The gauge/gravity duality allows us to gain insights into various properties of strongly coupled gauge theories both at zero and non-zero temperature. In particular, the transport coefficients of strongly coupled gauge theories, which are hard to compute otherwise, can now be computed using gauge/gravity duality. Furthermore, for many cases, in the low frequency limit, at the level of linear response, the horizon geometry of the gravity dual determines the behavior of the gauge theory. This can, in particular, be used to show that the shear viscosity to entropy density ratio for strongly coupled gauge theories at finite temperature with a gravity dual is universal and takes value $\frac{1}{4\pi}$. One can further show that, the electrical conductivity of the gauge theory at finite temperature but zero chemical potential can be determined in terms of geometrical quantities evaluated at the horizon. This is so because the response function in the low frequency limit evolves in a very simple manner as we go away from the horizon along the radial direction. However, the introduction of a chemical potential primarily brings in several non-trivialities in the evolution of response function from the horizon to the boundary. Although the shear viscosity can still be computed solely in terms of horizon data, for the computation of electrical conductivity, horizon data is not enough. Nevertheless, our analysis reveals that if the stress-energy tensor related to the matter content of the bulk satisfies a compact relation among its space and time components, the boundary conductivity at low frequencies is universal and can be written in terms of geometrical quantities evaluated at the horizon and thermodynamic quantities. In this thesis, we also have shown that at any radial position outside the horizon, the conductivity is given by a simple expression which interpolates smoothly be-
between the one computed at the horizon and at the boundary. We also computed
the electrical conductivity in the presence of more than one chemical potentials
for several models. What we observe is that, in the presence of multiple chemical
potentials, there is a nontrivial mixing between current operators which, from the
bulk point of view, can be understood to be arising because of the interactions
through graviton. We have also shown that one can write a general expression for
conductivity matrix in the presence of multiple chemical potentials provided dual
gravity background satisfies some constraints. By using the relation with electrical
conductivity, we have also computed the thermal conductivity and observed that
thermal conductivity to shear viscosity ratio ($\frac{\kappa_T}{\eta} \sum_{n=1}^{\infty} \nu_n^2$) is independent of the num-
ber of chemical potentials turned on. This ratio remains same even in the limit of
zero chemical potential. We also discussed, how for CFT’s with gravity dual, this
ratio can be expressed in terms of central charges of the CFT. Using these results,
we could express the electrical conductivity solely in terms of the thermodynamic
quantities of the gage theory. We then turn our attention to study of transport
coefficients of gauge theories at zero temperature which corresponds to extremal
black hole in the bulk. We have shown that electrical conductivity goes as $\omega^2$.
We have also seen that analytic expression for shear viscosity and the viscosity to
entropy ratio remain same as that of many non-extremal black holes where near
horizon geometry is radically different.

We hope that our explorations regarding the universalities of various trans-
port coefficients will be useful in understanding generic behaviour of the strongly
coupled quantum field theories at zero and non-zero temperature.
Membrane paradigm

To an external observer, a black hole appears as dynamical fluid membrane sitting at the horizon, with mechanical and electrical properties. They also show dissipation and one can compute quantities such as conductivity, shear viscosity. In the following we shall give a brief introduction to membrane paradigm in the spirit of [4, 96]. See [5] and [97, 98, 99, 100, 101, 102] for discussion on same topic. Classically an outside observer does not see inside the horizon. Effectively, for an external observer one can write

\[ S_{\text{eff}} = S_{\text{out}} + S_{\text{surf}}, \quad (A.1) \]

where \( S_{\text{out}} \) is the part of action defined out side the horizon where as \( S_{\text{surf}} \) represents effectively the effect of black hole to external universe. \( S_{\text{surf}} \) is a boundary term to the horizon, and can be determined by demanding \( S_{\text{eff}} \) to be stationary with respect to solution to the equation of motion. Rather than putting the membrane exactly at the horizon, one can put it slightly away and thus avoiding complexity that arises due to null hypersurface. In the following we shall discuss briefly electrical and mechanical properties of the membrane.

### A.1 Electrical properties of the membrane

Let us consider the metric of the form

\[ ds^2 = g_{tt}(r)dt^2 + g_{rr}(r)dr^2 + g_{xx}(r) \sum_{i=1}^{d-1} (dx^i)^2, \quad (A.2) \]
where $r$ is the radial coordinate. We have assumed full rotational symmetry in $x^i$ directions so that $g_{ij} = g_{xx}\delta_{ij}$, where $i, j$ run over all the indices except $r, t$. We also assume that metric components depend on radial coordinate only. We shall work with the metric which has an event horizon, where $g_{tt}$ has a first order zero and $g_{rr}$ has a first order pole. We also assume that all the other metric components are finite as well as non vanishing at the horizon. Consider a bulk $U(1)$ gauge field for which the action is of the form

$$S_{\text{Out}} = - \int_{r > r_h} d^{d+1}x \sqrt{-g} \frac{1}{4g_{d+1}^2(r)} F_{MN} F^{MN}. \quad (A.3)$$

Now varying this action we get,

$$\delta S_{\text{out}} = -2 \int_{r > r_h} d^{d+1}x \sqrt{-g} \frac{1}{4g_{d+1}^2(r)} \delta F_{MN} F^{MN}$$

$$= -4 \int_{r > r_h} d^{d+1}x \sqrt{-g} \nabla_M (\frac{1}{4g_{d+1}^2(r)} \delta A_N F^{MN})$$

$$+ 4 \int_{r > r_h} d^{d+1}x \sqrt{-g} \delta A_N \nabla_M (\frac{1}{4g_{d+1}^2(r)} F^{MN}). \quad (A.4)$$

Using Maxwell equation

$$\nabla_M (\frac{1}{4g_{d+1}^2(r)} F^{MN}) = 0, \quad (A.5)$$

and the fact that for any vector $V^A$, we have

$$\nabla_M V^M = \frac{1}{\sqrt{-g}} \partial_A (\sqrt{-g} V^A), \quad (A.6)$$

we get

$$\delta S_{\text{out}} = - \int d^{d}x \sqrt{-g} \frac{1}{g_{d+1}} \delta A_M F^{rM} \bigg|_{r \to \infty}^{r = r_h}. \quad (A.7)$$
Appendix A. Membrane paradigm

Using the fact that at the boundary $\delta A_B = 0$, and staying slightly away from the horizon we get,

$$\delta S_{\text{out}} = - \int d^dx \sqrt{-h} \left( \frac{\sqrt{-g}}{\sqrt{-h} g^2_{d+1}} \delta A \mathcal{F}^{\mathcal{M}} \right)_{r = r_h + \epsilon}$$

$$= - \int d^dx \sqrt{-h} \delta A \mathcal{F}^{\mathcal{M}}_{\text{membrane}}(x). \quad (A.8)$$

where $h_{\mu\nu}$ is the induced metric at the stretched horizon and

$$J^B_{\text{membrane}} = \left. \frac{\sqrt{g_{rr}}}{g^2_{d+1}} F^{\mathcal{M}} \right|_{r = r_h + \epsilon}. \quad (A.9)$$

In order to have a well defined variational principle, we need to cancel the boundary term. For that purpose we add $S_{\text{Surf}}$ such that

$$\delta S_{\text{Surf}} = - \delta S_{\text{Out}}; \quad (A.10)$$

One can write $S_{\text{Surf}}$, as

$$S_{\text{Surf}} = \int d^dx \sqrt{-h} \delta A \mathcal{F}^{\mathcal{M}}_{\text{membrane}}(x). \quad (A.11)$$

Let us note that Maxwell equation can be written as

$$\nabla_M \left( \sqrt{-g} \frac{1}{g^2_{d+1}} F^{\mathcal{M}} \right) = 0$$

$$\Rightarrow \nabla_M J^{\mathcal{M}}_{\text{membrane}} = 0, \quad (A.12)$$

where $J^{\mathcal{M}}_{\text{membrane}}$ can be interpreted as the membrane current. Total integral of $J^{\mathcal{M}}_{\text{membrane}}$ over the horizon will give charge of the black hole. The spatial component of the membrane current is given by

$$J^i_{\text{membrane}} = \left. \frac{\sqrt{g_{rr}}}{g^2_{d+1}} F^{\mathcal{M}}_{ir} \right|_{r = r_h + \epsilon}. \quad (A.13)$$

In order to proceed further, let us choose the gauge $A_r = 0$. Since horizon is a regular place for an in falling observer, the $A_i$ should be regular at the horizon. This implies that, gauge field should only depend on a non singular combination
Appendix A. Membrane paradigm

\[ dv = dt + \sqrt{\frac{g_{rr}}{-g_{tt}}} dr. \]  \hspace{1cm} \text{(A.14)}

This gives,

\[ (\partial_r - \sqrt{\frac{g_{rr}}{g_{tt}}}) A_i = 0 \]

\[ \Rightarrow F_{ri} = \sqrt{\frac{g_{rr}}{g_{tt}}} F_{ti}. \]  \hspace{1cm} \text{(A.15)}

Plugging it in Eq.(A.13) we get,

\[ J_{\text{mem}}^i = \frac{1}{g_{d+1}} \sqrt{-g^{tt}} F_t^i = \frac{1}{g_{d+1}} \tilde{E}_i, \]  \hspace{1cm} \text{(A.16)}

where \( \tilde{E}^i \) is the electric field measured in an orthonormal frame of a physical observer hovering just outside of the black hole. \( J_{\text{mem}}^i \) can be interpreted as the response of the membrane to electric field \( \tilde{E}^i \). Now comparing with \( \tilde{J} = \sigma \tilde{E} \) we get

\[ \sigma_{\text{mem}} = \frac{1}{g_{d+1}(r_h)}, \]  \hspace{1cm} \text{(A.17)}

where \( \sigma \) is the electrical conductivity of the membrane.

A.2 Mechanical properties of the membrane

Fluctuation of gravitational field will induce energy momentum tensor \( T^{\mu \nu} \) in the membrane. To illustrate this with an example, let us consider a metric fluctuation \( h_2^1(x) \). The action of this to the quadratic order is that of a free mass less scalar field

\[ S_{\text{grav}}^{\text{grav}} = \frac{1}{2} \int d^{d+1} x \sqrt{-g} \frac{1}{16\pi G_N} (\nabla \phi)^2, \]  \hspace{1cm} \text{(A.18)}

with \( \phi = h_2^1 \). Following previous discussion, we need to add a surface term

\[ S_{\text{surf}} = \int_{\text{horizon}} d^d x \sqrt{-h} \frac{\Pi_r(x)}{\sqrt{-h}} \phi(x), \]  \hspace{1cm} \text{(A.19)}

\[ 136 \]
Appendix A. Membrane paradigm

with \( \Pi_r = \sqrt{-\frac{g_{rr}}{16\pi G}} \). This will induce a current \( J(x) \) in the membrane \( J(x) \propto T^1_2 \). Regularity implies

\[
\partial_r \phi = \sqrt{-g} \partial_t \phi,
\]

so that one can write

\[
\Pi_{\text{mem}} = \frac{\Pi_r(x)}{\sqrt{-h} r_h} = -\frac{1}{\sqrt{g_{tt}}} \frac{16\pi G}{16\pi G} \partial_t \phi = -\frac{1}{16\pi G} \partial_t \phi.
\]

In the last line, we again have passed to orthonormal basis. As in the electromagnetic case (see Eq.(A.11)), we can interpret \( \Pi_{\text{mem}} \) in Eq.(A.18) as the membrane response of the field \( \phi \), with response function \( \eta = \frac{1}{16\pi G} \), the shear viscosity since \( \Pi_{\text{mem}} = (T_{\text{mem}})_{\eta} \). Since the entropy density \( (s) \) per unit volume of membrane fluid is \( s_{\text{mem}} = \frac{1}{4G} \), we get

\[
\frac{\eta}{s} = \frac{1}{4\pi}.
\]

So we see that one can consider horizon as fluid with response fluctuations such as \( \eta_{\text{mem}}, \sigma_{\text{mem}} \). Let us note that computation done using gauge gravity duality for boundary fluid also shows

\[
\frac{\eta}{s} = \frac{1}{4\pi}.
\]
R-charged black holes in various dimensions

Here we collect all the relevant information about four and seven dimensional R-charged black hole [33]. The case of five dimensional black hole was already discussed in the introduction. The R-charged black hole solutions in asymptotically AdS$_4$ and AdS$_7$ can be obtained by doing dimensional reduction of rotating M2 brane and M5 branes on S$^7$ and S$^4$ respectively. The relevant part of the Lagrangian is

$$\mathcal{L} = R - \frac{1}{4} G_{ij} F_{\mu \nu}^i F^{\mu \nu j} - G_{ij} \partial_\mu X^i \partial^\mu X^j + \ldots$$  \hspace{1cm} (B.1)

### B.1 Four dimensional black hole

Metric and gauge fields in this case are

$$ds^2_4 = \frac{16(\pi T_0 L)^2}{9 u^2} \mathcal{H}^{1/2} \left( -\frac{f}{\mathcal{H}} dt^2 + dx^2 + dz^2 \right) + \frac{L^2}{f u^2} \mathcal{H}^{1/2} du^2 ,  \hspace{1cm} (B.2)$$

$$A_i^t = \frac{4}{3} \pi T_0 \sqrt{2\kappa_i \prod_{i=1}^4 (1 + \kappa_i)} \frac{u}{H_i}, \quad H_i = 1 + k_i u ,  \hspace{1cm} (B.3)$$

$$\mathcal{H} = \prod_{i=1}^4 H_i, \quad f = \mathcal{H} - \prod_{i=1}^4 (1 + \kappa_i) u^3. \hspace{1cm} (B.4)$$
Appendix B. R-charged black holes in various dimensions

Thermodynamic quantities are given by

\[
\epsilon = \sqrt{2} \pi \left(\frac{2}{3}\right)^4 N^{3/2} T_0^3 \prod_{i=1}^{4} (1 + \kappa_i), \\
P = \frac{\sqrt{2} \pi^2}{3} \left(\frac{2}{3}\right)^3 N^{3/2} T_0^3 \prod_{i=1}^{4} (1 + \kappa_i), \\
\rho_i = \sqrt{2} \pi \left(\frac{1}{3}\right)^3 N^{3/2} T_0^2 \sqrt{2 k_i \prod_{j=1}^{4} (1 + \kappa_j)}, \\
\mu_i = \frac{4\pi T_0}{3} \frac{1}{1+k_i} \sqrt{2 k_i \prod_{i=1}^{4} (1 + \kappa_i)},
\]

(B.5a)

\[
T = \frac{T_0 \left(3 + 2 \sum_{j=1}^{4} k_i + \sum_{j>i,j=1}^{4} k_i k_j - \prod_{i=1}^{4} k_i\right)}{3 \prod_{i=1}^{4} (1 + \kappa_i)}
\]

(B.5b)

Other relevant expressions are

\[
G_{ij} = \frac{L^2}{2} \text{diag} \left[(X^1)^{-2}, (X^2)^{-2}, (X^3)^{-2}, (X^4)^{-2}\right], \quad X^i = \frac{\mathcal{H}_1^{1/4}}{H_1(u)}.
\]

(B.6)

and \(\frac{1}{16\pi G_4} = \frac{N^2}{24\sqrt{2} \pi^2}\). As was discussed in section (1.5.2), in this case as well, one can go to a case where one has diagonal \(U(1)\) of the group \(U(1)^4\). In this case, all the scalar field vanishes and one is left with the action of the form

\[
S_4 = \frac{1}{16\pi G_4} \int d^4x \sqrt{-g} (R - \frac{1}{4} F^2 + ...)
\]

(B.7)

which is exactly same as with Re\‘ssner-Nordstrom black hole in four dimension. Now comparing this action Eq.(2.66) gives us \(\frac{1}{2\kappa^2} = \frac{1}{16\pi G_4}\) and \(\gamma^2 = 8\).
B.2 Seven dimensional black hole

\[ ds^2 = \frac{4(\pi T_0 L)^2}{9u} \mathcal{H}^{1/5} \left( -\frac{f}{\mathcal{H}} dt^2 + dx_1^2 + \cdots + dx_4^2 + dz^2 \right) + \frac{L^2}{4fu^2} \mathcal{H}^{1/5} \, du^2 , \quad (B.8) \]

\[ A_t = \frac{2}{3} \pi T_0 \sqrt{2 \kappa_i \prod_{i=1}^{2} \frac{u^2}{H_i} } , \quad H_i = 1 + \kappa_i u^2 , \quad (B.9) \]

\[ H_i = 1 + \kappa_i u^2 , \quad \mathcal{H} = \prod_{i=1}^{2} H_i , \quad f = \mathcal{H} - \prod_{i=1}^{2} (1 + \kappa_i) u^3 , \quad (B.10) \]

Thermodynamic quantities are given by

\[ \epsilon = \frac{5 \pi^3}{2} \left( \frac{2}{3} \right)^7 N^3 T_0^6 \prod_{i=1}^{2} (1 + \kappa_i) , \quad P = \frac{\pi^3}{2} \left( \frac{2}{3} \right)^7 N^3 T_0^6 \prod_{i=1}^{2} (1 + \kappa_i) , \quad (B.11) \]

\[ s = 3 \pi^3 \left( \frac{2}{3} \right)^7 N^3 T_0^5 \sqrt{\prod_{i=1}^{2} (1 + \kappa_i) } , \quad T = \frac{T_0}{3} \left( 3 + \kappa_1 + \kappa_2 - \kappa_1 \kappa_2 \right) , \quad (B.12) \]

\[ \rho_i = \frac{\pi^2}{2} \left( \frac{2}{3} \right)^6 N^3 T_0^5 \sqrt{2 \kappa_i \prod_{i=1}^{2} (1 + \kappa_i) } , \quad \mu_i = \frac{2 \pi T_0}{3(1 + \kappa_i)} \sqrt{2 \kappa_i \prod_{i=1}^{2} (1 + \kappa_i) } . \quad (B.13) \]

Other relevant results are

\[ G_{ij} = \frac{L^2}{2} \text{diag} [(X_1)^2, (X_2)^2] , \quad X^i = \frac{\mathcal{H}^{2/5}}{H_i(u)} , \quad (B.14) \]

and \( \frac{1}{16 \pi G_t} = \frac{N^3}{6 \pi^3 L^5} \).

B.3 R-charged black holes at extremality

Above black holes at extremality was constructed in [85]. Take

\[ \bar{g}_{tt} = -f(u) A_1(u) , \quad \bar{g}_{uu} = A_2(u) f^{-1}(u) , \quad f(u) = (1 - u)^2 V(u) . \quad (B.15) \]
Appendix B. R-charged black holes in various dimensions

Here we just give relevant information about $f$.

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Extremality condition</th>
<th>$V(u)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>$2 + \kappa_1 + \kappa_2 + \kappa_3 - \kappa_1\kappa_2\kappa_3 = 0$</td>
<td>$(1 + \kappa_1\kappa_2\kappa_3 u)$</td>
</tr>
<tr>
<td>4</td>
<td>$3 + \sum_{j=1}^{4} k_i + \sum_{i&lt;j,i,j=1}^{4} k_i k_j - \prod_{i=1}^{4} k_i = 0$</td>
<td>$(1 + (2 + \sum_{j=1}^{4} k_i)u + \prod_{i=1}^{4} k_i u^2)$</td>
</tr>
<tr>
<td>7</td>
<td>$3 + \kappa_1 + \kappa_2 - \kappa_1\kappa_2 = 0$</td>
<td>$(1 + 2u + \kappa_1\kappa_2 u^2)$</td>
</tr>
</tbody>
</table>