Chapter V

ON GENERALIZED SPACES OF 
$B$-BOUNDED SEQUENCES

5.1. Definitions and notations

All relevant definitions and notations except those given here are the same as in Chapters 0 and I.

For any sequence $x = (x_n) \in w$ and for any given sequence $B = (B_i)$ of infinite matrices with $B_i = (b_{nk}(i))$, we write

\begin{equation}
(5.1.1) \quad t^B_{in}(x) = (B_i x)_n = \sum_k b_{nk}(i)x_k
\end{equation}

and let $x_n$ be the $n$th partial sum of a given series $\sum z_n$ (denoted by $z$), so that

\begin{equation}
(5.1.2) \quad x_n = z_0 + z_1 + \cdots + z_n,
\end{equation}

and

\begin{equation}
(5.1.3) \quad x_n - x_{n-1} = z_n.
\end{equation}

Then, for $n \geq 0$, we have

\[
\psi_{in}(x) = t^B_{in}(x) - t^B_{i,n-1}(x) = \sum_k = 0 \{b_{nk}(i) - b_{n-1,k}(i)\} x_k = \sum_k = 0 \Delta b_{nk}(i) x_k = \sum_v = 0 \{\sum_k = v \Delta b_{nk}(i)\} z_v, = \sum_v = 0 g(n, v, i) z_v,
\]

(whenever the change of order of summation is justified),
where
\[ g(n, v, i) = \sum_{k=v}^{\infty} \Delta b_{nk}(i), \]
and
\[ \Delta b_{nk}(i) = \{b_{nk}(i) - b_{n-1,k}(i)\}. \]

Let \( p = (p_n) \) be a sequence of strictly positive real numbers, with \( \sup_n p_n < \infty \). Riazuddin ([72]; see also [73]) defined the sequence spaces:

\[ \mathcal{C}^{\infty} := \{x \in c^\mathcal{B} : \sup \psi_{in}(x) < \infty\}, \]
\[ \mathcal{C}^p := \{x \in c^\mathcal{B} : \sup \psi_{in}(x) |^p < \infty\}. \]

We generalize these spaces to define:

\[ \mathcal{C}^{p,s} := \{x \in c^\mathcal{B} : \sup \psi_{in}(x) |^{pn} < \infty, s \geq 0\} \]

so that, for \( s = 0, \mathcal{C}^{p,s} = \mathcal{C}^p = \mathcal{C}^0 \) (if \( p \) is constant for all \( n \)). If \( p_n = 1 \), for all \( n \), then it reduces to \( \mathcal{C}^{0,s} \).

In the special cases in which
\[
\begin{align*}
b_{nk}(i) &\equiv \hat{b}_{nk}(i) = \begin{cases} (n+1)^{-1}, & i \leq k \leq i+n, \\ 0, & \text{otherwise}; \end{cases} \\
b_{nk}(i) &\equiv \check{b}_{nk}(i) = \begin{cases} n^{-1}, & \sigma^1(i) \leq k \leq \sigma^n(i), \\ 0, & \text{otherwise}, \end{cases}
\end{align*}
\]
the space \( \mathcal{C}^{p,s} \) reduces to the spaces \( \mathcal{C}^{p,s} \) and \( \mathcal{C}^{p,s} \) which are the generalizations of the spaces \( \mathcal{C}^{p,s} \) and \( \mathcal{C}^{p,s} \) of almost-bounded and \( \sigma \)-bounded sequences respectively (see Nanda [59] and Saraswat [75]); of course, if \( B = (B_i) = I \), the identity matrix, then it reduces to the space \( \ell^{p,s}(p) \) defined by Basarir [12] as an extension of the space \( \ell^{p,s}(p) \) (defined by Simons [82]).
5.2. Introduction

Riazuddin ([70]; see also [73]) proved the following results concerning the topological properties and mapping theorems for the spaces $L^g_\infty$ and $L^g_\infty(p)$.

**Theorem 5.2.1.** If $\inf p_n > 0$, then $L^g_\infty(p)$ is a complete linear topological space over the complex field paranormed by $h$ defined by:

$$h(z) = \sup_{i,n} | \psi_{in}(z) |^{p_n/M}, \quad \text{for all } z \in L^g_\infty(p),$$

where $M = \max(1, \sup_n p_n)$.

In particular, $L^g_\infty$ is a Banach space normed by

$$\| z \| = \sup_{i,n} | \psi_{in}(z) | .$$

**Theorem 5.2.2.** Let $0 < p_n \leq q_n$. Then $L^g_\infty(q)$ is a closed subspace of $L^g_\infty(p)$.

**Theorem 5.2.3.** $L^g_\infty(p)$ is 1-convex.

**Theorem 5.2.4.** $A \in (\ell_\infty, L^g_\infty)$ if and only if

$$\sup_{i,n} \sum_k | \Delta g(n,k,i) | < \infty,$$

where

$$\Delta g(n,k,i) = \sum_{j=0}^{\infty} g(n,k,i) a_{jk}.$$

**Theorem 5.2.5.** $A \in (\ell_\infty(p), L^g_\infty)$ if and only if for every integer $L > 1$, 

$$\sup_{i,n} \sum_k | \Delta g(n,k,i) |^{1/p_k} < \infty,$$

where $\Delta g(n,k,i)$ is the same as in Theorem 5.2.4.

**Theorem 5.2.6.** Let $\inf p_n > 0$. Then $A \in (\ell_\infty, L^g_\infty(p))$ if and only if

$$\sup_{i,n} \left( \sum_k | \Delta g(n,k,i) | \right)^{p_n} < \infty.$$
Theorem 5.2.7. \( A \in (\ell(p), \mathcal{L}_\infty^g) \) if and only if

(i) there exists an integer \( K > 1 \) such that

\[
\sup_{i,n} \sum_k | \Delta g(n,k,i) |^{q_k} K^{-q_k} < \infty,
\]

for \( 1 < p_k < \infty, \ p_k^{-1} + q_k^{-1} = 1; \)

(ii) \( \sup_{i,n,k} | \Delta g(n,k,i) |^{p_k} < \infty, \) for \( 0 < p_k \leq 1. \)

Theorem 5.2.8. \( A \in (c_0(p), \mathcal{L}_\infty^g(p)) \) if and only if there exists an integer \( K > 1 \) such that

\[
\sup_{i,n} \left\{ \sum_k | \Delta g(n,k,i) |^{K^{-1/p_k}} \right\}^{p_n} < \infty.
\]

In the present chapter we study the topological properties and mapping theorems for our new spaces \( \mathcal{L}_\infty^g(s) \) and \( \mathcal{L}_\infty^g(p,s) \) and prove theorems which generalize all the above-mentioned results and yield them as special cases and also the results corresponding to \( \hat{\mathcal{L}}_\infty^g(s), \hat{\mathcal{L}}_\infty^g(p,s), \mathcal{L}_\infty^g(s) \) and \( \mathcal{L}_\infty^g(p,s). \)

5.3. Topological properties

We prove the following theorems concerning the topological properties of the space \( \mathcal{L}_\infty^g(p,s). \)

Theorem 5.3.1. If \( \inf p_n > 0, \) then \( \mathcal{L}_\infty^g(p,s) \) is a complete linear topological space over the complex field \( \mathbb{C} \) paranormed by:

\[
g(z) = \sup_{i,n} n^{-s/M} | \psi_{i,n}(z) |^{p_n/M},
\]

for all \( z \in \mathcal{L}_\infty^g(p,s), \) where \( M = \max(1, \sup_n p_n). \) In particular, \( \mathcal{L}_\infty^g(s) \) is a Banach space normed by

\[
\| z \| = \sup_{i,n} n^{-s} | \psi_{i,n}(z) |.
\]

Theorem 5.3.2. Let \( 0 < p_n \leq q_n. \) Then \( \mathcal{L}_\infty^g(q,s) \) is a closed subspace of \( \mathcal{L}_\infty^g(p,s). \)
Theorem 5.3.3. $L^B_{\infty}(p, s)$ is 1-convex.

5.4. Lemmas

We need the following lemmas for the proof of Theorem 5.3.1.

Lemma 5.4.1. $L^B_{\infty}(p, s)$ is a linear space over the complex field $\mathbb{C}$.

The proof is easy and is based on the well-known inequalities given in 0.14.5 (last two inequalities).

Lemma 5.4.2. If \( \inf p_n > 0 \), then $L^B_{\infty}(p, s)$ is a linear topological space paranormed by $g$ defined by (5.3.1).

Proof. We can easily see that \( g(0) = 0 \), and \( g(z) = g(-z) \), for all \( z \in L^B_{\infty}(p, s) \). The subadditivity of $g$ follows from the following steps:

(i) apply the inequality 0.14.5 (ii) to

\[
| \psi_{in}(a + b) |^{p_n/M} \quad \text{(putting } z = a + b),
\]

that is,

\[
| \psi_{in}(a + b) |^{p_n/M} \leq | \psi_{in}(a) |^{p_n/M} + | \psi_{in}(b) |^{p_n/M}.
\]

(ii) then multiply by $n^{-s}$ and take supremum on both sides.

It remains to show that the scalar multiplication is continuous.

It follows from inequality 0.14.5 (iii) that, for \( \lambda \in \mathbb{C} \) and \( z \in L^B_{\infty}(p, s) \),

\[
g(\lambda z) \leq \max(1, |\lambda|)g(z).
\]

Therefore, \( \lambda \to 0, z \to 0 \Rightarrow \lambda z \to 0 \); and if \( \lambda \) is fixed, then \( z \to 0 \Rightarrow \lambda z \to 0 \).

Now, let \( \inf p_n = m > 0 \). Then, we have

\[
g(\lambda z) \leq \max(|\lambda|, |\lambda|^{m/M})g(z).
\]

Hence \( \lambda \to 0 \Rightarrow \lambda z \to 0 \) (\( z \) fixed).

This completes the proof of the lemma.
Lemma 5.4.3. For $\inf p_n > 0$, $\mathcal{L}_\infty^B(p, s)$ is complete with respect to the paranorm topology.

Proof. Let $(z^l_n)$ be a Cauchy sequence in $\mathcal{L}_\infty^B(p, s)$. Then $(z^l_n)$, for each $n$, is a Cauchy sequence in $\mathcal{C}$ and hence $z^l_n \to z^l_n$, for each $n$.

Given $\epsilon > 0$, there exists $N_0$ such that, for $l, l' > N_0$,

$$|\psi(z^l_n - z^l'_n)|^{p_n/M} < \epsilon.$$ for all $i, n$.

Taking limit, as $l' \to \infty$, we have

$$|\psi(z^l_n - z)|^{p_n/M} < \epsilon.$$ for all $i, n$.

In general $g$ is not a norm. But when $p_n = \lambda$, for all $n$, $g$ is a norm and then $\mathcal{L}_\infty^B(p, s)$ is a Banach space.

5.5. Proofs of Theorems 5.3.1-5.3.3

Proof of Theorem 5.3.1. This is obtained by combining Lemmas 1, 2 and 3.

Proof of Theorem 5.3.2. We may assume without loss in generality that $M = 1$. Let $z \in \mathcal{L}_\infty^B(q, s)$. Then, there exists a constant $K > 1$ such that

$$n^{-s/M} |\psi(z)|^{|q_n/M|} \leq K.$$ for all $i, n$.

Then, since $0 < q_n \leq p_n$, we have

$$n^{-s/M} |\psi(z)|^{p_n/M} \leq n^{-s/M} |\psi(z)|^{|q_n/M|} \leq K,$$

for all $i, n$. Thus $z \in \mathcal{L}_\infty^B(p, s)$.

To show that $\mathcal{L}_\infty^B(q, s)$ is closed, suppose that $(z^l) \in \mathcal{L}_\infty^B(q, s)$ with $z^l \to z \in \mathcal{L}_\infty^B(p, s)$. Then, for every $0 < \epsilon < 1$, there exists $N$ such that, for all $i, n$,

$$n^{-s} |\psi(z^l - z)|^{p_n/M} < \epsilon.$$ for all $l > N$.

$$\Rightarrow n^{-s} |\psi(z^l - z)|^{q_n/M} < n^{-s} |\psi(z^l - z)|^{p_n/M} < \epsilon,$$ for all $l > N$,.
This proves the theorem.

**Proof of Theorem 5.3.3.** It is easy to see that, for $0 < \delta < 1$, the set
\[ U = \{ z \in \mathcal{L}_\infty^g(p, s) : g(z) < \delta \} \]
is absolutely 1-convex. Hence the theorem.

### 5.6. Mapping Theorems

We write
\[
\psi_{in}(Az) = \sum_{j=0}^{\infty} \sum_{v=0}^{\infty} \sum_{k=1}^{\infty} \Delta b_{nk}(i)a_{vj}z_j
\]
\[ = \sum_{j=0}^{\infty} \sum_{v=0}^{\infty} g(n, v, i)a_{vj}z_j \]
\[ = \sum_{j=0}^{\infty} g^*(n, j, i)z_j, \]
where
\[ g^*(n, j, i) = \sum_{v=0}^{\infty} g(n, v, i)a_{vj}, \]
provided that the infinite sums involved exist.

We prove the following:

**Theorem 5.6.1.** $A \in (\ell_\infty, \mathcal{L}_\infty^g(s))$ if and only if
\[
\sup_{i,n} n^{-s} \sum_{j=0}^{\infty} |g^*(n, j, i)| < \infty.
\]

**Theorem 5.6.2.** $A \in (\ell_\infty(p), \mathcal{L}_\infty^g(s))$ if and only if
\[
\sup_{i,n} n^{-s} \sum_{j=0}^{\infty} |g^*(n, j, i)| L^{1/p_1} < \infty,
\]
for every integer $L > 1$.

**Theorem 5.6.3.** Let $\inf p_n > 0$. Then $A \in (\ell_\infty, \mathcal{L}_\infty^g(p, s))$ if and only if
Theorem 5.6.4. $A \in (\ell(p), \mathcal{L}_\infty^G(s))$ if and only if

(i) there exists an integer $K > 1$ such that

$$ \sup_{i,n} n^{-s} \left( \sum_{j=0}^\infty | g^*(n, j, i) |^p \right)^{1/p} < \infty. $$

(ii) $\sup_{i,n} n^{-s} | g^*(n, j, i) |^{p_k} < \infty, (0 < p_k \leq 1).$

Theorem 5.6.5. $A \in (c_0(p), \mathcal{L}_\infty^G(p, s))$ if and only if there exists an integer $K > 1$ such that

$$ \sup_{i,n} n^{-s} \left( \sum_k | g^*(n, j, i) |^{K^{-1/p_k}} \right)^{1/p_k} < \infty. $$

5.7. Proofs of mapping theorems

Proof of Theorem 5.6.1

Sufficiency: Suppose that (5.6.1) holds and that $z \in \ell_\infty$. Then

$$ \sup_{i,n} n^{-s} | \psi_n(Az) | \leq \sup_{i,n} n^{-s} \sum_j | g^*(n, j, i) z_j | $$

$$ \leq \| z \|_\infty \sup_{i,n} n^{-s} \sum_j | g^*(n, j, i) |. $$

Necessity: Suppose that $A \in (\ell_\infty, \mathcal{L}_\infty^G(s))$. We write

$$ F_i(z) = \sup_n n^{-s} | \psi_n(Az) |. $$

Now, $(F_i)_i$ is a sequence of continuous seminorms on $\ell_\infty$ such that $\sup_i F_i(z) < \infty$. Therefore, by Banach-Steinhaus theorem, there exists a constant $C$ such that

$$ F_i(z) \leq C \| z \|_\infty. \quad \text{for all } i \text{ and } z \in \ell_\infty. $$
Define a new sequence 

\[ z_j^r = \begin{cases} 
\text{sgn } n^{-s}g^*(n, j, i), & 0 \leq j \leq r, \\
0, & j > r.
\end{cases} \]

Now, since \( z = (z^r) \in \ell_\infty \) and by (5.7.1),

\[ \sum_{j=0}^{r} n^{-s} |g^*(n, j, i)| \leq C, \text{ for each } n \text{ and } r.
\]

Therefore, (5.6.1) holds.

**Proof of Theorem 5.6.2**

**Sufficiency:** Suppose that (5.6.2) holds. If \( L > \max(1, \sup |z_j|^p) \), then

\[ n^{-s} |\psi_n(Az)| \leq n^{-s} \sum_{j} |g^*(n, j, i)| L^{1/p}, \text{ for all } i, n.
\]

**Necessity:** Suppose that \( A \in (\ell_\infty(p), \mathcal{L}^B_\infty(s)) \) but there exists an integer \( L > 1 \) such that (5.6.2) does not hold. Then, by Theorem 5.6.1, the matrix

\[ B = (b_{nk}) = (a_{nk}L^{1/p_k}) \notin (\ell_\infty, \mathcal{L}^B_\infty(s)),
\]

that is, there exists \( z \in \ell_\infty \) such that \( Bz \notin \mathcal{L}^B_\infty(s) \).

Now, define a sequence \( y = (y_k) = (z_kL^{1/p_k}) \). Then \( (z_kL^{1/p_k}) \in \ell_\infty(p) \), but \( Ay = Bz \notin \mathcal{L}^B_\infty(s) \) which contradicts that \( A \in (\ell_\infty(p), \mathcal{L}^B_\infty(s)) \).

This completes the proof of Theorem 5.6.2.

**Proof of Theorem 5.6.3**

**Sufficiency:** It follows from Theorem 5.6.2.

**Necessity:** Suppose that \( A \in (\ell_\infty, \mathcal{L}^B_\infty(p, s)) \). Write

\[ F_i(z) = \sup_n n^{-s} |\psi_n(Az)|^p.
\]

Now, \( (F_i)_i \) is a sequence of continuous real functions on \( \ell_\infty \).
and further \(\sup_i F_i(z) < \infty\). Then the result follows by applying uniform boundedness principle.

**Proof of Theorem 5.6.4**

* Sufficiency: We only consider the case \(1 < p_k < \infty\). Suppose that the conditions hold and that \(z \in \ell(p)\). Then, by the inequality 0.14.5 (i), for any \(K > 0\),

\[
|\psi_n(Az)| \leq \sum_{j=0}^{\infty} K(|g^*(n, j, i)|^{p_k} K^{-q_k} + |z_j|^{p_k}).
\]

Therefore, \(A \in (\ell(p), L^B_{\infty}(s))\).

* Necessity: Suppose that \(A \in (\ell(p), L^B_{\infty}(s))\) and that \(z \in \ell(p)\). Put

\[
S_n(z) = n^{-z} |\psi_n(Az)|
\]

and

\[
F_i(z) = \sup_n S_n(z).
\]

For each \(i, (S_n)_n\) being a sequence of continuous real functions on \(\ell(p)\), \((F_i)\) is a sequence of continuous real functions on \(\ell(p)\) and further \(\sup_i F_i(z) < \infty\). Then the result follows by arguing as in uniform boundedness principle. (see Lascarides and Maddox [34], Theorem 1).

This completes the proof of the theorem.

**Proof of Theorem 5.6.5**

The sufficiency can be obtained by an analysis similar to (Lascarides [33], Theorem 10). For the necessity, suppose that \(A \in (c_0(p), L^B_{\infty}(p, s))\) and \(z \in c_0(p)\). Put

\[
F_n(z) = n^{-z} |\psi_n(Az)|^{\ln n},
\]

and

\[
F_i(z) = \sup_n F_n(z).
\]

Since \((F_n)_n\) is a sequence of continuous real functions on \(c_0(p)\), each \(F_i\) is a continuous real function on \(c_0(p)\), and further \(\sup_i F_i(z) < \infty\). Therefore, the result follows by an application of uniform boundedness principle.

This completes the proof.