CHAPTER 3

METHODOLOGY OF EVALUATION OF CINEMA PLAN - (MODEL 1 AND MODEL 2)

3.0. Introduction:

In Chapter 1, we stressed the need for the use of formal methods in the planning of cinema medium. In Chapter 2, we have pointed out that in the literature, there are no methods available that can be used for the resolution of the planning problem in the medium of cinema. It is, therefore, necessary to develop a methodology suited to the resolution of this problem. We develop this methodology in chapters three, four and five.

In Chapter 1, we had pointed out that in view of the practice currently being followed in the practising world, the basis of our planning will be the reach and frequency that the advertiser wishes to achieve. In this and the next chapter, we develop the methodology for computing the reach and frequency that will be achieved by a given cinema plan. In Chapter 5, we will develop the methodology for preparing a plan that will achieve the desired reach and frequency at the lowest possible cost.
3.1. **Approach to Development of Models for Computing Reach and Frequency**

The development of the model is done to aid the planning process involved in the cinema medium. The models, therefore, are developed keeping in view the process followed for planning in this medium. Cinema is a highly localized medium and hence planning is done at the smallest geographical unit, namely, a town.

The planning process comprises the steps listed below:

1. **Listing of the theatres,** which in the planner's judgement, are suitable for advertising the product in the town under consideration. The list of such theatres will be called list of candidate theatres.

2. **Choosing from the list of candidate theatres a group of theatres** where the advertisement is to be screened,

3. **Deciding on the number of screening weeks to be taken in each of selected theatres.**

In the problem of evaluation the decisions involved in Steps (1), (2) and (3) have already been made. The planner is now interested in knowing the reach and the frequency
that will be achieved by the plan that the planner has prepared. The approaches that are developed for obtaining the reach and the frequency of a plan formally take into account the factors that determine the reach and the frequency. These factors are:

a. The cinema-going habits of the target audience/population, and
b. The manner in which the members of the target audience select a theatre from the candidate theatres, to see a movie when they do decide to see a movie.

All the models assume that the cinema-going habits are known and can be expressed in terms of probability distribution of the number of movies seen in a given time-span, namely a year. The essential difference in the models developed arises with respect to (b). We have developed these different models by changing the assumption regarding the manner in which the members of the target audience select the theatre when they decide to visit a theatre. Two models based on comparatively strong assumptions are presented in this chapter. A third one based on still weaker assumptions is presented in the fourth chapter.
In the development of these models our endeavour has been to make as weak and as realistic an assumption as possible and to ensure that the data required for the use of the models are such as can be easily obtained.

3.2. **Model 1**

3.2.1. **Assumptions:**

1. A person visits a given theatre at the most once a week.

2. A person's choice of a theatre in a given week will not affect his choice of another theatre in the same week. In addition, for a person in the target audience, whatever may be the choice of theatre week combination in the first $r$ visits, each of the remaining theatre week combination has the same probability of being selected at $(r + 1)$ th visit.

3. All the theatre-week combinations have equal chances of being selected.

4. The probability distribution of the number of visits to a theatre made by a member of the target audience is known.

5. Without loss of generality, we can assume that planning horizon is a year.
With these assumptions, we will develop formulae to compute the reach and the frequency of a given plan. Some notations are introduced to simplify the development of the formulae.

3.2.2. Notations:

Let

\( M \) denote the number of candidate theatres,

\( T \) denote the number of theatres chosen for screening the advertisement,

\( W_j \) denote the number of weeks for which the advertisement is screened in the \( j \)th theatre during the planning horizon,

\( W = \sum_{j=1}^{T} W_j \) denote the total number of theatre week combination for screening the advertisement,

\[ N(W) = \{ W_{gh} | W_{gh} = 1 \}, \]

where

\[ W_{gh} = \begin{cases} 
1 & \text{if the advertisement is screened in the } \text{h th week in the } \text{g th theatre} \\
0 & \text{Otherwise.} 
\end{cases} \]

\( g = 1, 2, \ldots, M, \) and \( h = 1, 2, \ldots, 52, \)

\[ |N(W)| = \text{Number of elements in the set } N(W). \text{ Since } W \text{ screening weeks have been selected we have } |N(W)| = W. \]
i denote the random variable, the number of visits to a theatre made by a person in the target audience \( i = 1, 2, \ldots, n \).

Note that \( n \leq 52M \) by virtue of assumption (1).

Finally let

\[ f_i \]

\[ f_i \]

\[ \sum_{i=0}^{n} f_i = 1. \]

Note that in cinema medium theatre week represents a vehicle. A theatre week is a specified week in a specified theatre. Note that if we have \( M \) candidate theatres, we have \( 52M \) theatre week combinations.

3.2.3. Development of Model 1:

Let \( A_j \) denote the event that a member of the target audience visits \( j \)th theatre week combination. \( j = 1, 2, \ldots, W \).

We have thus \( W \) events which are not independent. Our interest is in computing the probability that exactly 'k' among these 'W' events occur simultaneously and in computing the probability that at least one of these events occurs. The event that exactly 'k' among the 'W' events
occur corresponds to the event, that a person in the target group has exactly \( k \) opportunities to see the advertisement. This follows from our assumption that a person visits a given theatre at the most once a week.

Thus the problem of obtaining OTS distribution is equivalent to finding the probability distribution of random variable \( k \) that exactly \( k \) events among the \( W \) events occur simultaneously. The probability that at least one of these \( W \) events occurs corresponds to the problem of finding the reach of the cinema plan. This follows from the definition of reach which is the probability that a person has at least one opportunity to see the advertisement. Clearly reach can be obtained by summing up the probabilities \( P_{[k]} \), that a person has exactly \( k \) opportunities to see the advertisement. The summation will extend over the value of \( k \) ranging from 1 to \( W \).

The solution of our problem, therefore reduces to that of finding the value of \( P_{[k]} \).

The solution to this problem is already available in the literature on theory of probability. It is wellknown that

\[
P_{[k]} = S_k - \binom{k+1}{k} S_{k+1} + \binom{k+2}{k} S_{k+2} - \cdots + (-1)^{W-k} \binom{W}{k} S_W
\]
where $S_k$ is the sum of the probabilities of $k$ events.

i.e. $S_k = \Sigma \Pr(A_{i_1}, A_{i_2}, \ldots, A_{i_k}) = \Sigma \Pr(A_{i_1}, A_{i_2}, \ldots, A_{i_k})$.

We now, therefore, obtain the values of $S_j$ and those of $P[k]$.

To illustrate the methodology we first compute $S_1$ which is equal to $\Sigma \Pr(A_j)$.

Let $E_i$ denote the event that a person in the target group makes exactly 'i' visits during the planning horizon.

$i = 1, 2, \ldots, n$.

It can be seen that

$A_j = \bigcup_{i=1}^{n} (A_j \cap E_i)$. Since $E_i \cap E_j = \emptyset$ for $i \neq j$.

it follows that

$\Pr(A_j) = \Sigma_{i=1}^{n} \Pr(A_j \cap E_i) = \Sigma_{i=1}^{n} \Pr(A_j | E_i) \times \Pr(E_i)$.

...........(3.1)

We now want to compute $P(A_j | E_i)$.

$P(A_j | E_i)$ - Probability that a person visits in the specified theatre week combination given that he makes $i$ visits to theatre.
To compute \( P(A_j|E_1) \), we take the following steps:

**Step 1:** Given the information that a person has made \( i \) visits, we know by virtue of assumption (1) that he has chosen \( i \) distinct theatre-week combinations.

From assumption (2) probability of choosing \( i \) theatre week combination is \( \frac{1}{52M \choose i} \).

**Step 2:** We now seek the probability that his 'i' visits include a visit to the \( j \)th theatre week combinations. The number of ways in which the \( i \) visits can be made so as to include a visit to the \( j \)th theatre week combination is \( \frac{52M - 1}{i - 1} \). This is because one of his visits must be to the \( j \)th theatre week combination and remaining \( i - 1 \) can be chosen from remaining \( 52M - 1 \) theatre week combinations.

Thus we have

\[
P(A_j|E_1) = \frac{\frac{52M - 1}{i - 1}}{\frac{52M}{i}} &= \frac{i}{(52M)} = \frac{\binom{52M}{1}}{\binom{52M}{i}} \quad \text{...(3.2)}
\]

(Using Fermi-Dirac Statistics).

\(^{14}\)The notation \( \binom{52M}{i} \) stands for \( \frac{52M!}{i!(52M-1)!} \).
\[ P(E_1) = f_1 \]

Substituting these values in (3.1) we have
\[
P_j = P(A_j) = \sum_{i=1}^{W} \frac{\binom{52M-1}{i-1}}{\binom{52M}{i}} f_i = \sum_{i=1}^{W} \frac{\binom{1}{i}}{\binom{52M}{i}} f_i.
\]

Thus
\[
S_1 = \sum_{j=1}^{W} p_j = \sum_{j=1}^{W} p_r(A_j).
\]

Note that \( p(A_j) \) is independent of \( j \). Hence
\[
\sum_{j=1}^{W} P(A_j) = P(A_j) \times \text{No. of terms. No. of terms} = W.
\]

\[ \ldots \ldots \ldots \ldots (3.3) \]

(The choice of one visit can be made in \( \binom{W}{1} \) ways).

Hence
\[
S_1 = \sum_{i=1}^{n} \frac{\binom{52M-1}{i-1}}{\binom{52M}{i}} f_i = \sum_{i=1}^{n} \frac{\binom{1}{i}}{\binom{52M}{i}} f_i \binom{W}{1}
\]

\[ \ldots \ldots \ldots \ldots (3.4) \]
where \( S_{1} \) - Sum of all the probabilities that a person visits in 'a' specified theatre week combination out of \( W \) theatre week combinations for screening the advertisement.

Having developed the expression for \( S_{1} \), let us develop the expression for \( S_{j} \).

Suppose we are concerned with the probability that a person makes visits in \( i_{1}, i_{2}, \ldots, i_{j} \) specified theatre week combinations, given that he has made \( i \) visits, we have the by using earlier notation and logic

\[
Pr(\bigcap_{s=i_{1}}^{i_{j}} A_{s}|E_{i}) = Pr(A_{i_{1}} \cap A_{i_{2}} \ldots \cap A_{i_{j}} | E_{i})
\]

\[
= \binom{52M-j}{i-j} \quad \ldots \ldots \quad (3.5)
\]

Notice that (3.5) is the conditional probability that a person sees the advertisement in \( j \) specified theatre-week combinations given that he makes \( i \) visits in a year.

In order to get the unconditional probability that a person makes visits in \( j \) specified theatre-week combinations,
\[ P_{i_1, i_2, \ldots, i_j} = \Pr \left( \bigcap_{s=i_1}^{i_j} A_s \right) = \sum_{i \geq j} \Pr \left( \bigcap_{s=i_1}^{i_j} A_s | E_1 \right) P(E_1) \]

\[ = \sum_{i \geq j} \frac{(52M - j)!}{(i - j)! (52M - i)!} f_i. \]

\[ P_{i_1, i_2, i_3, \ldots, i_j} = \frac{n}{\binom{52M}{i}} f_i \quad \ldots \quad (3.6) \]

\[ S_j = \sum_{i_1 < i_2 < \ldots < i_j} P_{i_1, i_2, \ldots, i_j} \]

= Sum of all the probabilities that a person visits in \( j \) specified theatre-week combinations out of \( W \) theatre-week combinations chosen for screening the advertisement.
The choice of j theatre weeks out of \( \binom{W}{j} \) ways. Therefore summing (3.6), \( \binom{W}{j} \) times using the similar logic of (3.3).

\[
S_j = \sum_{i_1 < i_2 \ldots < i_j} p_{i_1, i_2, \ldots, i_j} = \sum_{i_1 < i_2 \ldots < i_j} \Pr \left[ \bigcap_{i=1}^{i_j} A_i \right] \\
= \sum_{\{i_1 < i_2 \ldots < i_j\} \subseteq N(W)} \Pr_{s=i_1}^{(\bigcap A_s)} \\
= \binom{W}{j} \sum_{i \geq j} \binom{52M}{i} \binom{1}{i-j} f_1 \ldots (3.7) \\
= \binom{W}{j} \sum_{i \geq j} \binom{52M}{j} f_1 \binom{W}{j} \ldots (3.8)
\]

Now substituting the expression developed for \( S_j \) in the expression of \( P[k] \), where

\[
P[k] = S_k - \binom{k+1}{k} S_{k+1} + \binom{k+2}{k} S_{k+2} + \ldots + (-1)^{W-k} \binom{W}{k} S_W \\
= \sum_{j=k}^{W} (-1)^{j-k} \binom{j}{k} S_j \ldots (3.9).
\]
Substituting the value of $S_j$ given in (3.7),

$$P[k] = \sum_{j=k}^{W} (-1)^{j-k} \binom{j}{k} \sum_{i \geq j}^{n} \frac{\binom{52M-j}{i-j}}{52M} f_i \left( \frac{W}{j} \right) \quad \ldots (3.10)$$

By the earlier arguments, we know that

$$\text{Reach} = \sum_{k=1}^{W} P[k]$$

$$\text{Reach} = \sum_{k=1}^{W} \left\{ \sum_{j=k}^{W} (-1)^{j-k} \binom{j}{k} \right\} \sum_{i \geq j}^{n} \frac{\binom{52M-j}{i-j}}{52M} f_i \left( \frac{W}{j} \right) \quad \ldots (3.11)$$

Expression (3.11) can be also written as follows by using (3.8)

$$\text{Reach} = \sum_{k=1}^{W} \left\{ \sum_{j=k}^{W} (-1)^{j-k} \binom{j}{k} \right\} \sum_{i \geq j}^{n} \frac{\binom{i}{j}}{52M} f_i \left( \frac{W}{j} \right)$$

In the next section, we shall illustrate the use of these formulae through a numerical example.
3.2.1. **Explanation of the Mathematics Through a Hypothetical Numerical Problem**

Let us assume that the theatre visiting habits of a specified target audience are as follows:

<table>
<thead>
<tr>
<th>$i$</th>
<th>$f_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.523</td>
</tr>
<tr>
<td>1</td>
<td>0.120</td>
</tr>
<tr>
<td>2</td>
<td>0.211</td>
</tr>
<tr>
<td>3</td>
<td>0.138</td>
</tr>
<tr>
<td>4</td>
<td>0.008</td>
</tr>
</tbody>
</table>

$\sum_{i=0}^{4} f_i = 1.000$

Candidate theatres $= M = 5$

Theatres chosen for screening the advertisement $= T = 2$

Total theatre week combination $= W_1 + W_2 = W = 15 + 15 = 30$.

Potential theatre week combination $= 5 \times 52 = 260$ weeks

expression

Using the (3.14) we first compute $S_1$,

$$S_1 = \sum_{i \geq 1} \frac{1}{i} \left( \begin{array}{c} 1 \\ i \\ \end{array} \right) \frac{f_i}{\left( \begin{array}{c} 260 \\ 1 \\ \end{array} \right)} \left( \begin{array}{c} 30 \\ 1 \\ \end{array} \right)$$

$$= \frac{30}{260} \left[ .12 + .422 + .414 + .032 \right]$$
Using (3.8) we next calculate $S_2$, $S_3$ and $S_4$.

$$S_2 = \sum_{i \geq 2} \binom{1}{i} \binom{3}{2} \frac{f_1(2)}{260} \binom{30}{2}$$

$$= \frac{435}{33670} \left[0.211 + 0.414 + 0.048\right]$$

$$= 0.008695$$

Similarly

$$S_3 = \sum_{i \geq 3} \binom{1}{i} \binom{30}{3} \frac{f_1(3)}{260} \binom{3}{2}$$

$$= \frac{170}{2895620} \left[0.170\right]$$

$$= 0.00025$$

$$S_4 = \sum_{i = 4} \binom{4}{i} \binom{30}{4} \frac{f_1(4)}{260} \binom{260}{4}$$

$$= 0.000011.$$ 


$$P[1] = S_1 - \binom{2}{1} S_2 + \binom{3}{1} S_3 - \binom{4}{1} S_4$$

$$= 0.1139999 - 2(0.008695) + 3(0.00025) - 4(0.000011)$$

$$= 0.09736.$$
We know from the previous section that

\[ \text{Reach} = \sum_{k=1}^{W} P[k] \]


Note that \( P[5] \ldots \ldots \ldots P[30] \) will be zero, as \( f_{5}, f_{6} \ldots \ldots \) etc. are zero.

Therefore reach of the plan is 0.1056.

<table>
<thead>
<tr>
<th>OTS distribution</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.097360</td>
</tr>
<tr>
<td>2</td>
<td>0.007981</td>
</tr>
<tr>
<td>3</td>
<td>0.000246</td>
</tr>
<tr>
<td>4</td>
<td>0.000001</td>
</tr>
</tbody>
</table>
3.3. Mean and the Variance Of OTS Distribution:

In addition to the complete distribution of OTS, the advertiser is also interested in the mean and the variance of the distribution. Fortunately these can be obtained in terms of $S_1$ and $S_2$. In this section we obtain these formulae.

**Average Frequency:**

The average number of opportunities -to-see (OTS) for a member of the target audience is defined as the average frequency. This is the mean of the OTS distribution.

$$\text{Average frequency} = \sum_{k=1}^{W} k P [k] \quad \quad (3.1.2)$$

For the example stated in 3.2.4, we have

$$\text{Average frequency} = \sum_{k=1}^{30} k P [k] = \sum_{k=1}^{4} k P [k].$$

Since $P [k] = 0$ for $k = 5$ onwards.

$$\begin{align*}
\text{Average frequency} &= 1 \times (.09736) + 2 \times (.00796) \\
&\quad + 3 \times (.00024) + 4 \times (.00001) \\
&= .1140
\end{align*}$$

**Lemma 1:**  To show that $\sum_{k=1}^{W} k P [k] = S_1$.

We know that $P [k] = \sum_{a=k}^{W} (-1)^{a-k} \binom{a}{k} S_a$. 

\[ W \sum_{k=1}^{\alpha} k P[k] = W \sum_{\alpha=1}^{\alpha} k(-1)^{\alpha-k} \binom{\alpha}{k} S_{\alpha} \]

Interchanging the summation sign, we have

\[ = \sum_{\alpha=1}^{W} S_{\alpha} \sum_{k=1}^{\alpha} k(-1)^{\alpha-k} \binom{\alpha}{k} \]

\[ = S_{1} + \sum_{\alpha=2}^{W} S_{\alpha} \sum_{k=1}^{\alpha} (-1)^{\alpha-k} \frac{k(\alpha!)}{(\alpha-k)! k!} \]

\[ = S_{1} + \sum_{\alpha=2}^{W} S_{\alpha} \sum_{k=1}^{\alpha} (-1)^{\alpha-k} \frac{\alpha(\alpha-1)!}{(\alpha-k)! (k-1)!} \]

\[ = S_{1} + \sum_{\alpha=2}^{W} S_{\alpha} \left[ \sum_{k=1}^{\alpha} (-1)^{\alpha-k} \binom{\alpha-1}{k-1} \right] \]

Put \( k-1 = t \)

\[ = S_{1} + \sum_{\alpha=2}^{W} S_{\alpha} \left[ \sum_{t=0}^{\alpha-1} (-1)^{\alpha-1-t} \binom{\alpha-1}{t} \right] \]

\[ = S_{1} + \sum_{\alpha=2}^{W} S_{\alpha} (-1)^{\alpha-1} \left[ \sum_{t=0}^{\alpha-1} (-1)^{t} \binom{\alpha-1}{t} \right] \]

\[ = S_{1} + \sum_{\alpha=2}^{W} S_{\alpha} (-1)^{\alpha-1} (1-1)^{\alpha-1} \]

\[ = S_{1} \]
Hence \( \sum_k P[k] = S_1 \ldots \) (3.13)

The result (3.13) can also be obtained from the interpretation of \( S_1/W \). \( S_1/W \) denotes the probability that a person will visit a randomly chosen theatre week. To see this, let us denote \( B_j \) the event that the randomly chosen theatre week turns out to be the \( j \)th theatre-week. Let \( B \) denote the theatre week combination that is chosen as a result of our random selection.

Then \( B = \bigcup_{j=1}^{W} (A_j \cap B_j) \) (\( A_j \) denotes the same event as stated earlier).

Since \( B_j \) and \( B_j' \) are mutually exclusive for \( j \neq j' \)

\[
P_r(B) = P_r\left( \bigcup_{j=1}^{W} A_j \cap B_j \right)
\]

\[
= \sum_{j=1}^{W} P_r(A_j | B_j) P(B_j) \quad \text{(from section 3.2.3)}
\]

\[
= \sum_{j=1}^{W} P_r(A_j) / W = S_1/W.
\]

Since there are \( W \) possibilities to be chosen, the expected number will be \( \frac{S_1}{W} \times W = S_1 \).

In the numerical illustration, we see that \( S_1 = 0.1140 \) which is equal to the average frequency \( = 0.1140 \).
\[ \text{GTS per person reached} = \frac{\text{Average frequency}}{\text{Reach}} \]

\[ = \frac{\sum_{k=1}^{W} k P[k]}{\text{Reach}} = \frac{S_1}{1 - P[0]} \]

In the numerical problem illustrated GTS per person reached \[ \frac{0.1140}{1.056} = 1.08 \]

where \( P[0] \) is the probability of zero exposure (OTS).

**Lemma 2:**

Variance of the OTS distribution - The variance of the OTS distribution is the spread of OTS distribution.

We know that

\[ \text{Variance} = \sum_{k=1}^{W} k^2 P[k] - \left( \sum_{k=1}^{W} k P[k] \right)^2 \]

We know that \( \sum_{k=1}^{W} k P[k] = S_1 \)

\[ \text{Variance} = \sum_{k=1}^{W} k^2 P[k] - S_1^2 \quad \cdots \quad (3.14) \]

\[ \sum_{k=1}^{W} k^2 P[k] = \sum_{k=1}^{W} \sum_{\alpha=k}^{W} k^2 (-1)^{\alpha-k} \binom{\alpha}{k} S_\alpha \]

\[ \cdots \quad (3.15) \]

Interchanging the summation sign R.H.S of (3.15) becomes
\[
\begin{align*}
\sum_{\alpha=1}^{W} s_{\alpha} \sum_{k=1}^{a} k^2 (-1)^{a-k} \binom{a}{k} \\
\text{Now } k^2 = k(k-1) + k
\end{align*}
\]

\[
\begin{align*}
\sum_{\alpha=1}^{W} s_{\alpha} \sum_{k=1}^{a} [k(k-1) + k] (-1)^{a-k} \binom{a}{k} \\
= \sum_{\alpha=1}^{W} s_{\alpha} k(k-1)(-1)^{a-k} \binom{a}{k} + \sum_{\alpha=1}^{W} s_{\alpha} k(-1)^{a-k} \binom{a}{k}
\end{align*}
\]

From Lemma (1) we know that
\[
S_1 = \sum_{\alpha=1}^{W} s_{\alpha} \sum_{k=1}^{a} k(-1)^{a-k} \binom{a}{k}
\]

Expression (3.16) becomes
\[
\begin{align*}
\sum_{\alpha=1}^{W} s_{\alpha} \sum_{k=1}^{a} k(k-1)(-1)^{a-k} \binom{a}{k} + S_1 \\
= \sum_{\alpha=2}^{W} s_{\alpha} k(k-1)(-1)^{a-k} \binom{a}{k} + S_1 \\
= 2S_2 + \sum_{\alpha=3}^{W} s_{\alpha} \sum_{k=2}^{a} k(k-1)(-1)^{a-k} \binom{a}{k} + S_1 \\
= S_1 + 2S_2 + \sum_{\alpha=3}^{W} s_{\alpha} \sum_{k=2}^{a} k(k-1)(-1)^{a-k} \binom{a}{k}
\end{align*}
\]

Consider
\[
\sum_{\alpha=3}^{W} s_{\alpha} \sum_{k=2}^{a} k(k-1)(-1)^{a-k} \binom{a}{k}
\]
From the above expression it is clear that the coefficients of $S_i$ become zero for $i = 3, \ldots, W$.

Hence substituting value of expression (3.16) in (3.14) we get

$$\text{Variance} = S_1 + 2S_2 - S_1^2 \quad \ldots\ldots\ldots (3.17)$$

This can be verified through a numerical problem.

We know $S_1 = 0.113999$

$S_2 = 0.008695$

$$\text{Variance} = S_1 + 2S_2 - S_1^2 = 0.1140 + 2(0.008695) - (0.1140)^2 = 0.1184.$$
3.4. Computational Schemes

The numerical example presented in Section 3.2.4 gives an idea about the degree of the computational effort involved in the computation of reach, frequency and OTS distribution. In this numerical example, the maximum value of i, the number of visits to theatres made by a person in the target audience, was only four, and hence in expression (3.8), we had to add only four terms. From the expression it is clear that the number of non-zero terms will be equal to minimum of n and W.

In most of the practical examples n will be of the order of hundred and W will be of the order 150. Thus even for a town the computational effort involved will be quite substantial. For a moderately large cinema plan the number of towns involved is of the order of 50. In a large cinema plan the number of towns involved will be of the order of thousands. We, therefore, need a very efficient computational scheme to obtain the reach, frequency and the OTS distribution. In this section we simplify the equation (3.7) obtained in (3.2.3), so that the computation involved can be easily carried out. So far we assumed that $f_1$ is known. In Chapter 6 we discuss the method for obtaining $f_1$ for the number of visits made by a person in the target audience.
To simplify the expression (3.7) we note that
\[ W \leq n \leq 52M \quad \text{or} \quad n \leq W \leq 52M. \]

We will simplify the expression under both of these conditions.

**Case 1:** Assume that \( W \leq n \leq 52M \)

Since \( W \) is a positive integer, we have
\[
\binom{W}{j} = \begin{cases} 
0 & \text{if } j > W \\
\frac{W!}{j!(W-j)!} & \text{if } j \leq W.
\end{cases} \tag{3.18}
\]

We know
\[
P[k] = \sum_{j=k}^{W} (-1)^{j-k} \binom{j}{k} \left[ \sum_{i=j}^{n} \binom{52M-j}{i} \binom{W}{i} \right] \tag{3.19}
\]

Using (3.18), expression (3.19) can be re-written as:
\[
P[k] = \sum_{j=k}^{n} (-1)^{j-k} \binom{j}{k} \left[ \sum_{i=j}^{n} \binom{52M-j}{i} \frac{(i-j)}{(52M)} \frac{W}{i} \right] \tag{3.20}
\]

Now interchanging the summation sign, (3.20) becomes
Let $j-k = t$

$$
\frac{n}{\sum_{i=k}^{52M}} \binom{W}{k} (-1)^{i-k} \sum_{t=0}^{i-k} \binom{W-k}{t} \binom{-52M+1-l}{i-t-k}
$$

[Applying $\sum_{x=0}^{n} \binom{a}{x} \binom{b}{n-x} = \binom{a+b}{n}$]

$$
= \sum_{i=k}^{52M} \binom{W}{i} (-1)^{i-k} \binom{-52M+1-l+W-k}{i-k}
$$

$$
= \sum_{i=k}^{52M} \binom{W}{i} \binom{52M - W}{i-k}
$$

$$
P[k] = \sum_{i=k}^{52M} \binom{W}{i} \binom{52M - W}{i-k}
$$

Case 2:

In this case we have $n \leq W \leq 52M$

We have

$$
P[k] = \sum_{j=k}^{W} (-1)^{j-k} \binom{j}{k} \sum_{i=j}^{52M} \binom{52M - j}{i-j} \binom{W}{j}
$$

$$
\text{......... (3.22)}
$$
Now since $W \geq n$ the right hand side of (3.22)
can be written as

$$\sum_{i=k}^{n} \frac{f_i}{\binom{52M}{i}} \sum_{j=k}^{i} (-1)^{i-k} \binom{i}{k} \binom{52M-j}{i-j} W$$

By the same logic used in Case 1 this can be
simplified to the same expression as obtained in (3.21).

Therefore

$$P[k] = \sum_{i=k}^{n} \frac{f_i}{\binom{52M}{i}} \binom{i}{k} \binom{52M-W}{i-k}$$

We compute OTS distribution using (3.21) whichever may be
the case, then as stated earlier we obtain reach by using
the expression

$$\text{Reach} = \sum_{k=1}^{W} P[k]$$

$$\text{Reach} = \sum_{k=1}^{W} \left\{ \sum_{i=k}^{n} \frac{f_i}{\binom{52M}{i}} \binom{i}{k} \binom{52M-W}{i-k} \right\} \text{ by using (3.21)}$$

Reversing the summation sign (using (3.18)) we get

$$= \sum_{i=1}^{n} \frac{f_i}{\binom{52M}{i}} \sum_{k=1}^{i} \binom{i}{k} \binom{52M-W}{i-k}$$

$$= \sum_{i=1}^{n} \frac{f_i}{\binom{52M}{i}} \left\{ \sum_{k=0}^{i} \binom{i}{k} \binom{52M-W}{i-k} - \binom{0}{i} \right\}$$
The expression (3.21) for computing OTS distribution and (3.23) for computing the reach are very useful in the following ways:

1. Computational complexities are reduced to a great extent.
2. The efficiency of the program can be increased.
3. This method takes care of the problem of underflow and overflow.
4. We can avoid the damage caused by finite arithmetic.

The terms in the right hand side of (3.23) have an interesting interpretation. Clearly $1 - f_o$ denotes the
probability that a person makes at least one visit to a theatre. The expression
\[ \sum_{i=1}^{n} \binom{52M-W}{i} \] denotes the joint probability that a person makes at least one visit to the theatre and visits only the theatres where the advertisement is not screened. This can be seen from the argument given below:

We define the following:

Let

A denote the event that a person makes at least one visit to a theatre,

B denote the event that he sees the advertisement at least once, and

B' denote the event that he does not see the advertisement.

Recall that \( E_i \) denotes the event that a person in the target group makes exactly 'i' visits.

\[ A \cap B' = \bigcup [E_i \cap B'] \]

\[ P[A \cap B'] = \sum Pr[E_i \cap B'] \]

(since \( E_i \) and \( E_i' \) are mutually exclusive)

\[ P[E_i \cap B'] = P(B' | E_i) P(E_i) = P(E_i) P(B' | E_i) \]
A \cap B' is the joint event that a person makes at least one visit to the theatre where the advertisement is not screened.

Clearly \( A = (A \cap B) \cup (A \cap B') \)

\[
P(A) = P(A \cap B) + P(A \cap B')
\]

\[
P(A) = 1 - f_0, \quad P(A \cap B') = \sum_{i=1}^{n} f_i \left( \frac{52M-W}{52M} \right) \]

\[
P(A \cap B) = P(A) - P(A \cap B')
\]

\[
P(A \cap B) = 1 - f_0 - \sum_{i=1}^{n} f_i \left( \frac{52M-W}{52M} \right)
\]

Now \( P(A \cap B) = P(B) \cdot P(A|B) \)

Note that \( P(A|B) = 1 \)

\( P(A \cap B) = P(B) \). By definition of B, \( \therefore P(A \cap B) = P(B) \) is the reach of the plan.
**Corollary 1:** To show that maximum reach = 1 - f_0.

**Maximum Reach:** Maximum reach is defined by the reach achieved by including all the candidate theatres for all the weeks in the planning horizon for screening the advertisement. That is W = 52M (for a planning horizon of one year).

We have the following expression for reach

\[
\frac{1 - f_0 - \sum f_i}{52M - W}
\]

Substituting W = 52M, we have

\[
\text{Reach} = 1 - f_0 - \sum f_i \frac{1}{52M} = 1 - f_0 \quad \cdots \cdots (3.24)
\]

We know that \( \binom{n}{r} = 0 \) if r is an integer and n > r.

where f_0 is the probability of 0 visit.

In the numerical example shown in section 3.2.4

\[
\text{Maximum Reach} = 1 - f_0 = 1 - 0.523 = 0.477.
\]

**Corollary 2:** To show that maximum frequency = \( \sum f_i \)

Maximum frequency can be defined in the same manner as maximum reach. That is if the advertiser screens the advertisement for all the candidate theatres and for all the weeks in the planning horizon, the frequency attainable is the maximum frequency.
Average frequency = \(
\sum_{k=1}^{W} k P[k] = S_1
\)

\[ S_1 = \frac{n}{\sum_{i \geq 1} f_1(i)} \left( \frac{W}{52M} \right) \left( \frac{1}{l} \right) \]

Now \( W = 52M \)

\[ = \frac{n}{\sum_{i=1}^{l} f_1(i)} \left( \frac{W}{52M} \right) \left( \frac{1}{l} \right) \]

\[ = \frac{n}{\sum_{i=1}^{l} i f_1(i)} \quad \ldots \ldots \quad (3.25) \]

In the numerical example discussed in Section 3.2.4 we have,
maximum frequency = \( \sum_{i=1}^{n} i f_1(i) = 0.988 \).

Therefore maximum OTS per person reached = \( \frac{\text{maximum frequency}}{\text{maximum reach}} \)

\[ = \frac{n}{\sum_{i=1}^{l} i f_1(i)} = \frac{1}{1 - f_1} \]

For the numerical example discussed in Section 3.2.4,
max OTS per person reached

\[ = \frac{\text{Maximum frequency}}{\text{Maximum reach}} = \frac{0.988}{0.477} = 2.071. \]
Corollary 3:

When all $52M$ (i.e. total available screening weeks) weeks are taken for screening the advertisement, $P[k]$ - the OTS distribution is same as that of $f_1$ - the distribution of visits to theatres.

We have the following expression for $P[k]$ from the earlier section

$$P[k] = \sum_{i=k}^{n} \frac{f_1}{\binom{52M}{i}} \binom{W}{k} \binom{52M-W}{i-k}$$

This expression can be rewritten as

$$P[k] = \sum_{i=k}^{\infty} \frac{f_1}{\binom{52M}{i}} \binom{W}{k} \binom{52M-W}{i-k} + \sum_{i=k+1}^{n} \frac{f_i}{\binom{52M}{i}} \binom{W}{k} \binom{52M-W}{i-k}$$

The expression (3.26) will be simplified by
putting \( W = 52M \) and using the result

\[
\binom{r}{n} = 0 \text{ if } r \text{ is an integer and } n > r.
\]

\[
P[k] = \frac{f_k}{\binom{52M}{k}} (-1)^{k-k} \binom{k}{52M-k} \binom{52M}{k} + 0
\]

\[
P[k] = f_k.
\]

Therefore, if we take all the theatre weeks for screening the advertisement, the OTS distribution will be the same as that of the cinema going habits of the people.

For the numerical illustration discussed in section 3.2.4, by taking \( W = 52M \) we have

\[
S_1 = \sum_{i \geq 1} \binom{i}{1} f_1 \binom{260}{1} = 0.988
\]

\[
S_2 = \sum_{i \geq 2} \binom{i}{2} f_1 \binom{260}{2} = 0.673
\]

\[
S_3 = \sum_{i \geq 3} \binom{i}{3} f_1 \binom{260}{3} = 0.170
\]
\[
S_4 = \frac{1}{\binom{4}{4}} \binom{260}{4} = 0.008.
\]

\[
P[1] = S_1 - \binom{2}{1} S_2 + \binom{3}{1} S_3 - \binom{4}{1} S_4
\]

\[
= 0.988 - 2(0.673) + 3(0.170) - 4(0.008)
\]

\[
P[1] = 0.12
\]

Similarly

\[
P[2] = 0.211
\]

\[
P[3] = 0.138
\]

\[
P[4] = 0.008
\]

3.5. Sensitivity of Reach and Frequency to Selected Parameters of a Plan:

In many practical applications an advertiser is interested in knowing the sensitivity of reach and frequency of the plan with respect to the number of candidate theatres and the number of screening weeks used in the plan.

\[\text{Let } W_j, \text{ for } j = 1, 2, \ldots, T \text{ denote the number of weeks for which the } j^{th} \text{ theatre is chosen for screening the advertisement.}\]

\[W = \sum_{j=1}^{T} W_j\] denote the number/screening weeks.
The interest in the sensitivity with respect to the candidate theatres arises from the fact that there is no hard and fast rule for deciding the number of candidate theatres that should be used. How much effort should be expended in determining the number of candidate theatres is therefore a matter of practical interest. Should the reach and frequency be very insensitive to the number of candidate theatres used, then of course a great deal of effort need not be spent on determining the number of candidate theatres that should be used. On the other hand, if the reach and frequency do turn out to be sensitive to this parameter then of course it is necessary to exercise a great deal of care in the determination of the number of candidate theatres.

Moreover, sometimes after evaluating a plan an advertiser is interested in knowing the extra screening weeks he may have to take in order to achieve a certain minimum level of reach and frequency. Should it be possible to get this answer quickly, then the advertiser is able to decide whether or not to implement the plan that has been prepared.

For these reasons, we examine the sensitivity of reach and frequency with respect to change in candidate theatres and change in theatre weeks selected for screening the advertisement.
3.5.1. Change in the Candidate Theatres:

Impact on Frequency

In Section 3.3 we have shown that average frequency \( \Sigma k P[k] = S_1. \)

\[
S_1 = \sum_{i=1}^{n} \frac{f_i W}{52M + \Delta}
\]

Let the candidate theatres be increased from \( M \) to \( M + \Delta. \)

Then \( S'_1 = \sum_{i=1}^{n} \frac{f_i W}{52M + \Delta} \) ....... (3.27)

where \( S'_1 \) - denotes the average frequency achieved if the candidate theatres are \( M + \Delta. \)

(3.27) reduces

\[
S'_1 = \sum_{i=1}^{n} \frac{f_i W}{52M + 52 \Delta}
\]

\[
= \sum_{i=1}^{n} \frac{f_i W}{52M (1 + 52 \Delta / 52M)}
\]
\[ S' = \frac{\sum_{i=1}^{n} i f_i \cdot W}{\bar{M}} \left( \frac{\bar{M}}{\bar{M} + \Delta} \right) \]

\[ S' = \frac{\sum_{i=1}^{n} i f_i \cdot W}{\bar{M}} \left( \frac{M}{M + \Delta} \right) \]

Let \( M' = M + \Delta \)

Then \( S'_1 = S_1 \left( \frac{M}{M'} \right) \) \( \cdots \cdots \) (3.28).

From (3.28) we see that the average frequency \( (S'_1) \) is proportional to \( S_1 \), where constant of proportionately is \( (M / M') \).

This can be also verified through the example given in Sec.3.2.4. (See Table 3.1). Hence if the candidate theatres are increased the frequency will be decreased by \( S_1 - S_1 \left( \frac{M}{M'} \right) \) where \( M \) is the original number of candidate theatres and \( M' \) is the number of theatres after increasing the candidate theatres.
\( f_0 = 0.523, \ f_1 = 0.120, \ f_2 = 0.211, \ f_3 = 0.138, \ f_4 = 0.008 \)

\( W = 30 \) weeks

<table>
<thead>
<tr>
<th>Candidate theatre</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0.1140</td>
</tr>
<tr>
<td>6 (20%)</td>
<td>0.0950 (-16.66%)</td>
</tr>
<tr>
<td>7 (40%)</td>
<td>0.0812 (-28.57%)</td>
</tr>
</tbody>
</table>

Notice that frequency

when \( M = 6 \) is \( s_1 (5/6) = 0.1140 \left(\frac{5}{6}\right) = 0.0949 \).

The figures in the brackets represent per cent increase/decrease. We see that 20% increase in the candidate theatres reduces the average frequency by 16.66%.

3.5.2. Impact on Reach by Changing the Candidate Theatres:

From Section 3.4 we have the following expression for reach:

\[
\text{Reach} = 1 - f_0 - \sum_{i=1}^{n} \frac{f_i}{\binom{52M-W}{i}}
\]
\[
\phi(M) = 1 - f_0 - \frac{n}{\sum_{i=1}^{52M} f_i} \binom{52M-W}{i}
\]

Let the candidate theatres be increased to \(M + 1\)

\[
\phi(M+1) = 1 - f_0 - \frac{n}{\sum_{i=1}^{52M+1} f_i} \binom{52M+1-W}{i}
\]

\[
\phi(M+1) - \phi(M) = -\left[ \sum_{i=1}^{n} f_i \left( \frac{52 \ M + 1 - W}{(52 M+1)_{i}} - \frac{(52M - W)_{i}}{(52M)_{i}} \right) \right]
\]

(Using the notation \(\binom{n}{m} = \frac{(n)_m}{m!}\) where

\((n)_m = n(n-1) \ldots (n-m+1)\) ................. \((n-m+1)\)

Let \(c_i = \frac{(52 \ M + 1 - W)_{i}}{(52 M+1)_{i}} - \frac{(52M - W)_{i}}{(52M)_{i}}\)

\[
\phi(M+1) - \phi(M) = -\left[ \sum_{i=1}^{n} f_i (c_i) \right] \quad (3.29)
\]

\[
c_i = \frac{(52 \ M + 1 - W)_{i}}{(52 M+1)_{i}} - \frac{(52M - W)_{i}}{(52M)_{i}}
\]
\[
\begin{align*}
= \left\{ \frac{52 \cdot M + 1 - W}{52 \cdot M + 1} \right\} \left( \frac{52 \cdot M + 1 - W - 1}{52 \cdot M + 1 - 1} \right) \cdots \left( \frac{52 \cdot M + 1 - 1 + 1 - W}{52 \cdot M + 1 - 1 + 1} \right) \\
- \left( \frac{52M - W}{52M} \right) \left( \frac{52M - W - 1}{52M - 1} \right) \cdots \left( \frac{52M - 1 - W + 1}{52M - 1 + 1} \right) \right\}
\end{align*}
\]

\[
= \left\{ \left( 1 - \frac{W}{52 \cdot M + 1} \right) \left( 1 - \frac{W}{52M + 1 - 1} \right) \cdots \left( 1 - \frac{W}{52 \cdot M + 1 - 1 + 1} \right) \\
- \left( 1 - \frac{W}{52M} \right) \left( 1 - \frac{W}{52M - 1} \right) \cdots \left( 1 - \frac{W}{52M - 1 + 1} \right) \right\}
\]

We know \(52(M+1) > 52M; \quad \frac{W}{52 (m+1)} < \frac{W}{52M}\)

\[
1 - \frac{W}{52 \cdot M + 1} > 1 - \frac{W}{52 \cdot M}
\]

using similar logic.

\[
1 - \frac{W}{52 \cdot M + 1 - 1 + 1} > 1 - \frac{W}{52M - 1 + 1}
\]

Therefore

This implies that $c_i$ is positive for all $i = 1, 2, \ldots n$.

The expression (3.29) is negative (since $f_i > 0$). This can be verified from Table 3.2. Therefore any increase in the number of candidate theatres decreases the reach.

Our next interest is to study the behaviour of the second difference namely $\phi(M+2) - 2 \phi(M+1) + \phi(M)$. It so happens that no general conclusion can be drawn regarding the behaviour of the second difference. For the numerical example discussed in Section 3.2.4, we change the number of candidate theatres. The result indicates that the second difference is positive for some value of $M$ and negative for some other value of $M$. Thus, for some value of $M$ the reach decreases at an increasing rate; for some other value of $M$ reach decreases at a decreasing rate. This can be verified from Table 3.2 shown overleaf.
Table 3.2

\[ f_0 = 0.523, \ f_1 = 0.120, \ f_2 = 0.211, \ f_3 = 0.138, \ f_4 = 0.008 \]

\[ W = 30 \]

<table>
<thead>
<tr>
<th>M</th>
<th>Reach = ( \phi(M) )</th>
<th>( \Delta \phi(M) )</th>
<th>( \Delta^2 \phi(M) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5 (-)</td>
<td>0.1056 (-)</td>
<td>-0.0165</td>
<td>0.0045</td>
</tr>
<tr>
<td>6 (20%)</td>
<td>0.08910 (-15.6%)</td>
<td>-0.0120</td>
<td>-0.02814</td>
</tr>
<tr>
<td>7 (+0%)</td>
<td>0.07710 (-26.99%)</td>
<td>-0.0141</td>
<td>-0.03014</td>
</tr>
<tr>
<td>8 (60%)</td>
<td>0.036960 (-65.0%)</td>
<td>-0.0165</td>
<td>-0.0328</td>
</tr>
</tbody>
</table>

Note that \( \Delta \phi(M) = \phi(M+1) - \phi(M) \)

\[ \Delta^2 \phi(M) = [\phi(M+2) - \phi(M+1)] - [\phi(M+1) - \phi(M)] \]

The figures in the brackets denote the percentage increase. 20% increase in the number of candidate theatres decreases reach by 15.6%.

This numerical example illustrates the sensitivity of reach and frequency with respect to the number of candidate theatres. An example in the real life setting is presented in Chapter 6. This example also shows that the reach and frequency are sensitive to the number of candidate theatres used in the planning horizon.
3.5.3. Impact of Increase in the Screening Weeks on Average Frequency

Let \( W \) be increased by \( W + \Delta \)

\[
S_1' = \sum_{i \geq 1} \frac{\binom{i}{1}}{\binom{52M}{1}} f_i \left( W + \Delta \right)
\]

where \( S_1' \) is the frequency achieved, when there are \( W + \Delta \) weeks for screening the advertisement.

Expression (3.30) becomes

\[
= \sum_{i \geq 1} \frac{\binom{i}{1}}{\binom{52M}{1}} f_i (W + \Delta)
\]

\[
= \sum_{i \geq 1} \frac{\binom{i}{1}}{\binom{52M}{1}} f_i W(1 + \frac{\Delta}{W})
\]

\[
= \sum_{i \geq 1} \frac{\binom{i}{1}}{\binom{52M}{1}} f_i \left( \frac{W}{W + \Delta} \right)
\]
We note that is proportional to $S_1^\alpha$, constant of proportionality being $(w'/W)$.

Thus by increasing the screening weeks from $W$ to $W+\Delta$, frequency is increased by $S_1^\alpha - S_1$. This can be verified from the Table 3.3 below:

\[
\begin{align*}
S_1' &= S_1 \left( \frac{W + \Delta}{W} \right) \\
\end{align*}
\]

Let $W' = W + \Delta$

\[
\begin{align*}
S_1' &= S_1 \left( \frac{W'}{W} \right) \quad \ldots \ldots \quad (3.31)
\end{align*}
\]

We note that $S_1'$ is proportional to $S_1$, constant of proportionality being $(W'/W)$.

Thus by increasing the screening weeks from $W$ to $W+\Delta$, frequency is increased by $S_1(\frac{W'}{W}) - S_1$. This can be verified from the Table 3.3 below:

**Table 3.3**

<table>
<thead>
<tr>
<th>Theatre week combination</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>0.1140</td>
</tr>
<tr>
<td>40 (33.3%)</td>
<td>0.1520 (33.3%)</td>
</tr>
<tr>
<td>50 (66.6%)</td>
<td>0.1900 (66.6%)</td>
</tr>
</tbody>
</table>

Notice that when $W = 40$

\[
S_1 = \left( \frac{40}{30} \right) (0.1140) = 0.1520.
\]

Figures in the brackets denote the increase in the average frequency. We see that the average frequency is increased by 33% for an increase of 33% in the number of screening weeks.
3.5.4. **Impact of Increase in Screening Weeks on Reach:**

After examining the impact on average frequency, we will now examine the impact on reach by increasing the screening weeks.

From Section 3.4, we have the following expression for reach:

\[
\text{Reach} = 1 - f_0 - \sum_{i=1}^{n} f_i \left( \frac{52M-W}{i} \right)
\]

\[
\text{Reach} = \phi(W) = 1 - f_0 - \sum_{i=1}^{n} f_i \left( \frac{52M}{i} \right)
\]

\[
\phi(W+1) = 1 - f_0 - \sum_{i=1}^{n} f_i \left( \frac{52M - W - 1}{i} \right)
\]

\[
\phi(W) = 1 - f_0 - \sum_{i=1}^{n} f_i \left( \frac{52M-W}{i} \right)
\]
\[ \phi(W+1) - \phi(W) = \sum_{i=1}^{n} f_i \binom{52M-W}{i} - \sum_{i=1}^{n} f_i \binom{52M-W-1}{i} \]

\[ = \sum_{i=1}^{n} f_i \left[ \binom{52M-W}{i} - \binom{52M-W-1}{i} \right] \]

\[ = \sum_{i=1}^{n} f_i \binom{52M-W-1}{i-1} \quad \cdots \quad (3.32) \]

Since \( 52M-W-1 > 0 \)

\( 52M \) is larger than \( W \) or \( n \)

Therefore, expression (3.32) is positive for all values of \( M, W \) and \( i \).

Let us consider the second difference

\[ \phi(W+2) - \phi(W+1) = \]

\[ \sum_{i=1}^{n} f_i \binom{52M-W-2}{i} = \frac{f_1}{52M} + \sum_{i=2}^{n} f_i \binom{52M-W-2}{i-1} \]

\[ \phi(W+1) - \phi(W) = \sum_{i=1}^{n} f_i \binom{52M-W-1}{i-1} \]
\[
\phi(W+2) - \phi(W+1) - [\phi(W+1) - \phi(W)]
\]

\[
= \frac{f_1}{(52M)} + \sum_{i=2}^{n} f_i \left(\frac{52M - W - 1}{i - 1}\right)
\]

\[
= \sum_{i=2}^{n} f_i \left\{\frac{(52M - W - 2)}{i - 1} - \frac{(52M - W - 1)}{i - 1}\right\}
\]

\[
= -\sum_{i=2}^{n} f_i \frac{(52M - W - 2)}{i - 1} < 0. \quad \ldots (3.33)
\]

We notice that the second difference is negative. Therefore reach is increasing at a decreasing rate. This can be verified through the example discussed in 3.2.4 shown in Table 3.4. Maximum reach is attained if \( W \) is equal to 52M as we have seen in the earlier section.
Table 3.4

<table>
<thead>
<tr>
<th>Screening weeks W</th>
<th>Reach= $\phi(W)$</th>
<th>$\Delta \phi(W)$</th>
<th>$\Delta^2 \phi(W)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>30 (-)</td>
<td>0.1056</td>
<td>0.0314</td>
<td></td>
</tr>
<tr>
<td>40 (33.3%)</td>
<td>0.1370 (29.73%)</td>
<td>0.0296</td>
<td>-0.0018</td>
</tr>
<tr>
<td>50 (66.6%)</td>
<td>0.1666 (57.77%)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The figures in the brackets denote the percentage increase. A 33.3% increase in screening weeks results in an increase of 29.73% in reach. We note that the first difference is positive and the second difference is negative.

Thus we see that the reach and the average frequency are sensitive to the total screening weeks. Therefore, the decision regarding the total number of screening weeks has to be taken very carefully. Moreover, the cost of screening weeks is also high. Therefore, we will develop the methods to find the total number of screening weeks in Chapter 5.
3.5.5. Impact of Increase in the Number of Screening Weeks on the Right Cumulative OTS Distribution:

In the evaluation of cinema plan, reach and frequency are not only the summary measures of interest. In many situations the advertiser may also be interested in knowing the proportion of people in the target audience that will get at least \( k_0 \) OTS from the cinema plan. This interest stems from the fact that for an advertisement to be effective a person must be exposed to it at least a specified number of times. We are, therefore, interested in studying the impact of increasing the number of screening weeks on the right cumulative OTS distribution.

Mathematically the problem can be stated as follows:

Let \( G(W) = \sum_{k_0}^{n} P[k](W) \)

We are interested in studying the behaviour of \( G(W) \). To study this let us define

\[
\Delta G(W) = G(W+1) - G(W)
\]
\[
\Delta G(W) = \sum_{k_0}^{n} P[k](W+1) - \sum_{k_0}^{n} P[k](W)
\]

\( P[k](W) \) — denotes the probability of the \( k \)th exposure when there are \( W \) screening weeks.
\[ 10^+ \]

\[ = \sum_{k_0}^{n} \left[ P_{[k]}(W+1) - P_{[k]}(W) \right] \]

We will, therefore, obtain an expression for

\[ P_{[k]}(W+1) - P_{[k]}(W) = \Delta P_{[k]}(W) \]

\[ P_{[k]}(W) = \sum_{i=k}^{n} f_i \left( \begin{array}{c} W \\ k \end{array} \right) \left( \begin{array}{c} 52M - W \\ i - k \end{array} \right) \]

\[ = \left( \begin{array}{c} W \\ k \end{array} \right) \sum_{i=k}^{n} f_i \left( \begin{array}{c} 52M \\ i \end{array} \right) \left( \begin{array}{c} 52M - W \\ i - k \end{array} \right) \]

Let \( U(W) = \left( \begin{array}{c} W \\ k \end{array} \right) \) and \( V(W) = \sum_{i=k}^{n} \left( \begin{array}{c} 52M - W \\ i - k \end{array} \right) \frac{f_i}{52M} \)

Thus

\[ \Delta P_{[k]}(W) = U(W+1) V(W+1) - U(W) V(W) \]

\[ = U(W+1) V(W+1) - U(W) V(W+1) \]

\[ + U(W) V(W+1) - U(W) V(W) \]

\[ = V(W+1) [U(W+1) - U(W)] + U(W) [V(W+1) - V(W)] \]

Now \( U(W+1) - U(W) = \left( \begin{array}{c} W+1 \\ k \end{array} \right) - \left( \begin{array}{c} W \\ k \end{array} \right) = \left( \begin{array}{c} W \\ k-1 \end{array} \right) \)
\[ V(W+1) - V(W) = \sum_{i=k}^{n} \frac{f_i}{\binom{52M}{i}} \left[ \binom{52M-W-1}{i-k} - \binom{52M-W}{i-k} \right] \]

\[ = \sum_{i=k}^{n} \frac{f_i}{\binom{52M}{i}} \left[ -\binom{52M-W-1}{i-k-1} \right] \]

Hence

\[ \Delta P[k](W) = \binom{W}{k-1} \sum_{i=1}^{n} \frac{f_i}{\binom{52M}{i}} \left( \binom{52M-W-1}{i-k} - \binom{52M-W}{i-k} \right) \]

Now

\[ P[k](W+1) = \binom{W+1}{k} \sum_{i=k}^{n} \frac{f_i}{\binom{52M}{i}} \left( \binom{52M-W-1}{i-k} \right) \]

and

\[ P[k+1](W+1) = \binom{W+1}{k+1} \sum_{i=k+1}^{n} \frac{f_i}{\binom{52M}{i}} \left( \binom{52M-W-1}{i-k-1} \right) \]

Hence \[ \Delta P_k(W) = \frac{\binom{W}{k-1}}{\binom{W+1}{k}} P[k](W+1) - \frac{\binom{W}{k}}{\binom{W+1}{k+1}} P[k+1](W+1) \]
\[
\binom{W}{k} \binom{W+1}{k-1} = \frac{W!}{(k-1)! (W-k+1)!} \times \frac{k! (W+1-k)!}{(W+1)!} = \frac{k}{W+1}
\]

\[
\binom{W}{k} \binom{W+1}{k+1} = \frac{k+1}{W+1}
\]

Hence \( \Delta P[k] (W) = \frac{1}{W+1} \left[ k \ P[k] (W+1) - (k+1) \ P[k+1] (W+1) \right] \)

\[\cdots \cdots (3.34)\]

Hence

\[
\sum_{k=k_0}^{n} \Delta P[k] (W) = \frac{1}{W+1} \left[ k_0 \ P[k_0] (W+1) - n+1 \ P[n+1] (W+1) \right]
\]

We know that \( P[n+1] = 0 \).

\[
= \frac{1}{W+1} k_0 \ P[k_0] (W+1) \quad \cdots \cdots (3.35)
\]

Since \( k_0 > 0 \), \( P[k_0] \geq 0 \), \( (W+1) > 0 \),

\[
\sum_{k=k_0}^{n} \Delta P[k] (W) \geq 0, \text{ for all values of } W.
\]

Therefore \( G(W) \) is a monotonic non-decreasing function of \( W \).

Note that \( \sum_{k=k_0}^{n} P[k] (W) \) will be zero if \( \sum_{k=k_0}^{n} f_1 = 0 \).

Therefore if \( \sum_{k=k_0}^{n} f_1 \neq 0 \), \( g(W) \) is a monotone increasing function of \( W \).
Corollary 4:

Impact of Increase in Screening Weeks on $P_k(W)$, the Probability of a Person Receiving Exactly $k$ OTS and Its Behaviour with Respect to the Total Screening Weeks.

By increasing the screening weeks the probability of exactly $k$ OTS either increases or decreases.

It can be seen from (3.34) that

$$\Delta P_k[W] = \frac{1}{W+1} \left[ k \cdot P_k(W+1) - (k+1) \cdot P_{k+1}(W+1) \right].$$

The quantity $k \cdot P_k(W+1) - (k+1) \cdot P_{k+1}(W+1)$ may be either positive or negative and hence the first difference may be either positive or negative. Consequently no generalization is possible regarding the behaviour of $P_k(W)$ with respect to $W$.

For the numerical illustration discussed in Section 3.2.4, we check the value of expression (3.36)

<table>
<thead>
<tr>
<th>$i$</th>
<th>$f_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.523</td>
</tr>
<tr>
<td>1</td>
<td>0.120</td>
</tr>
<tr>
<td>2</td>
<td>0.211</td>
</tr>
<tr>
<td>3</td>
<td>0.138</td>
</tr>
<tr>
<td>4</td>
<td>0.008</td>
</tr>
<tr>
<td></td>
<td>1.000</td>
</tr>
</tbody>
</table>
Reach = .1056

<table>
<thead>
<tr>
<th>OTS</th>
<th>Distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.09736</td>
</tr>
<tr>
<td>2</td>
<td>0.007981</td>
</tr>
<tr>
<td>3</td>
<td>0.000246</td>
</tr>
<tr>
<td>4</td>
<td>0.000001</td>
</tr>
</tbody>
</table>

Taking $W = 31$

Reach =

<table>
<thead>
<tr>
<th>OTS</th>
<th>Distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.099997</td>
</tr>
<tr>
<td>2</td>
<td>0.0085108</td>
</tr>
<tr>
<td>3</td>
<td>0.0002586</td>
</tr>
<tr>
<td>4</td>
<td>0.0000013</td>
</tr>
</tbody>
</table>

Let $k = 2$

The expression (3.36) becomes

$$2 P[2](W+1) - 3 P[3](W+1) = 2(0.0085108) - 3(0.0002586)$$

$$= 0.0170216 - 0.0007758 > 0.$$  

**Corollary 5:**

To show that the reach is a strictly monotonic function of $W$: This result is proved in Section 3.5.4. Reach can be also considered as a special case of right cumulative OTS distribution.

We know that $Reach = \sum_{k=1}^{n} P[k]$
This expression is obtained by putting \( k_o = 1 \) in \( \sum_{k=k_o}^{n} P[k] \).

Therefore from 3.35 we have the following expression:

\[
\sum_{k=k_o}^{n} \Delta P[k](W) = \frac{1}{W+1} k_o P[k_o](W+1)
\]

Substituting \( k_o = 1 \)

\[
\sum_{k=1}^{n} \Delta P[k](W) = \frac{1}{W+1} P[1](W+1)
\]

\[
= \frac{1}{W+1} \sum_{i=1}^{n} f_i \frac{(W+1)}{1} \frac{52M-W-1}{i-1} \frac{52M}{i} \quad \cdots (3.37)
\]

\[
= \frac{W+1}{W+1} \sum_{i=1}^{n} f_i \frac{(i-1)}{52M} \frac{52M-W-1}{i}
\]

\[
= \sum_{i=1}^{n} f_i \frac{(52M-W-1)}{52M} > 0.
\]

This shows that \( \sum_{k=1}^{n} \Delta P[k](W) > 0 \). Hence reach is strictly a monotonic function of \( W \).
3.6. **Model 2:**

In Section 3.1 we have already stated that the approach to the development of this model is the same as that used in the development of Model 1 except for the assumptions.

Model 2 is based on the assumption that people select movies but not theatres. It was, therefore, believed that the cinema-going habits should be expressed in terms of the choice that the cinema-goers make with regard to the movies. The model 2 attempts to translate the cinema-going habits in terms of the choice people make in selecting movies.

3.6.1. **Assumptions:**

The basic assumptions in cinema model 2 are as follows:

1. A person sees any particular movie at most once.
2. The distinct movies are chosen independently in the sense that whichever may be the choice of the movies by a person in the target audience in the first \( r \) visits, each of the remaining movies has the same probability of being selected at \( (r + 1) \) th visit.
3. All movies have equal chances of being screened in the candidate theatre.

4. The average life of a movie is specified.

3.6.2. Approach of Model 2:

The approach discussed in Section 3.1 is valid for model 2 also. In this model the number of screening weeks is interpreted in a different fashion.

Suppose we have M candidate theatres as discussed earlier.

T theatres are chosen for screening the advertisement.

\[ W_1, W_2, \ldots, W_T \] be the screening weeks chosen with T theatres.

\[ W = \sum_{i=1}^{T} W_i \]

Suppose the life of a movie is assumed to be \( M_0 \) weeks (i.e., the movie runs for \( M_0 \) weeks in a particular theatre).

\[ W_j \] screening weeks in \( j^{th} \) theatre will be interpreted in conjunction with the movies as follows:

Screen \( M_0 \) weeks with movie 1

\( M_0 \) weeks with movie 2
\[ W_j = \left\lfloor \frac{W_j - 1}{M_0} \right\rfloor \times M_0 \text{ weeks with the } [W_j / M_0] + 1 \text{ th movie.} \]

For example if an advertisement is screened for 26 weeks in a theatre and moreover if we assume that the life of a movie is 8 weeks, then the screening plan is 8 weeks with movie 1, 8 weeks with movie 2, 8 weeks with movie 3, and 2 weeks with movie 4.

\[ N = \left\lfloor \frac{52 \times M - 1}{M_0} \right\rfloor + 1 \quad \ldots \ldots \quad (3.38) \]

where \( N \) denotes the potential number of distinct movies available for a person in the target audience to make the selection of a movie.

Our next task is to develop an expression for \( S_j \).

Let \( A \) - be the event that he sees a specific movie.

\( B \) - be the event that the movie runs for \( M_0 \) weeks.

\( C \) - be the event that the person belongs to \( i \) th crowd, meaning he sees \( i \) movies in a year.

Next we seek, the probability that the person sees specific movie = \( P\left\{ \bigcup_{B, C} (A \cap B \cap C) \right\} \)

\[ \left\lfloor \frac{W_j - 1}{M_0} \right\rfloor \] denotes the largest integer contained in the fraction \( W_j / M_0 \). This subtraction of \( .1 \) is made to adjust, when numerator is an exact multiple of denominator.
\[ \sum_{B,C} P(A \cap B \cap C) \]
\[ = \sum_{B,C} P(A | B \cap C) P(B \cap C) \]
\[ = \sum_{B,C} P(A | B \cap C) P(B).P(C) \ldots \ldots (3.39) \]

Since we can assume that B and C are independent.

The probability that a person sees a specified movie given that he sees i movie is

\[ \binom{N - 1}{i - 1} \binom{N}{i} \] (Using the logic discussed in 3.3)

Therefore the probability that he sees the advertisement along with a specified movie in a week

\[ = \text{Prob\{that he sees the movie\} \ Prob\{sees the advertisement\}} \]
\[ \text{Prob\{sees the advertisement\}} \]

\[ \text{movie runs for } M_0 \text{ weeks} \]
\[ = \sum_{i \geq 1} \binom{N - 1}{i - 1} \binom{N}{i} f_1 \left( \frac{1}{M_0} \right) \ldots \ldots (3.40) \]

'Suppose we screen the advertisement for \( X_1 \) weeks with
the first movie, \( X_2 \) weeks with the second movie and \( X_3 \) weeks with the \( i \)th movie, the sum of the probabilities that he sees the advertisement with one specified movie \((S_1)\).

\[
S_1 = \sum_{i=1}^{N-1} \binom{i-1}{N-1} \binom{1}{N} f_1 \left( \frac{1}{M_0} \right) \sum_{j=1}^{i-1} X_j = \sum_{i=1}^{N} \binom{1}{N} f_1 \left( \frac{\sum X_i}{M_0} \right)
\]

\[\ldots \quad (3.41)\]

where \( i = \left[ \frac{W_1 - 1}{M_0} \right] + 1 + \left[ \frac{W_2 - 1}{M_0} \right] + 1 + \ldots + \left[ \frac{W_T - 1}{M_0} \right] + 1 \]

\[= \left[ \frac{W_1 - 1}{M_0} \right] + \left[ \frac{W_2 - 1}{M_0} \right] + \ldots + \left[ \frac{W_T - 1}{M_0} \right] + T \quad (3.42)\]

Therefore the probability that a person in the target audience sees the advertisement in \( j \) specified movies

\[
\rho_{i_1, i_2, \ldots, i_j} = \sum_{i_1}^{N} \binom{i_1}{j} f_1 \left( \frac{X_1, X_2, \ldots, X_j}{M_0 j} \right)
\]
and we also know that

\[ S_j = \sum p_i, i_2, \ldots, i_j \]

\[ = \sum_{i \geq j} \binom{i}{j} f_{i-1} \left[ \frac{\sum x_i x_2 \ldots x_j}{W_0} \right] \]

\[ \ldots \ldots \ldots \ldots \text{(3.43)} \]

where we know \( \lambda = \text{the total number of distinct movies chosen for screening the advertisement in } W \text{ screening weeks.} \)

\[ \lambda = \left[ \frac{W_1-1}{M_0} \right] + 1 + \left[ \frac{W_2-1}{M_0} \right] + 1 + \ldots + \left[ \frac{W_T-1}{M_0} \right] + 1. \]

In (3.43) we have obtained the expression for \( S_j \). Now substituting the values of \( S_j \) in the expression of \( P[k] \) we get

\[ P[k] = \sum_{j=k} \binom{j}{k} (-1)^{j-k} \sum_{i \geq j} \binom{i}{j} f_{i-1} \left[ \frac{\sum x_1 x_2 \ldots x_j}{M_0^j} \right] \]

\[ \ldots \ldots \ldots \ldots \text{(3.44)} \]

By the earlier argument, discussed in Section 3.3
Reach = \frac{1}{2} \sum_{k=1}^{5} P[k]

3.6.3. Explanation of the Mathematics Through a Hypothetical Example

We will consider the same problem as discussed in Section 3.2.4, to show the computations involved in using model 2.

<table>
<thead>
<tr>
<th>i</th>
<th>f_i</th>
<th>Candidate theatres M = 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.523</td>
<td>Theatres chosen T = 2</td>
</tr>
<tr>
<td>1</td>
<td>0.120</td>
<td>Total theatre week combination</td>
</tr>
<tr>
<td>2</td>
<td>0.211</td>
<td>W_1 = 15, W_2 = 15</td>
</tr>
<tr>
<td>3</td>
<td>0.138</td>
<td>W_1 + W_2 = 15 + 15 = 30.</td>
</tr>
<tr>
<td>4</td>
<td>0.008</td>
<td></td>
</tr>
<tr>
<td>l.000</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Average life of a movie = 8 weeks

The total number of distinct movies available in the candidate theatres

N = \left[ \frac{5 \times 52 - 1}{8} \right] + 1 = 33 distinct movies.

W_1 = 15 weeks, \ X_1 = 8, \ X_2 = 7. That is the advertisement is screened for 8 weeks with the first movie and 7 weeks with the second movie in the first theatre chosen.

W_2 = 15 weeks, \ X_3 = 8, \ X_4 = 7
That is the advertisement is screened with the first movie for 8 weeks and the subsequent movie for 7 weeks in the second theatre. Thus the advertisement will be screened with 4 distinct movies in two theatres.

\[ S_1 = \sum_{i=1}^{4} \frac{1}{N} \cdot f_i \left( \frac{X_j}{M_0} \right) \]

\[ = \left( \frac{30}{33} \right) \frac{1}{8} \left\{ f_1 + 2f_2 + 3f_3 + 4f_4 \right\} \]

\[ = \left( \frac{30}{33} \right) \frac{1}{8} \times (0.988) \]

\[ S_1 = 0.1227273 \]

\[ S_2 = \sum_{i \geq 2} \frac{1}{N} \cdot f_i \left( \frac{X_j X_k}{M_0^2} \right) \]

\[ = \frac{f_2 + 3f_3 + 6f_4}{33} \cdot \frac{[(8) (7) + (8) (8) + (8) (7) + (7) (8) + (7) (7) + (8) (7)]}{8^2} \]

\[ = \left( \frac{-337}{64} \right) \left( \frac{1}{528} \right) \times 0.673 \]
\[
S_3 = \frac{1}{\binom{33}{3}} \cdot \left\{ f_3 + 4f_4 \right\} \cdot \frac{(8.7.8 + 8.7.7 + 7.8.7 + 8.8.7)}{8^3}
\]

\[
= \frac{1}{5446} \cdot (0.17) \left( \frac{1680}{512} \right)
\]

\[
= 0.00010224
\]

\[
S_4 = \frac{1}{\binom{33}{4}} \cdot (0.008) \left( \frac{8.8.7.7}{8^4} \right)
\]

\[
= \frac{1}{40920} \cdot (0.008) \left( \frac{3136}{6096} \right)
\]

\[
= 0.00000015
\]

\[
P[1] = S_1 - \left( \begin{array}{c} 2 \\ 1 \end{array} \right) S_2 + \left( \begin{array}{c} 3 \\ 1 \end{array} \right) S_3 - \left( \begin{array}{c} 4 \\ 1 \end{array} \right) S_4
\]

\[
= 0.11227273 - 2(0.00671168) + 3(0.00010224) - 4(0.00000015)
\]

\[
= 0.09915549
\]

\[
P[2] = S_2 - 3S_3 + 6S_4
\]

\[
= (0.00671168) - 3(0.00010224) + 6(0.00000015)
\]

\[
= 0.00640586
\]
\[ P[3] = S_3 - \varepsilon_4 \]
\[ = 0.00010224 - 4(0.00000015) \]
\[ = 0.00010164 \]

\[ P[4] = 0.00000015 \]


Average Frequency = \[ \sum k P[k] = 0.11227273 \]

OTS distribution

<table>
<thead>
<tr>
<th>k</th>
<th>P[k]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.09915549</td>
</tr>
<tr>
<td>2</td>
<td>0.00640586</td>
</tr>
<tr>
<td>3</td>
<td>0.00010164</td>
</tr>
<tr>
<td>4</td>
<td>0.00000015</td>
</tr>
</tbody>
</table>

3.7. Comparison of Model 1 and Model 2:

In this section we will illustrate the results obtained by the two models for the hypothetical numerical problem illustrated in Section 3.2.4.

\[ f_0 = 0.523 \]
\[ f_1 = 0.120 \]
\[ f_2 = 0.211 \]
\[ f_3 = 0.138 \]
\[ f_4 = 0.008 \]
Average frequency in model 1 is always higher than or equal to that of model 2. This can be verified mathematically.

We know that $S_1$ is the average frequency. Therefore

$$S_{\text{Model 1}} = \frac{\sum i f_i W}{52M}$$

$$S_{\text{Model 2}} = \frac{\sum i f_i (X_1 + X_2 + \ldots + X_n)}{N M_0}$$

where $\ell = \left[ \frac{w_1 - 1}{M_0} \right] + \left[ \frac{w_2 - 1}{M_0} \right] + \ldots \left[ \frac{w_n - 1}{M_0} \right] + T$

$$N = \left\{ \left[ \frac{52xM - 1}{M_0} \right] + 1 \right\} M_0$$
We know $W = X_1 + X_2 + \ldots + X_k$.

by the definition of $X_i$'s.

Therefore, numerator of $S_{\text{Model 1}}$ and $S_{\text{Model 2}}$ is the same.

Next we shall show that denominator of $S_{\text{Model 2}}$ is higher than the $S_{\text{Model 1}}$.

$$52M = \left( \frac{52 \times M}{M_0} \right) \frac{M_0}{1} \leq \left\lfloor \frac{52 \times M - 1}{M_0} \right\rfloor + 1 M_0$$

because $$\frac{52 \times M}{M_0} \leq \left\lfloor \frac{52 \times M - 1}{M_0} \right\rfloor + 1$$

Hence average frequency of model 2 is always smaller than the average frequency of model 1. No generalized statements can be made regarding the reach. In the above illustration the reach obtained by using model 2 is higher than that obtained by using model 1. There might be some situations in which the reach obtained by using model 1 will be higher than the reach obtained by using model 2.

Similarly no conclusive statements can be made regarding the OTS per person reached and the OTS distribution obtained by using model 1 and model 2.