CHAPTER – III

THE q-HERMITE POLYNOMIALS
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3.1. INTRODUCTION:

Heine [127] give the following representation for the Legendre polynomials \( \{P_n(x)\}_{n=0}^{\infty} \)

\[
P_n(\cos \theta) = \frac{4}{\pi} \frac{2 \cdot 4 \cdots 2n}{3 \cdot 5 \cdots (2n+1)} \sum_{k=0}^{\infty} f_{k,n} \sin (n + 2k + 1) \theta
\]

where \( f_{0,n} = 1 \) and

\[
f_{k,n} = \frac{1.3 \cdots (2k-1)}{2 \cdot 4 \cdots 2k} \frac{(n+1) \cdots (n+k)}{(n+\frac{3}{2}) \cdots (n+\frac{5}{2}) \cdots (n+k+\frac{1}{2})}
\]

Szego [127] generalized this result to the ultraspherical polynomial set \( \{C_n^\lambda(x)\}_{n=0}^{\infty} \) and obtained

\[
\frac{(\sin \theta)^{2\lambda-1}}{2^\lambda} C_n^\lambda(\cos \theta) = \sum_{k=0}^{\infty} f_{k,n}^\lambda \sin (n + 2k + 1) \theta \tag{3.1.1}
\]

where

\[
f_{k,n}^\lambda = \frac{2^{2-2\lambda} \Gamma(n+2\lambda)\Gamma(1-\lambda)\Gamma(n+1)\Gamma(n+\lambda+1)K!}{\Gamma(\lambda)\Gamma(n+\lambda+1)K! (n+\lambda+1)_k} , \quad \lambda > 0
\]

Equation (5.1.1) is the Fourier sine series expansion of \( (\sin \theta)^{2\lambda-1} C_n^\lambda(\cos \theta) \). Because for each non-negative integer \( n \), \( f_{k,n}^\lambda \) is eventually monotonic in \( K \) and \( \lim_{k \to \infty} f_{k,n}^\lambda = 0 \), it follows from classical Fourier
analysis. That (5.1.1) converges pointwise in \((0, \pi)\) and uniformly on \([\varepsilon, \pi - \varepsilon]\) for \(0 < \varepsilon < \frac{\pi}{2}\).

It is well known [113] that \(\left\{C_n^0(\cos \theta)\right\}_{n=0}^{\infty}\) is orthogonal on \([0, \pi]\) with weight function \((\sin \theta)^{2n-1}\). In [6] Allaway identified a large class of orthogonal polynomial sets that satisfy an equation of the form (3.1.1).

One of these polynomial sets turned out to be \(\{R_n(x; q)\}_{n=0}^{\infty}\) designed by three term recursion relation

\[
\begin{align*}
R_0(x; q) &= 1 \\
R_1(x; q) &= 2x \\
R_{n+1}(x; q) &= 2xR_n(x; q) - (1 - q^n) R_{n-1}(x; q), \quad (n \geq 1)
\end{align*}
\]

where \(|q| < 1\). From this three term recursion formula it is easy to show that

\[
\lim_{q \to 1} \frac{R_n\left(\frac{1-q}{2}\right)^{\frac{1}{2}} x; q}{\left(\frac{1-q^n}{2}\right)^{\frac{n}{2}}} = H_n(x),
\]

where \(\{H_n(x)\}_{n=0}^{\infty}\) is the Hermite polynomial set [113]. It is for this reason that \(\{R_n(x; q)\}_{n=0}^{\infty}\) is called the q-Hermite polynomial set. This polynomial set was first introduced by Rogers [109] in 1894.

In paper [10] Allaway studied some of the properties of \(\{R_n(x; q)\}\). He showed that \(\{R_n(x; q)\}_{n=0}^{\infty}\) is characterized by a Fourier sine series similar to (5.1.1) in which the coefficients satisfy a very simple recursion formula. From
this fact has been able to deduce that \( \{R_n(\cos \theta ; q)\}_{n=0}^{\infty} \) is orthogonal on \([0, \pi]\) with respect to the weight function \( \theta_1 \left( z; q^2 \right) \), where \( \theta_1 (z ; q) \) is one of the Theta Functions [113], defined by

\[
\theta_1 (z, q) = 2 \sum_{n=0}^{\infty} (-1)^n q^{\left( \frac{n+1}{2} \right)^2} \sin (2n+1)z
\] (3.1.3)

It is interesting to note that

\[
R_n (x; q) = V^n H_n \left( \frac{u}{v}; q \right) = U^n H_n \left( \frac{v}{u}; q \right)
\] (3.1.4)

where \( u = x - \sqrt{x^2 - 1}, \quad v = x + \sqrt{x^2 - 1} \) and \( H_n (x ; q) \) is the polynomial set first introduced by Szego [128]. \( H_n (x ; q) \) is defined by

\[
H_n (x ; q) = \sum_{k=0}^{n} \binom{n}{k} x^k
\] (3.1.5)

Carlitz [40] has made a detailed study of \( \{H_n (x ; q)\}_{n=0}^{\infty} \). Also, Al-Salam and Chihara [8] have studied generalization of \( \{R_n (x ; q)\}_{n=0}^{\infty} \).

### 3.2. ORTHOGONALITY OF \( \{R_n (x ; q)\}_{n=0}^{\infty} \):

For \( q \), a real number such that \( |q| < 1 \),

\[
\sum_{n=1}^{\infty} \left| q^{n(n+1)/2} \right|^2 < \infty.
\]
Thus by Riesz-Fischer theorem there exists $w(\cos \theta; q) \in L^2[0, \pi]$ such that for all non-negative integer $n$,

$$
\int_0^\pi w(\cos \theta; q) \sin((n+1)\theta) \, d\theta = \begin{cases} (-1)^k q^{\frac{k(k+1)}{2}}, & n = 2k \\ 0, & n = 2k + 1 \end{cases}
$$

(3.2.1)

In (3.2.1), Allaway [10] substituted $x = \cos \theta$ to obtain

$$
\int_{-1}^1 w(x; q) U_n(x) = \begin{cases} (-1)^k q^{\frac{k(k+1)}{2}}, & n = 2k \\ 0, & n = 2k + 1 \end{cases}
$$

(3.2.2)

where $\{U_n(x)\}_{n=0}^\infty$ is the Chebyshev polynomial of second kind (see [113]), defined by

$$
u_n(\cos \theta) = \frac{\sin((n+1)\theta)}{\sin \theta} \quad (n \geq 0).
$$

He extended this definition of $\{U_n(x)\}_{n=0}^\infty$ to all integers $n$ by defining

$$
\begin{align*}
U_{-1}(x) &= 0 \\
U_n(x) &= -U_{n-2}(x)
\end{align*}
$$

(3.2.3)

It is easy to show that these extended Chebyshev polynomials of the second kind satisfy a three term recursion relation of the form

$$
\begin{align*}
U_0(x) &= 1 \\
U_1(x) &= 2x \\
U_n(x) &= 2x U_{n-1}(x) - U_{n-2}(x)
\end{align*}
$$

(3.2.4)

Both $\{R_n(x; q)\}$ and $\{U_n(x; q)\}_{n=-\infty}^\infty$ are examples of symmetric orthogonal polynomial sets and thus for all $n \geq 0$ and $0 \leq n + 2k$, we have
By using this equation and equation (3.2.2) we obtain for \( n \geq 0 \) and \( n + 2k \geq 0 \)

\[
\int_{-1}^{1} w(x;q) R_n(x;q) U_{n+2k+1}(x) \, dx = 0 \quad (3.2.5)
\]

Let us define for \( n \geq -1 \) and all integers \( k \)

\[
f_{k,n} = \begin{cases} 
  \int_{-1}^{1} w(x;q) R_n(x;q) U_{n+2k}(x) \, dx, & \text{if } n + 2k \geq 0 \text{ and } n \neq -1 \\
  0, & \text{if } n + 2k < 0 \text{ or } n = -1
\end{cases}
\]

(3.2.6)

It follows directly from this definition and three term recursion formulas for \( \{R_n(x;q)\}_{n=0}^{\infty} \) and \( \{U_n(x;q)\}_{n=-\infty}^{\infty} \) that, for all integer values \( k \),

\[
f_{k,n+1} = f_{k+1,n} + f_{k,n} - (1 - q^n) f_{k+1,n-1} \quad (3.2.7)
\]

for \( n \geq 0 \).

Allaway proved by mathematical induction on \( n \) that for all non-negative integers \( k \)

\[
f_{k,n} = (-1)^k q^{\frac{k(k+1)}{2}} [q]_{n+k}
\]

(3.2.8)

For \( n = 0 \), he obtained from definition (3.2.6) and equation (5.2.2)

\[
f_{k,0} = \int_{-1}^{1} w(x) U_{2k}(x) \, dx = (-1)^k q^{\frac{k(k+1)}{2}},
\]

and for \( n = 1 \), he obtained in the same manner
\[ f_{k,1} = \int_{-1}^{1} w(x; q) R_1(x; q) U_{1+2k}(x) \, dx. \]

\[ = \int_{-1}^{1} w(x) (U_{2+2k}(x) + U_{2k}(x)) \, dx \]

\[ = \left( -1 \right)^{k+1} q^{\frac{(k+1)(k+2)}{2}} + (-1)^k q^{\frac{k(k+1)}{2}} \]

\[ = (-1)^k q^{\frac{k(k+1)}{2}} \left( 1 - q^{k+1} \right). \]

Thus equation (3.2.8) is true for all non-negative integers \( k \), and \( n = 0 \) or 1. Now let us make the induction hypothesis that equation (3.2.8) is true for all non-negative integers \( k \), and \( n = 0, 1, 2, \ldots, m \). By equation (3.2.7) and the induction hypothesis we obtain for all non-negative integers \( k \).

\[ f_{k,m+1} = \frac{(-1)^{k+1} q^{\frac{(k+1)(k+2)}{2}}}{[q]_{m+1+1}} + (-1)^k q^{\frac{k(k+1)}{2}} \frac{[q]_{m+1+1}}{[q]_k} \]

\[ \quad - \frac{(1 - q^m) (-1)^{k+1} q^{\frac{(k+1)(k+2)}{2}}}{[q]_{m+1}} \frac{[q]_{m+1+1}}{[q]_{m+1+k}} \]

\[ = \frac{(-1)^k q^{\frac{k(k+1)}{2}}}{[q]_k} \frac{[q]_{m+1+k}}{[q]_{m+1+1+k}} \]

It is an easy exercise to show that equation (3.2.8) is equivalent to

\[ \int_{-1}^{1} R_n(x; q) U_m(x) w(x; q) \, dx = \begin{cases} 0, & 0 \leq m < n \\ \frac{[q]_m}{[q]_n}, & m = n \end{cases} \quad (3.2.9) \]

From equation (3.2.5) we know that for \( n \geq 0 \) and \( n + 2k \geq 0 \)
\[ \int_{-1}^{1} R_n(x; q) U_{n+2k+1}(x) w(x; q) dx = 0, \]

and from equations (3.2.8) and (3.2.6) we have that for \( n \geq 0, \)
\[
\int_{-1}^{1} R_n(x; q) U_n(x) w(x; q) dx = [q]^n.
\]

By the definition of \( f_{k,n} \) as given by equation (3.2.6)

\[ f_{k,0} = 0 \] (3.2.10)

for \( k, \) a negative integer. Also for the case \( n = 1, \) we have from the definition of
\( f_{k,n} \) that \( f_{-1,1} = 0, \) and from equations (3.2.7) and (3.2.10) that

\[ f_{k,1} = 0, \]

for \( k, \) a negative integer. Now let us make the induction hypothesis that for all
negative integer \( k, \)

\[ f_{k,n} = 0. \]

for \( n = 0, 1, 2, \ldots, m. \) By equation (3.2.7), we have

\[ f_{k,m+1} = f_{k+1,m} + f_{k,m} - (1-q^m) f_{k+1,m-1} \] (3.2.11)

Thus from the induction hypothesis

\[ f_{k,m+1} = 0 \quad \text{for} \quad k = -2, -3, \ldots. \]

For \( k = -1, \) we obtain from equations (3.2.11) and (3.2.8), and the induction hypothesis

\[ f_{-1,m+1} = f_{0,m} - (1-q^m) f_{0,m-1} \]

\[ = [q]^m - (1-q^m)[q]_{m-1} \]

\[ = 0. \]
Therefore for all negative integers \( k \) and non-negative integers \( n \), \( f_{k,n} = 0 \).

Therefore \( \{R_n(x;q)\}_{n=0}^{\infty} \) is orthogonal on \([-1, 1]\) with respect to the weight function

\[
 w(x;q) = \frac{2}{\pi} \sqrt{1-x^2} \sum_{k=0}^{\infty} (-1)^k q^{k(k+1)/2} U_{2k}(x) \quad (3.2.12)
\]

### 3.3. A CHARACTERIZATION OF \( \{R_n(x;q)\}_{n=0}^{\infty} \):

Let the polynomial set \( \{E_n^\lambda(x)\}_{n=0}^{\infty} \) be defined by

\[
 E_n^\lambda(x) = \frac{n!}{(1+\lambda)_n} C_n^\lambda(x) \quad (n \geq 0),
\]

where \( \{C_n^\lambda(x)\}_{n=0}^{\infty} \) is the ultraspherical polynomial set. It follows directly from equation (3.1.1) that

\[
 \left(1-x^2\right)^{-\frac{1}{2}} E_n^\lambda(x) = \frac{2^{2\lambda}(n+2\lambda)}{\Gamma(\lambda) \Gamma(n+\lambda+1)} \sqrt{1-x^2} \sum_{k=0}^{\infty} g_{k,n} U_{n+2k}(x)
\]

(3.3.1)

where

\[
 g_{k,n}^\lambda = \frac{k-\lambda}{k} g_{k-1,n+1}^\lambda \quad (3.3.2)
\]

Equations (3.3.1) and (3.3.2) suggest studying the polynomial set \( \{A_n(x)\}_{n=0}^{\infty} \) such that there exists a function \( w(x) \) and a sequence of real numbers \( \{\alpha_n\}_{k=0}^{\infty} \) having the property that the Fourier Chebychev expansion of \( w(x) A_n(x) \) is
\[ w(x)A_n(x) = \frac{2}{\pi} \sum_{k=0}^{\infty} h_{k,n} U_{n+2k}(x) \]  

(3.3.3)

where

\[ h_{0,n} \neq 0 \]

and

\[ h_{k,n} = \alpha_k h_{k-1,n+1} \quad (k \geq 1, \ n \geq 0) \]  

(3.3.4)

In [6] Allaway found the three recursion relation for all these polynomial sets and studied some of their properties.

It is easy to show (see [6]) that all polynomials sets \( \{A_n(x)\}_{n=0}^{\infty} \) that satisfy equation (5.3.3) are symmetric and orthogonal on \([-1, 1]\) with respect to the weight functions \( w(x) \).

It is well known (see [6]) that such symmetric orthogonal polynomial sets satisfy a three term recursion formula of the form

\[
\begin{align*}
A_0(x) &= 1, \\
A_1(x) &= 2b_1(x), \\
A_n(x) &= 2b_n x \ A_{n-1}(x) - \lambda_n A_{n-2}(x), \quad (n \geq 2)
\end{align*}
\]

(3.3.5)

where \( \{b_n\}_{n=0}^{\infty} \) and \( \{\lambda_n\}_{n=0}^{\infty} \) are real non-zero sequences.

We note from equation (3.1.2) that in order for equation (3.3.5) to the three term recursion relation for \( \{R_n(x; q)\}_{n=0}^{\infty} \), we require \( b_1 = b_2 \) and \( 2b_1 b_2 > \lambda_2 \).

Allaway [10] proved the following theorem:
Theorem (3.3.1): Let \( \{A_n(x)\}_{n=0}^\infty \) be any polynomial set satisfying equations (3.3.3), (3.3.4) and (3.3.5). Also let \( \{R_n(x; q)\}_{n=0}^\infty \) be defined by (3.1.2).

\[
R_n(x; q) = \frac{A_n(x)}{b_1^n}
\]

if and only if \( b_1 = b_2 \) and \( 2b_1b_2 > \lambda_2 > 0 \).

3.4. THE WEIGHT FUNCTION \( w(x; q) \):

From equation (3.2.11) we see that the Fourier sine series expansion of \( w(\cos \theta; q) \) is given by

\[
w(\cos \theta; q) = \frac{2}{\pi} \sum_{k=0}^{\infty} (-1)^k q^{\frac{k(k+1)}{2}} \sin(2k+1)\theta
\]

\[
= \frac{2}{\pi} q^\frac{1}{8} \sum_{k=0}^{\infty} (-1)^k \left( q^\frac{1}{2} \right)^{k^2+k+\frac{1}{4}} \sin(2k+1)\theta.
\]

By comparing this with the theta function \( \theta_1(z, q) \) as described in [113] by

\[
\theta_1(z, q) = 2 \sum_{n=0}^{\infty} (-1)^n q^{\frac{n+\frac{1}{2}}{2}} \sin(2n+1)z. \tag{3.4.1}
\]

Allaway [10] obtained

\[
w(\cos z; q) = \frac{-1}{\pi} q^\frac{1}{8} \theta_1 \left( z, q^{\frac{1}{2}} \right) \tag{3.4.2}
\]

\( \theta_1(z, q) \) has an infinite product representation (see [113]).
\[ \theta_i(z, q) = 2q^i \sin z \prod_{n=1}^{\infty} \left( 1 - q^{2n} \right) \left( 1 - 2q^{2n} \cos 2z + q^{4n} \right). \]

Therefore

\[ w(\cos z; q) = \frac{2}{\pi} \sin z \prod_{n=1}^{\infty} \left( 1 - q^n \right) \left( 1 - 2q^n \cos 2z + q^{2n} \right) \quad (3.4.3) \]

Equation (3.4.3) agrees with results obtained by Al-Salam and Chihara [8, P. 28].

### 3.5. MEHLER’S FORMULA FOR q-HERMITE POLYNOMIALS:

Bressoud [33] denoted the q-Hermite polynomials by the symbol \( H_n(x \mid q), |q| < 1. \) He pointed out that \( H_n(x \mid q) \) can be defined by their generating function:

\[ \prod_{n=0}^{\infty} \frac{1}{\left( 1 - 2xr q^n + r^2 q^{2n} \right)} = \sum_{n=0}^{\infty} \frac{H_n(x \mid q)r^n}{(q)_n} \quad (3.5.1) \]

Or by their recurrence relation:

\[
\begin{align*}
2x H_n(x \mid q) &= H_{n+1}(x \mid q) + \left( 1 - q^n \right) H_{n-1}(x \mid q) \\
H_1(x \mid q) &= 0 \\
H_0(x \mid q) &= 1
\end{align*}
\]

Or by their Fourier expansion:

\[ H_n(\cos \theta \mid q) = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right]_q \cos (n - 2k)\theta \\
= \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right]_q e^{i(n-2k)\theta} \quad (3.5.3) \]
These polynomials have been studied by Rogers [108], [109], [110], [111], Sezgo [128], Carlitz [40], [41] and Askey and Ismail [9]. The Rogers-Ramanujan identities first arose in the study of these polynomials (see Rogers [109], Bressoud [35]).

The q-Hermite polynomials are also of formal interest, for they have been found to satisfy analogs of many of the formulae known for the Hermite polynomials. Among these is a q-analogue of Mehler's formula:

\[
\left( \frac{\left( r e^{i0} \right)_n}{\left( r e^{i0-\phi} \right)_n} \right)^2 = \sum_{n=0}^{\infty} \frac{H_n(\cos \theta | q) H_n(\cos \phi | q) r^n}{(q)_n} \tag{3.5.4}
\]

For simplicity of notation, \( \left( r e^{i0} \right)_n \) will be used to denote \( (r e^{i0}) \infty \), whether or not \( r \) and \( \theta \) are both real.

This formula was known to Rogers, and is a corollary of a more general result, first proved in [110] and then again in [108]:

\[
\left( \frac{\left( \lambda e^{i\phi} \right)_n}{\left( \mu e^{i\phi} \right)_n} \right)^2 = \sum_{n=0}^{\infty} \frac{H_n(\cos \theta | q) \sum_{r=0}^{\infty} \frac{\lambda^r \mu^{n-r}}{(q)_r (q)_{n-r}}}{\sum_{n=0}^{\infty} \lambda^{n-r}} \tag{3.5.5}
\]

Equation (3.5.4) follows if \( \lambda = re^{i\phi} \), \( \mu = re^{i\theta} \), and equation (3.5.3) is used to sum the inner sum of the right hand side.

Neither of Roger's proofs of (3.5.5) is particularly simple. Bressoud [33] gave a proof of (3.5.5) which relies only on the recurrence relation (3.5.2). His proof is given below:
Proof of (3.5.5): Let

\[ f(\lambda, \mu; x) = \sum_{n=0}^{\infty} H_n(x|q) \sum_{r=0}^{n} \frac{\lambda^r \mu^{n-r}}{(q)_r (q)_{n-r}} \]

\[ = \sum_{r,s \geq 0} \frac{\lambda^r \mu^s}{(q)_r (q)_s} H_{rs}(x|q). \]

We use (3.5.2) to obtain the following;

\[ 2x f(\lambda, \mu; x) = \sum_{r,s \geq 0} \frac{\lambda^r \mu^s}{(q)_r (q)_s} 2x H_{rs}(x|q). \]

\[ = \sum_{r,s \geq 0} \frac{\lambda^r \mu^s}{(q)_r (q)_s} H_{rs+1}(x|q) + \sum_{r,s \geq 0} \frac{\lambda^r \mu^s}{(q)_r (q)_s} (1-q^r s) H_{rs+1}(x|q) \]

\[ = \frac{1}{\lambda} \sum_{r,s \geq 0} \frac{\lambda^r \mu^s}{(q)_r (q)_s} (1-q^r s) H_{rs}(x|q) + \sum_{r,s \geq 0} \frac{\lambda^r \mu^s}{(q)_r (q)_s} (1-q^r s) H_{rs+1}(x|q) \]

\[ + \sum_{r,s \geq 0} \frac{(\lambda q)^r \mu^s}{(q)_r (q)_s} (1-q^r s) H_{rs+1}(x|q). \]

\[ = \frac{1}{\lambda} \sum_{r,s \geq 0} \frac{\lambda^r \mu^s}{(q)_r (q)_s} H_{rs}(x|q) - \frac{1}{\lambda} \sum_{r,s \geq 0} \frac{(\lambda q)^r \mu^s}{(q)_r (q)_s} H_{rs}(x|q) \]

\[ + \lambda \sum_{r,s \geq 0} \frac{\lambda^r \mu^s}{(q)_r (q)_s} H_{rs}(x|q) + \mu \sum_{r,s \geq 0} \frac{(\lambda q)^r \mu^s}{(q)_r (q)_s} H_{rs}(x|q) \]

\[ = \frac{1}{\lambda} f(\lambda, \mu; x) - \frac{1}{\lambda} f(\lambda q, \mu; x) + \lambda f(\lambda, \mu; x) + \mu f(\lambda q, \mu; x) \quad (3.5.6) \]

If (3.5.6) is solved for \( f(\lambda, \mu; x) \), we get

\[ f(\lambda, \mu; x) = \left( \frac{1 - \frac{\lambda \mu}{2 \lambda \mu + \lambda^2}}{1 - 2x \frac{\lambda}{\lambda + \lambda^2}} \right) f(\lambda q, \mu; x) \quad (3.5.7) \]

or equivalently,
\[ f(\lambda, \mu; \cos \theta) = (1 - \lambda \mu) f(\lambda q, \mu; \cos \theta) \] (3.5.8)

Since \( f \) is symmetric in \( \lambda \) and \( \mu \), we also have;

\[ f(\lambda, \mu; \cos \theta) = \frac{(1 - \lambda \mu)}{|1 - \mu e^{i\theta}|^2} f(\lambda, \mu q; \cos \theta) \] (3.5.9)

If equations (3.5.8) and (3.5.9) are now combined, we get

\[
\begin{align*}
\frac{\lambda \mu}{(1 - \lambda e^{i\theta})(1 - \mu e^{i\theta})} & f(\lambda q^2, \mu q^2; \cos \theta) \\
\vdots & \vdots \\
\frac{\lambda \mu}{(1 - \lambda e^{i\theta})^k (1 - \mu e^{i\theta})^k} & f(\lambda q^k, \mu q^k; \cos \theta) \\
\vdots & \vdots \\
\frac{\lambda \mu}{(1 - \lambda e^{i\theta})^\infty (1 - \mu e^{i\theta})^\infty} & f(0, 0; \cos \theta) \quad (3.5.10)
\end{align*}
\]

But \( f(0, 0; \cos \theta) = 1 \), and so equation (3.5.5) is proved.

As Rogers observed, equation (3.5.5) has many interesting corollaries, one of which is a formula for the product of two q-Hermite polynomials. In the following equation, we use the generating function for the q-Hermite polynomials equation (3.5.5), and an elementary result of Euler.
When the coefficients of $\lambda^n$, $\mu^m$ are compared, we see that

$$H_n(\cos \theta | q) H_m(\cos \theta | q) = \sum_{t=0}^{\min(m, n)} \frac{(q)_n (q)_m}{(q)_t (q)_{m-t} (q)_{n-t}} H_{m+n-2t}(\cos \theta | q)$$

(3.5.12)

### 3.6 FURTHER PROPERTIES OF q-HERMITE POLYNOMIALS:

Al-Salam and Ismail [15], in 1988, used continuous q-Hermite polynomials to give a new proof of a q-beta integral which is an extension of the Askey-Wilson integral. Multilinear generating functions, some due to Carlitz were also established by them.

The continuous q-Hermite polynomials $\{H_n(x | q)\}$ are given by

$$H_n(\cos \theta | q) = \sum_{k=0}^{n} \frac{(q)_k}{(q)_n (q)_{n-k}} e^{i(n-2k)\theta}$$

(3.6.1)

(see [8]). Their orthogonality [8, 6] is

$$\int_0^\pi w(\theta) H_m(\cos \theta | q) H_n(\cos \theta | q) = (q;q)_n \delta_{mn}$$

(3.6.2)

where
\[ w(\theta) = \frac{(q)_\infty}{2\pi} \left( e^{2i\theta} \right)_\infty \left( e^{-2i\theta} \right)_\infty \quad (3.6.3) \]

Rogers also introduced the continuous q-ultraspherical polynomials \( \{ C_n(x; \beta | q) \} \) generated by
\[
\sum_{n=0}^{\infty} C_n(x; \beta | q) t^n = \frac{(\beta t e^{i\theta})_\infty}{(t e^{i\theta})_\infty} \left( \beta t e^{-i\theta} \right)_\infty \quad (3.6.4)
\]
whose weight function was found recently [12, 14]. It is easy to see that
\[
C_n(x; 0 | q) = H_n(x | q)/(q)_n \quad (3.6.5)
\]

Rogers solved the connection coefficient problem of expressing \( C_n(x; \beta | q) \) in terms of \( C_n(x; \gamma | q) \) as a consequence of which we get
\[
C_n(x; \beta | q) = \sum_{k=0}^{\binom{n}{2}} (-\beta)^k q^{k(k-1)/2} \frac{(\beta)_{n-k}}{(q)_k (q)_{n-2k}} H_{n-2k}(x | q) \quad (3.6.6)
\]

Rogers evaluated explicitly the coefficients in the linearization of products of two q-Hermite polynomials. He proved
\[
H_m(x | q) H_n(x | q) = \sum_{k=0}^{\min(m,n)} \frac{(q)_m (q)_n}{(q)_k (q)_{n-k} (q)_{m-k}} H_{m+n-2k}(x | q) \quad (3.6.7)
\]
Which can be iterated to obtain the sum
\[
H_k(x | q) H_m(x | q) H_n(x | q) = \sum_{r,s} \frac{(q)_k (q)_m (q)_n}{(q)_{m-r} (q)_{n-r} (q)_{r} (q)_{k-s} (q)_{m+n-2r} (q)_{s}} H_{k+m+n-2r-2s}(x | q) \quad (3.6.8)
\]
We shall also need the formula
\[
\frac{H_n (x \mid q)}{(q)_m} C_n (x; \beta \mid q) = \sum_{k,j} \left( \begin{array}{c}
-\beta \\
q
\end{array} \right)^k q^{(k-1)/2} (\beta)_{n-k} (q)_k (q)_{m-j} (q)_{n-k-j} H_{m+n-2k-2j} (x \mid q)
\]
(3.6.9)

which follows from (5.6.6) and (5.6.7).

We shall also use the polynomials
\[
h_n (x \mid q) = \sum_{k=0}^{n} \frac{(q)_n}{(q)_k (q)_{n-k}} x^k,
\]
so that
\[
H_n (\cos \theta x \mid q) = e^{i n \theta} h_n (e^{-2i \theta} \mid q)
\]
(3.6.10)

It was shown in [1], [83] that \( \{ h_n (a \mid q) \} \) are moments of a discrete distribution \( d \psi_a (x) \), viz.,
\[
h_n (a \mid q) = \int_{-\infty}^{\infty} x^n d \psi_a (x), n = 0, 1, 2, \ldots
\]
(3.6.11)

where \( d \psi_a (x) \) is a step function with jumps at the points \( x = q^k \) and \( x = a q^k \)
for \( k = 0, 1, 2, \ldots \) given by
\[
d \psi_a (q^k) = \frac{q^k}{(a)_\infty (q)_k (q \mid a)_k},
\]
d \psi_a (a q^k) = \frac{q^k}{(1 / a)_\infty (q)_k (a q)_k}
(3.6.12)

where \( a < 0, 0 < q < 1 \).

Askey and Wilson [14] proved
\[
\frac{(q)_\infty}{2 \pi} \int_{0}^{\pi} \frac{(e^{2i \theta})_\infty (e^{-2i \theta})_\infty d \theta}{\prod_{1 \leq i < j \leq 4} (a_j e^{i \theta})_\infty (a_j e^{-i \theta})_\infty} = \frac{(a_1 a_2 a_3 a_4)_\infty}{\prod_{1 \leq i < j \leq 4} (a_i a_j a_s)_\infty}
\]
(3.6.13)
where $| a_r | < 1$, for $r = 1, 2, 3, 4$. They used this integral to prove the orthogonality of what is now known as the Askey–Wilson polynomials.

Ismail and Stanton [85] observed that the left hand side of (3.6.13) is a generating function of the integral of the product of four q-Hermite polynomials times the weight function $w(\theta)$. They used this observation, combined with (3.6.8) and (3.6.3), to give a new proof of (3.6.13). Other analytic proofs of (3.6.13) can be found in [13] and [115].

Furthermore a combinational derivation of (3.6.13) is given in [84].

Nasrallah and Rahman [107], proved the following generalization of (3.6.13).

**Theorem (3.6.1): (Nasrallah and Rahman).** If $| a_j | < 1$, $j = 1, 2, 3, 4$ and $| q | < 1$ then

$$
\int_0^\pi w(\theta) \left( \frac{(A a_1 e^{i\theta})_\infty}{(a_k e^{i\theta})_\infty} \right)^{n} \prod_{1 \leq i < j \leq 5} (A a_i a_j)_\infty d\theta
$$

$$
= \frac{(a_1 a_2 a_3 a_4 a_5)_\infty (A a_1 a_2 a_3 a_4 a_5)_\infty (A a_3 a_4 a_5)_\infty}{(A a_1 a_3 a_4 a_5)_\infty \prod_{1 \leq i < j \leq 5} (a_i a_j)_\infty} g w_7 \left( A a_1 a_3 a_4 a_5 q^{-1}; A a_5 / a_2, A, a_1 a_3, a_1 a_4, a_3 a_4 / a_2 a_5 \right) \tag{3.6.14}
$$

where

$$
g w_7 (a; b, c, d, e, f | z) = g \Phi_7 \left[ \frac{a, q \sqrt{a}, -q \sqrt{a}, b, c, d, e, f}{\sqrt{a}, -\sqrt{a}, qa / b, qa / c, qa / d, qa / e, qa / f} | z \right]
$$
Rahman [116] observed that the $\phi_7$ in (3.6.14) can be summed when $A = a_1, a_2, a_3, a_4$. In this case, we have

$$
\int_0^\pi w(\theta) \frac{a_1 a_2 a_3 a_4 a_5 e^{i\theta}}{\prod_{1 \leq j < 5} (a_k e^{i\theta})_{\infty}} \frac{a_1 a_2 a_3 a_4 a_5 e^{-i\theta}}{\prod_{1 \leq k < 5} (a_k e^{-i\theta})_{\infty}} \mathrm{d}\theta = \prod_{k=1}^5 \frac{a_1 a_2 a_3 a_4 a_5}{a_k}
$$

(3.6.15)

Askey [31] gave an elementary proof of (3.6.15) by showing the two sides of (3.6.15) satisfy the same functional equation.

### 3.7. GENERATING FUNCTIONS:

To illustrate their technique Al-Salam and Ismail [15] derived Carlitz [42] extension of Mehler formula:

$$
S = \sum_{n=0}^{\infty} h_n(a \mid q) h_{n+k}(b \mid q) \frac{z^n}{(q)_n} = \frac{(abz^2)_{\infty}}{(z)_{\infty} (bz)_{\infty} (az)_{\infty} (abz)_{\infty}} \sum_{r=0}^{k} \frac{(q)_r (bz)_r (abz)_r}{(q)_{k-r} (abz^2)_{k-r}} b^{k-r}
$$

(3.7.1)

They began by the generating function

$$
\sum_{n=0}^{\infty} h_n(a \mid q) \frac{z^n}{(q)_n} = \frac{1}{(z)_{\infty} (az)_{\infty}}
$$

(3.7.2)

Multiplying by $z^k$, then replacing $z$ by $xz$ and using (3.6.11), we get

$$
\sum_{n=0}^{\infty} h_n(a \mid q) h_{n+k}(b \mid q) \frac{z^n}{(q)_n} = \frac{1}{(b)_{\infty} (z)_{\infty} (az)_{\infty}} z^{\phi_1} \left[ \frac{z, az}{q | b | q^{-k+1}} \right]
$$
Now using a transformation formula of Sears [126] (see also [72]):

\[
\phi_{1}[a, b, c | z] = \frac{(b)_\infty}{(c)_\infty} \frac{(q/c)_\infty (c/a)_\infty (az/q)_\infty (q^2/az)_\infty}{(bq/c)_\infty (q/a)_\infty (az/c)_\infty (qc/az)_\infty} \phi_{1}[q^2 / c | z]
\]

we get that the left hand side of (3.7.1) is

\[
S = \frac{(az^2 q^k)}{(z)_\infty (az)_\infty (bq^k)_\infty (q/z)_\infty} \phi_{1}[q^2 / bz, q^{l-k} / abz^2 | abz]
\]

By Heine's transformation formula

\[
\phi_{1}[\alpha, \beta, \gamma | z] = \frac{(\beta)_\infty (\alpha z)_\infty}{(\gamma)_\infty (z)_\infty} \phi_{1}[\gamma / \beta, z | \alpha, \beta]
\]

we get that

\[
S = \frac{(ab^2 q^k)}{(z)_\infty (az)_\infty (bq^k)_\infty (abz)_\infty} \phi_{1}[q^{-k}, abz, \frac{q}{bz} | abz]
\]

Now by a transformation formula [72 ex. 1.14 (ii)]

\[
\phi_{1}[q^n, b, c | z] = b^n \frac{(c/b)_n}{(c)_n} \sum_{j=0}^{n} \frac{(q^n)_j (b)_j (q/z)_j (-1)^j q^{-j(j-1)/2}}{(q)_j (bq^{l-n/c})_j} \frac{(z/c)}{(q)_m (q)_n (q)_k}
\]

we get the right hand side of (3.7.1).

We next consider the sum

\[
G = G(a, b, x, y, z) = \sum_{m, n, k} x^m y^n z^k \frac{h_{m+k} (a \mid q) h_{n+k} (b \mid q)}{} \]

75
\begin{align*}
&= \sum_{m,n,k} \frac{x^m y^n z^k}{(q)_m (q)_n (q)_k} \, h_{n+k}(b \mid q) \int_{-\infty}^{\infty} u^{m+k} \, d\psi_a(u) \\
&= \int_{-\infty}^{\infty} \frac{1}{(xu)_\infty} \sum_{k,n} \frac{y^n(zu)^k}{(q)_n (q)_k} \, d\psi_a(u), \\
&= \int_{-\infty}^{\infty} \frac{1}{(xu)_\infty} \sum_{r=0}^{\infty} \frac{y^r}{(q)_r} h_r(b \mid q) \, h_r\left(\frac{zu}{y} \mid q\right) \, d\psi_a(u) \tag{3.7.5}
\end{align*}

using the q-Mehler's formula (formula (3.7.1) with \( k = 0 \))

\begin{align*}
G(a,b,x,y,z) &= \frac{1}{(y)_\infty (by)_\infty} \int_{-\infty}^{\infty} \frac{(byzu)_\infty}{(xu)_\infty (zu)_\infty (bzu)_\infty} \, d\psi_a(u) \\
&= \frac{(byz)_\infty}{(y)_\infty (by)_\infty (a)_\infty (x)_\infty (z)_\infty (bz)_\infty} \, _3\phi_2\left[ \begin{array}{c} x, z, bz \\ q/a \ bzy \end{array} | q \right] \\
&\quad + \frac{(abzy)_\infty}{(y)_\infty (by)_\infty (1/a)_\infty (ax)_\infty (az)_\infty (abz)_\infty} \, _3\phi_2\left[ \begin{array}{c} ax, az, abz \\ a q, abzy \end{array} | q \right] \tag{3.7.6}
\end{align*}

But Carlitz [42] showed that

\begin{align*}
G(a,b,x,y,z) &= \frac{(axz)_\infty (byz)_\infty}{(x)_\infty (ax)_\infty (y)_\infty (by)_\infty (z)_\infty (az)_\infty (bz)_\infty} \, _3\phi_2\left[ \begin{array}{c} x, y, z \\ axz, byz \end{array} | abz \right] \tag{3.7.7}
\end{align*}

Although (3.7.6) and (3.7.7) are the same, we shall nevertheless need to use (3.7.6) for the representation of \( G(a,b,x,y,z) \). Equating \( G \) in (3.7.6) and (3.7.7), we get the transformation formula
\[ 3 \phi_2 \left[ \frac{x, y, z}{ax, byz} \right] = \frac{(axz)_{\infty}}{(a)_{\infty}} \left[ \frac{3 \Phi_2}{x, y, z} \right] \]

\[ + \frac{(abz)_{\infty}}{(az)_{\infty}} \left[ \frac{3 \Phi_2}{x, az, abz} \right] \]

\[ + \frac{(abz)_{\infty}}{(az)_{\infty}} \left[ \frac{3 \Phi_2}{x, az, abz} \right] \] (3.7.8)

An interesting special case of (3.7.8) is

\[ x = q^{-n} \text{ for } n = 0, 1, 2, \ldots, \text{ we get} \]

\[ 3 \phi_2 \left[ \frac{q^{-n}, y, z}{azq^{-n}, byz} \right] = \frac{(q/a)_{n}}{(q/az)_{n}} \left[ \frac{q^{-n}, b, bz}{q/a, bzy} \right] \] (3.7.9)

which is due to Sears [125]. Formula (3.7.9) in turn implies Jackson's theorem for the summation of a terminating balanced (Saalschützian) \( 3 \Phi_2 \) with argument \( q \), viz.,

\[ 3 \phi_2 \left[ \frac{q^{-n}, a, b}{c, abq^{-n/c}} \right] = \frac{(c/a)_{n} (c/b)_{n}}{(c)_{n} (c/ab)_{n}} \] (3.7.10)