CHAPTER – V

A Q-EXTENSION OF THE GENERALIZED HERMITE POLYNOMIALS WITH THE CONTINUOUS ORTHOGONALITY PROPERTY ON ₹
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A q-EXTENSION OF THE GENERALIZED HERMITE POLYNOMIALS WITH THE CONTINUOUS ORTHOGONALITY PROPERTY ON \( \mathcal{R} \)

In this chapter we study in detail a q-extension of the generalized Hermite Polynomials of Szego. A continuous orthogonal property on \( \mathcal{R} \) with respect to the positive weight function is established, a q-difference equation and a three term recurrence relation are derived for this family of q-polynomials.

5.1 INTRODUCTION:

The generalized Hermite polynomials were introduced by Szegö [127] as

\[
H_{2n}^{(a)}(x) = (-1)^n \ 2^{2n} \ n! \ L_n^{\left(\frac{\mu-1}{2}\right)}(x^2),
\]

\[
H_{2n+1}^{(a)}(x) = (-1)^n \ 2^{2n+1} \ n! \ x L_n^{\left(\frac{\mu+1}{2}\right)}(x^2),
\]  

(5.1.1)

Where \( \mu > -\frac{1}{2} \), \( L_n^{(a)}(x) \) are the laguerre polynomials,

\[
L_n^{(a)}(z) = \frac{(\alpha+1)_n}{n!} {}_1F_1\left( \begin{array}{c} -n \\ \alpha+1 \end{array} \right) z
\]
\[
(a + l)^n = \frac{(\alpha + 1)_n}{n!} \sum_{k=0}^{n} \left(\frac{-n}{k}\right) z^k (\alpha + 1)_k k!
\]  
(5.1.2)

and \((a)_n = \frac{\Gamma(1+n)}{\Gamma(a)}\), \(n = 0, 1, 2, \ldots\), is the shifted factorial.

The zero value of the parameter \(\mu\) in (5.1.1) corresponds to the ordinary Hermite polynomials \(H_n(x)\), i.e. \(H_n^{(0)}(x) = H_n(x)\).

The generalized Hermite polynomials (5.1.1) are orthogonal with respect to the weight function

\[
|x|^{2\mu} e^{-x^2}, \quad x \in \mathbb{R}, \quad \text{i.e.}
\]

\[
\int_{-\infty}^{\infty} H_n^{(\mu)}(x) H_m^{(\mu)}(x) x^{2\mu} e^{-x^2} dx = 2^{2n} \left[\frac{n}{2}\right]! \Gamma\left[\left\lfloor\frac{n+1}{2}\right\rfloor + \mu + \frac{1}{2}\right] \delta_{nm}, \quad (5.1.3)
\]

Where \([x]\) denotes the greatest integer not exceeding \(x\). They satisfy a three term recurrence relation

\[
2xH_n^{(\mu)}(x) = H_n^{(\mu)}(x) + 2(n + 2\mu)H_{n-1}^{(\mu)}(x), \quad n \geq 0
\]  
(5.1.4)

and a second order differential equation

\[
\left[ x \frac{d^2}{dx^2} + 2(\mu - x^2) \frac{d}{dx} + 2n - 2\mu x^{-1}\right] H_n^{(\mu)}(x) = 0, \quad n \geq 0
\]

with \(\theta_n = n - 2\left[\frac{n}{2}\right]\) see [127, 43]

A detailed discussion of other properties of \(H_n^{(\mu)}(x)\) can be found in [106, 118].

The reason for interest in studying the generalized Hermite polynomials (5.1.1) is two fold. Pure Mathematically they are of interest as an explicit
example of the complete orthonormal set in $L^2_\mu (\mathbb{R})$, the Hilbert space of Lebesgue measurable function $f(x), x \in \mathbb{R}$ with

$$
\|f\|_\mu = \left( \int_{-\infty}^{\infty} |f(x)|^2 |x|^{\mu} \, dx \right)^{\frac{1}{2}} < \infty.
$$

Hence one can build the Bose-like oscillator calculus in terms of these polynomials, which generalized the well known calculus, based on the quantum-mechanical harmonic oscillator in Physics [34]. So we try to make one step further by considering a generalization of the classical Hermite polynomials $H_n(x)$ with two additional parameters, $\mu$ and $q$.

The aim of this chapter is to investigate in detail a q-extension of the generalized Hermite polynomials (5.1.1) with the continuous orthogonality property on $\mathbb{R}$ (the case of discrete orthogonality requires a different technique, see, for example [34]).

In section 5.2 we introduce this family $\{H_n^{(\mu)}(x;q)\}$ in terms of the q-Laguerre polynomials and find a relevant q-difference equation for it. In section 5.3 the continuous orthogonality property for $\{H_n^{(\mu)}(x;q)\}$ with respect to the positive weight function on $\mathbb{R}$ is explicitly formulated. Section 5.4 is devoted to the derivation of a three term recurrence relation for this family of q-polynomials.
5.2 DEFINITION AND Q-DIFFERENCE EQUATION FOR

THE SEQUENCE \{H_n^{(\nu)}(x;q)\}:

It is known \([76, 62, 105]\) that the q-Laguerre polynomials \(L_n^{(\alpha)}(x;q)\) are explicitly given as

\[
L_n^{(\alpha)}(x;q) = \left(\frac{q^{\alpha+1};q}{(q;q)_n}\right)\phi_1\left(\begin{array}{c} q^{-n} \\ q^{\alpha+1} \end{array} \mid q, -q^{n+\alpha+1}x \right)
\]

\[
= \frac{1}{(q;q)_n} \phi_1\left(\begin{array}{c} q^{-n} \end{array} \mid q, q^{n+\alpha+1} \right)
\]  

(5.2.1)

where \((a;q)_0 = 1\) and \((a;q)_n = \prod_{j=0}^{n-1} (1 - aq^j)\), \(n = 1, 2, \ldots\) is the q-shifted factorial, and

\[
\phi_p\left(\begin{array}{c} q, a_1, \ldots, a_r \\ b_1, b_2, \ldots, b_p \end{array} \mid q, z \right)
\]

\[
= \sum_{k=0}^{n} \frac{(q^{-n};q)_k (a_1;q)_k \ldots (a_r;q)_k}{(b_1;q)_k (b_2;q)_k \ldots (b_p;q)_k} \frac{z^k}{(q;q)_k} \left(\frac{(-1)^k}{q^{k(k-1)/2}}\right)^{p-r+1}
\]

(5.2.2)

is the basic hypergeometric polynomial of degree \(n\) in the variable \(z\) (throughout this chapter, we will employ the standard notations of the q-special functions theory, See \([70]\) or \([29]\)). The q-Laguerre polynomials (5.2.1) satisfy two kind of orthogonality relations, an absolutely continuous one and a discrete one. The former orthogonality relation, in which we are interested in the present chapter is given by
where \( E_q(x) \) is the Jackson q-exponential function,

\[
E_q(z) = \sum_{n=0}^{\infty} \frac{q^n}{(q;q)_n} z^n = (-z; q)_\infty
\]  

and the normalization constant \( d_n(\alpha) \) is equal to

\[
d_n(\alpha) = \frac{1}{\pi} \sin \pi (\alpha + 1) \frac{q^n (q; q)_n (q; q)_\infty}{(q^{\alpha+1}; q)_n (q^{-\alpha}; q)_\infty}
\]  

The q-Laguerre polynomials (5.2.1) are defined in such a way that in the limit as \( q \to 1 \) they reduce to the ordinary Laguerre polynomials \( L^{(a)}_n(x) \), i.e.,

\[
\lim_{q \to 1} L^{(a)}_n((1-q)x; q) = L^{(a)}_n(x)
\]  

We can now define in complete analogy with the relationship (5.1.1), a q-extension of the generalized Hermite polynomials \( H^{(\mu)}_n(x) \) of the form

\[
H^{(\mu)}_{2n}(x; q) = (-1)^n (q; q)_n \times L^{(\mu+1)/2}_n (x^2; q)
\]

\[
H^{(\mu)}_{2n+1}(x; q) = (-1)^n (q; q)_n \times L^{(\mu+1)/2}_n (x^2; q)
\]  

which are orthogonal on the real line \( \mathbb{R} \).

Indeed, since

\[
\lim_{q \to 1} \frac{(q^n - q^n)/(1-q^n)}{(a)_n} = (a)_n
\]  

with the aid of (5.2.6) one readily verifies that
\[
\lim_{q \to 1} (1 - q)^{-\frac{n}{2}} H_n^{(\mu)}(\sqrt{1 - qx}; q) = 2^{-n} H_n^{(\mu)}(x) \quad (5.2.9)
\]

Also the zero value of the parameter \(\mu\) in (5.2.7.) corresponds to the polynomials

\[
H_n(x; q) = H_n^{(0)}(x, q).
\]

The sequence \(\{H_n(x; q)\}\) can be expressed either in terms of the q-Laguerre polynomials

\[
L_n^{(\alpha)}(x; q), \quad \alpha = \pm \frac{1}{2} \quad (as it obvious from definition (5.2.7) itself), or through the discrete q-Hermite polynomials \(\tilde{h}_n(x; q)\) of type II:
\[
H_n(x; q^2) = q^{\frac{n(n-1)}{2}} \tilde{h}_n(x; q) \quad (5.2.10)
\]

A detailed discussion of the properties of the polynomials \(H_n(x; q)\) can be found in paper [30] on this subject.

A q-difference equation for the introduced polynomials \(H_n^{(\mu)}(x; q)\) is, in fact, an easy consequence of the known q-difference equation

\[
q^a (1 + x) L_n^{(\alpha)}(qx; q) + L_n^{(\alpha)}(q^{-1}x; q) = \left[1 + q^a(1 + q^n x)\right] L_n^{(\alpha)}(x; q) \quad (5.2.11)
\]

for the q-Laguerre polynomials (see, for example formula (3.21.6) in [98]). Indeed, from this q-difference equation and definition (5.2.7) is follows immediately that

\[
q^{-\frac{1}{2}} (1 + x^2) H_n^{(\mu)}(\frac{1}{q^2} x; q) + H_n^{(\mu)}(\frac{1}{q^2} x; q)
\]
\[
q^{n+1} + (e_i - e_j) + q^{n+1} \quad H^{(\mu)}(x; q) \quad (5.2.12)
\]

where, as before, \( \theta_n = n - \left[ \frac{n}{2} \right] \). Taking into account that the dilations \( x \to q^{x^1}x \) are represented by the operators \( q^{x^1} \frac{d}{dx} \), that is \( q^{x^1} \frac{d}{dx} f(x) = f(q^{x^1}x) \), one now readily verifies that the q-difference equation (5.2.12) coincides with the second order differential equation (5.1.5) in the limit as \( q \to 1 \).

5.3 ORTHOGONALITY RELATION:

We begin this section with the following theorem:

**Theorem:** The sequence of q-polynomials \( \{H^{(\mu)}(x; q)\} \), which are defined by the relation (5.2.7), satisfies the orthogonality relation

\[
\int_{-\infty}^{\infty} H^{(\mu)}(x; q) H^{(\mu)}(x; q) \frac{|x|^{2\mu}}{E_q(x^2)} dx
\]

\[
= \frac{\pi}{\cos \pi \mu} \frac{q^{\mu - 1}}{(q; q)_\infty} \left( q^{\frac{1}{2}} - q \right)^n \left( q^{\frac{1}{2}} q^{\mu - 1} \right) \delta_{mn} \quad (5.3.1)
\]

on the whole real line \( \mathbb{R} \) with respect to the continuous positive weight function \( w(x) = \frac{1}{E_q(x^2)} \).

**Proof:** Since the weight function in (5.3.1) is an even function of the independent variable \( x \) and \( H^{(\mu)}(-x; q) = (-1)^n H^{(\mu)}(x; q) \) by the definition.
(5.2.7), the \( q \)-polynomials of an even degree \( H_{2m}^{(\mu)}(x;q) \) and of an odd degree \( H_{2n+1}^{(\mu)}(x;q) \), \( m, n = 0, 1, 2, \ldots \), are evidently orthogonal to each other.

Consequently, it is sufficient to prove only those cases in (5.3.1), when degrees of polynomials \( m \) and \( n \) are either simultaneously even or odd. Let us consider first the former case. From (5.2.7) and (5.2.3) it follows that

\[
\int_{-\infty}^{\infty} H_{2m}^{(\mu)}(x;q) \frac{|x|^{2\mu}}{E_q(x^2)} H_{2n}^{(\mu)}(x;q) \frac{|x|^{2\mu}}{E_q(x^2)} dx
\]

\[
= (-1)^{m+n}(q;q)_m(q;q)_n \int_{-\infty}^{\infty} L_{m-1}^{\mu}(x^2;q) L_{n-1}^{\mu}(x^2;q) \frac{|x|^{2\mu}}{E_q(x^2)} dx
\]

\[
= 2(-1)^{m+n}(q;q)_m(q;q)_n \int_{0}^{\infty} L_{m-1}^{\mu}(x^2;q) L_{n-1}^{\mu}(x^2;q) \frac{|x|^{2\mu}}{E_q(x^2)} dx
\]

\[
= (-1)^{m+n}(q;q)_m(q;q)_n \int_{0}^{\infty} L_{m-1}^{\mu}(y;q) L_{n-1}^{\mu}(y;q) \frac{|y|^{2\mu}}{E_q(y)} dy
\]

\[
= (q;q)_n^2 d_n^{-1}(\mu - 1/2) \delta_{mn},
\]

where the normalization constant \( d_n(\alpha) \) is defined in (5.2.5). Thus

\[
\int_{-\infty}^{\infty} H_{2m}^{(\mu)}(x;q) \frac{|x|^{2\mu}}{E_q(x^2)} \frac{|x|^{2\mu}}{E_q(x^2)} dx
\]

\[
= \frac{\pi}{\cos \mu} \left( \frac{1}{q^{1/2} - 1}; q \right)_{\infty} q^{-n} (q;q)_n \left( \frac{q^{1/2}}{q^{1/2} - 1}; q \right)_n \delta_{mn} \quad (5.3.2)
\]

In the same way we find that in the latter case
\[ \int_{-\infty}^{\infty} H_{2m+1}^{(\mu)}(x;q) H_{2n+1}^{(\mu)}(x;q) \frac{|x|^{2\mu}}{E_q(x^2)} \, dx \]

\[ = 2(-1)^{m+n}(q;q)_m(q;q)_n \int_0^\infty \binom{\mu+1}{2} \binom{1}{n+1} \left( x^2 ; q \right)^{\frac{\mu+1}{2}} \frac{x^{2(\mu+1)}}{E_q(x^2)} \, dx \]

\[ = (-1)^{m+n}(q;q)_m(q;q)_n \int_0^\infty \binom{\mu+1}{2} \binom{1}{n+1} \left( y^2 ; q \right)^{\frac{\mu+1}{2}} \frac{y^{\mu+1}}{E_q(y)} \, dy \]

\[ = (q;q)_n^2 \delta_{n} \delta_{n+1} \left( \mu + \frac{1}{2} \right) \delta_{mn}. \]

Consequently

\[ \int_{-\infty}^{\infty} H_{2m+1}^{(\mu)}(x;q) H_{2n+1}^{(\mu)}(x;q) \frac{|x|^{2\mu}}{E_q(x^2)} \, dx \]

\[ = \frac{\pi}{\cos \pi \mu} \left( \frac{1}{q^{2} ; q} \right)_\infty \left( q^{1-\mu} ; q \right)_\infty q^{-n-\frac{1}{2}} (q;q)_n \left( q^{\mu+1} ; q \right)_{n+1} \delta_{mn}. \tag{5.3.3} \]

Putting (5.3.2) and (5.3.3) together results in the orthogonality relation (5.3.1).

The positivity of Jackson’s q-exponential function \( E_q(x^2) \) for \( x \in \mathbb{R} \) and \( q \in (0,1) \) is obvious from its definition (5.2.4) for it is represented as an infinity sum of positive terms (or an infinite product of positive factors). This completes the proof.

To conclude this section, we note the obvious fact that in the limit as \( q \to 1 \) the (5.3.1) reduces to the orthogonality relation (5.1.3) for the generalized Hermite polynomials (5.1.1). This follows immediately from the limit relations (5.2.8) and (5.2.9), upon using the fact that
\[
\lim_{q \to 1} E_q (q^{-1}z) = e^z \quad (5.3.4)
\]

Also, in the event the parameter \( \mu \) is zero, then the (5.3.1) coincides with the orthogonality relation for the polynomials (5.2.10) derived in [30].

### 5.4 Recurrence Relation:

In this section we derive a three term recurrence relation for the \( q \)-extension of the generalized Hermite polynomials (5.2.7). Since an arbitrary family of orthogonal polynomials \( p_n(x) \) satisfies a recurrence relation of the form (see [43, p. 19]).

\[
(a_n x + b_n) p_n(x) = p_{n+1}(x) + c_n p_{n-1}(x), \quad n \geq 0, \quad (5.4.1)
\]

we need to find coefficients \( a_n, b_n, c_n \) which correspond to the case under discussion.

Before starting this derivation we note that in what follows it proves convenient to use the following form

\[
L^{(\alpha)}_n(x;q) = \left( \frac{q^{\alpha+1};q}{(q;q)_n} \right) \sum_{k=0}^{n} \frac{q^{k(k+\alpha)}}{(q^{\alpha+1};q)_k} \left[ \frac{n}{k} \right]_q (-1)^k
\]

of the \( q \)-Laguerre polynomials \( L^{(\alpha)}_n(x;q) \), which comes from the first line definition (5.2.1), upon using the relation

\[
\frac{(q^{-n};q)_k}{(q;q)_k} = (-1)^k q^{\frac{k(k-1)}{2}} \left[ \frac{n}{k} \right]_q \quad (5.4.3)
\]

we first consider the case when \( n \) in (5.4.1) is even. Then from (5.2.7) and (5.4.2) we find that

\[
H_{2n+1}^{(\mu)}(x;q) + c_{2n}(q) H_{2n-1}^{(\mu)}(x;q)
\]
\[
= (-1)^n x \left( q^\frac{\mu+3}{2}; q \right)_n \sum_{k=0}^{n} q^k \binom{k\left( k+\mu+\frac{1}{2} \right)}{\mu+\frac{3}{2}} \binom{n}{k} (-x^2)^k \\
+ (-1)^{n-1} c_{2n}(q)x \left( q^\frac{\mu+3}{2}; q \right)_n \sum_{k=0}^{n-1} q^k \binom{k\left( k+\mu+\frac{1}{2} \right)}{\mu+\frac{3}{2}} \binom{n-1}{k} (-x^2)^k
\] (5.4.4)

The next step is to employ the relation
\[
(1-q^a) \left( q^{a+1}; q \right)_n = (1-q^{n+a}) \left( q^a; q \right)_n
\] (5.4.5)

In order to rewrite the quotient \( \left( q^{\mu+\frac{3}{2}}; q \right)_n / \left( q^{\mu+\frac{3}{2}}; q \right)_k \) from the first term in the right side of (5.4.4) as
\[
\left( q^{\mu+\frac{3}{2}}; q \right)_n / \left( q^{\mu+\frac{3}{2}}; q \right)_k = \frac{1-q^{n+\mu+\frac{1}{2}}}{1-q^{k+\mu+\frac{1}{2}}}
\] (5.4.6)

In the second term in the right side of (5.4.4) we can use the evident relation
\[
\left( q^{\mu+\frac{3}{2}}; q \right)_n / \left( q^{\mu+\frac{3}{2}}; q \right)_k \quad \text{and the same formula (5.4.5) for the factor}
\]
\[
\left( q^{\mu+\frac{3}{2}}; q \right)_n / \left( q^{\mu+\frac{3}{2}}; q \right)_k
\] Also from the property of the q-binomial co-efficient
\[
\left[ \frac{n-k}{k} \right]_q = \frac{1-q^{n-k}}{1-q^k} \binom{n}{k}
\] (5.4.7)
Putting this all together, we obtain

\[ H^{(\mu)}_{2n+1}(x;q) + c_{2n}(q) \ H^{(\mu)}_{2n-1}(x;q) = (-1)^n x \left( q^{\mu+\frac{1}{2}} \right)_n \sum_{k=0}^{n} \frac{k^{(k+\mu+1)}_{\mu+\frac{1}{2}}}{q_{\mu+\frac{1}{2}}^k} \left[ \begin{array}{c} n \\ k \end{array} \right] \frac{(-x^2)^k}{1-q^{k+\mu+\frac{1}{2}}} \times \left\{ 1 - q^{k+\mu+\frac{1}{2}} c_{2n}(q) \frac{1-q^{n-k}}{1-q^n} \right\} \]  

(5.4.8)

The right hand side of (5.4.8) should match with

\[ H^{(\mu)}_{2n}(x;q) = (-1)^n \left( q^{\mu+\frac{1}{2}} \right)_n \sum_{k=0}^{n} \frac{k^{(k+\mu-1)}_{\mu+\frac{1}{2}}}{q_{\mu+\frac{1}{2}}^k} \left[ \begin{array}{c} n \\ k \end{array} \right] \frac{(-x^2)^k}{1-q^{k+\mu+\frac{1}{2}}} \]  

(5.4.9)

multiplied by \( a_{2n}(q)x + b_{2n}(q) \).

This means that the coefficient \( c_{2n}(q) \) can be found from the equation

\[ 1 - q^{n+\mu+\frac{1}{2}} - c_{2n}(q) \frac{1-q^{n-k}}{1-q^n} = d_n(q) q^{-k} \left( 1 - q^{n+\mu+\frac{1}{2}} \right) \]  

(5.4.10)

where \( d_n(q) \) is some \( k \)-independent factor. It is easy to verify that the only solution of the equation (5.4.10) is \( c_{2n}(q) = 1 - q^n \) and \( d_n(q) = q^n \).

Thus

\[ H^{(\mu)}_{2n+1}(x;q) + \left( 1 - q^n \right) H^{(\mu)}_{2n-1}(x;q) = q^n x H^{(\mu)}_{2n}(x;q) \]  

(5.4.11)

Similarly, in the case of an odd \( n \) from (5.4.9), we have

\[ H^{(\mu)}_{2n+2}(x;q) + c_{2n+1}(q) H^{(\mu)}_{2n}(x;q) \]
\[
\begin{align*}
&= (-1)^{n+1} \left( q^{\mu+\frac{1}{2}}; q \right)_{n+1} \sum_{k=0}^{n+1} \frac{q^{\frac{k(k+\mu-\frac{1}{2})}{2}}}{\left( q^{\mu+\frac{1}{2}}; q \right)_k} \left[ \frac{n+1}{k} \right]_q (-x^2)^k \\
&\quad + (-1)^n c_{2n+1}(q) \left( q^{\mu+\frac{1}{2}}; q \right)_n \sum_{k=0}^{n} \frac{q^{\frac{k(k+\mu-\frac{1}{2})}{2}}}{\left( q^{\mu+\frac{1}{2}}; q \right)_k} \left[ \frac{n}{k} \right]_q (-x^2)^k \\
&\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \}
\[ n + n^+ = q^{n+\mu+1/2} \times H_{2n+1}^{(\mu)}(x; q), \]

upon using the q-pascal identity
\[ \left[ \begin{array}{c} n + 1 \\ m + 1 \end{array} \right]_q = \left[ \begin{array}{c} n \\ m + 1 \end{array} \right]_q = q^{n-m} \left[ \begin{array}{c} n \\ m \end{array} \right]_q \]  \hspace{1cm} (5.4.15)

for the q-binomial co-efficient \[ \left[ \begin{array}{c} n \\ k \end{array} \right]_q. \]

From (5.4.11) and (5.4.14) it thus follows that the q-polynomials \( H_{n}^{(\mu)}(x; q) \)
satisfy a three term recurrence relation of the form \( \left( \theta_n = n - 2 \left[ \frac{n}{2} \right] \right) \)
\[ H_{n+1}^{(\mu)}(x; q) + \left( 1 - q^2 \right) H_{n-1}^{(\mu)}(x; q) \]
\[ = q^{n+\mu}_x \times H_{n}^{(\mu)}(x; q) \]  \hspace{1cm} (5.4.16)

With the aid of (5.2.9) we now readily verify that the (5.4.16) coincides with
the three-term recurrence relation (5.1.4) for the generalized Hermite polynomials \( H_{n}^{(\mu)}(x) \) in the limit as \( q \to 1. \)

5.5 CONCLUSION:

We conclude this exposition with the following remark. It is well known that
the Hermite functions \( H_n(x) e^{-\frac{x^2}{2}} \) (or the wave function of the linear harmonic oscillator in quantum mechanisms) are eigen functions of the fourier integral transform (with respect to the kernel \( e^{2\pi x y} \)) with eigen value \( i^n. \)
We can introduce the generalized Fourier transform operator

\[
F^{(i)}_\mu f(x) = c^{(-1)}_\mu \int_{-\infty}^{\infty} e^{(-i\mu t)} f(t) |t|^\mu dt
\]  \hspace{1cm} (5.5.1)

with a kernel

\[
e^\mu_{\mu+\frac{1}{2}}(x) = \frac{J_{\mu+\frac{1}{2}}(x) - iJ_{\mu-\frac{1}{2}}(x)}{2\sqrt{x}}
\]  \hspace{1cm} (5.5.2)

where the constant \( c^{(i)}_\mu = 2^{\mu+\frac{1}{2}} \Gamma\left(\mu + \frac{1}{2}\right) \) and \( J_\alpha(x) \) is the Bessel function. The generalized Hermite polynomials (5.1.1) are the eigen functions of generalized Fourier transform operator (5.5.1) [118].

It is of interest to find a q-extension of (5.5.1) and (5.5.2). This study is under way.