H = \sup p_k < \infty\),

\[|\lambda|^{p_k} \leq \max(1, |\lambda|^H).\]