Part II

Hagedorn Phase Transition and Matrix Model for Strings
4.1 Construction of Long D-strings

We will consider the classical configuration of the BFSS action (eqn.(2.1)) given by,

\[ X^9 = \tilde{L}^9 p \]

(4.1)

where \( X^9 \) is a compact direction of the BFSS model (See Section (2.2)), and \( p \) is defined in Section (2.4). This configuration is a solution of classical equation of motion and describes a membrane wrapped around \( X^9 \) with other edges free. If we consider \( X^i, i \neq 9 \) as periodic functions of \( q \), all of form \( \exp(i m q) \), then we get a closed string. Let us construct this string action in matrix model.

Consider T-duality (Section (2.3.1)) of BFSS matrix model along \( X^9 \), then the dual action is given by Eqn.(2.24), which is a dimensionally reduced 1 + 1 SYM theory describing \( N \) Type IIB D-strings. Let us assume that the radius of the compact direction \( X^9 \) is \( \tilde{L}^9 \), assumed to be small. A described in Section (2.3.1), the type IIB F-string length and coupling is given by \( \tilde{l}_s \) (same as Type IIA F-string, string tension= \( \frac{1}{2 \pi l_s^2} \) and \( \tilde{\alpha}' = \tilde{l}_s^2 \)) and \( \tilde{g}^{IIB}_s = \tilde{g}_s \frac{\tilde{L}^9}{\tilde{l}_9} = \frac{R_{11}}{L_9} \). Now the radius of the compact direction is \( \tilde{L}^9_o = \frac{\tilde{L}^9}{L_9} \). The D-string tension is \( \frac{1}{2 \pi \tilde{\beta}'} \), where \( \tilde{\beta}' = \tilde{l}_s^2 \beta' = \left( \frac{R_{11}}{L_9} \right)^3 = R_{11} \tilde{L}_9 \). Now consider a S-duality (Sec. (2.3.2)) transformation. Under S-duality type IIB theory is mapped to itself where the D-string and F-string interchanges and the string coupling is inverted. So we get a theory of \( N \) type IIB
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F-strings string length \((\tilde{g}_{s}^{IIB})^{1/2} = 1/2\) and coupling \(g_{s} = 1/2\). The compact direction is still given by \(\tilde{L}_9\). Now consider again a T-duality along this circle of radius \(\tilde{L}_9\), and the theory gets mapped to Type IIA matrix string theory (Section (2.5)) on the dual circle of radius \(\tilde{L}_9 = \frac{\tilde{L}_9}{\tilde{L}_9} = R_{11}\). The F-string inverse tension is given by \(\tilde{\beta} = \tilde{L}_9^3 = R_{11} \tilde{L}_9^3 = (\tilde{L}_9^3)^{1/2}\) (String tension does not change under T-duality) and the string coupling \(\tilde{g}_{IA} = \tilde{g}_{s}^{IIB} = \frac{\tilde{L}_9^3}{\tilde{L}_9^5} = \frac{(\tilde{L}_9^3)^{1/2}}{\tilde{L}_9^5\tilde{L}_9^3}\). So we get the relation \(\tilde{g}_{IA} = \tilde{L}_9\) (Compared to the original BFSS relation \(\tilde{g}_{IA} = R_{11}\)).

As shown in [20], a 1 + 1-dimensional matrix model action (after T-dual along \(X^9\)), at zero temperature, with background configuration given by membranes constructed in the above way, matches exactly with the string theory action in light cone frame. Similar calculation was done in [1] to show that Green-Schwarz IIA string can be constructed from BFSS matrix model. Consider the bosonic part of the 1 + 1-dimensional action given by Eqn.(2.24) (\(A_0 = 0\) gauge),

\[
S = \frac{1}{2R_{11}} \int dt \int \frac{dx}{2\pi \tilde{L}_9} T \left[ (\partial_x \chi)^2 - (D_x \chi)^2 + \frac{1}{(2\pi \tilde{L}_9)^2} \sum \left[ \chi^i, \chi^j \right]^2 \right] + (2\pi \tilde{L}_9^2)^2 (\partial_t \mathcal{A}_0)^2 + \cdots
\]  

(4.2)

where \(\cdots\) denotes other terms which is not purely bosonic. We have also used the relation \(\tilde{g}_{s} = R_{11}\). \(\mathcal{A}_0\) is related to original 0 + 1 matrix model by Taylor’s construction, given by (Sec.(2.3.1)),

\[
\mathcal{A}_0 = \frac{1}{2\pi \tilde{L}_9} \sum_{n=-\infty}^{\infty} \exp \left( i n x \frac{L_0}{\tilde{L}_9} \right) X_n^0
\]  

(4.3)

where \(X_0^0\) is the original 0 + 1 matrix of uncompactified theory representing D0-brane co-ordinate along the compact 9-th direction. \(D_x = \partial_x - i [\mathcal{A}_0, \bullet]\) is the covariant derivative in a direction \(X^9\), which is T-dual to \(X_9\), and has a radius \(\tilde{L}_9 = \tilde{L}_9\). \(x\) is the co-ordinate along a D1 brane wound around \(X_9\). Then if we choose the configuration (eqn.(4.1)), then,

\[
\mathcal{A}_0 = \frac{1}{2\pi \tilde{L}_9} p + \mathcal{A}_0^0(x,t,q)
\]  

(4.4)
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As mentioned we choose the configuration such that all the fields $\mathcal{X}^i$ are independent of $p$ except $\mathcal{A}^i$ which has $p$-dependence in above fashion. Now for large $N$, we can use Matrix/Membrane correspondence (Sec. (2.4.1)) for the action (4.2), which gives,

$$ S = \frac{1}{2R_{11}} \int dt \int_0^{2\pi \tilde{L}_9} \frac{dx}{2\pi \tilde{L}_9} N \int_0^{2\pi} \frac{dp}{2\pi} \int_0^{2\pi} \frac{dq}{2\pi} \frac{1}{N^2} \sum_{i>j} \{\mathcal{X}^i, \mathcal{X}^j \}^2_{P.B} $$

$$ + \frac{4\pi^2}{N^2} \sum_{i>j} \{\mathcal{X}^i, \mathcal{X}^j \}^2_{P.B} $$

$$ + (2\pi \tilde{t}_s^2)^2 (\partial_\tau \mathcal{A}_0)^2 + \cdots $$

(4.5)

Now as the fields are independent of $p$, the Poisson brackets will vanish except $\{p, \mathcal{X}_i \} = \partial_\tau \mathcal{X}_i$. So we get,

$$ S = \frac{1}{2R_{11}} \int dt \int_0^{2\pi \tilde{L}_9} \frac{dx}{2\pi \tilde{L}_9} N \int_0^{2\pi} \frac{dq}{2\pi} \frac{1}{N^2} \sum_{i>j} \{\mathcal{X}^i, \mathcal{X}^j \}^2_{P.B} $$

$$ + (2\pi \tilde{t}_s^2)^2 (\partial_\tau \mathcal{A}_0)^2 + \cdots $$

(4.6)

So in the large $N$ limit, $D_x$ is given by

$$ D_x = \partial_x \otimes I + I \otimes \frac{1}{N \tilde{L}_9} \partial_\tau $$

(4.7)

which acts on eigenfunctions,

$$ e^{i \frac{\pi}{\tilde{L}_9} e^{i \pi q}} $$

(4.8)

with eigenfunctions values $\frac{r^{N+n}}{NL^5}$. Fig.(4.1) illustrates the $x - q$ space. Thus the effective radius is $N \tilde{L}_9$. These are the long strings described in [116, 117, 26, 104].

We can replace $\tilde{D}_x$ by $\partial_\tau$ and eigenfunctions (eqn.(4.8)) by $e^{\frac{\pi r}{NL^5}}$. Here $\sigma$ has range $0 - 2\pi N \tilde{L}_9$. Also $\int \frac{dx}{2\pi \tilde{L}_9} \int_0^{2\pi} \frac{dq}{2\pi} \frac{1}{2\pi} \rightarrow \int_0^{2\pi N \tilde{L}_9} \frac{ds}{2\pi N \tilde{L}_9}$. So we have,

$$ S =\frac{1}{4\pi \beta'} \int dt \int_0^{2\pi N \tilde{L}_9} d\sigma \{ (\partial_\tau \mathcal{X}_i)^2 - (\partial_\sigma \mathcal{X}_i)^2 + (2\pi \tilde{t}_s^2)^2 (\partial_\tau \mathcal{A}_0)^2 + \cdots \} $$

(4.9)

where $\tilde{L}_9 = \frac{\alpha'}{\tilde{L}_9} = \frac{\beta'}{R_{11}}$. In the limit $\tilde{L}_9 \rightarrow 0$, $\mathcal{A}_0$ can be gauged away, and the action
becomes,
\[ S = \frac{1}{4\pi \beta'} \int dt \int_0^{2\pi \beta' P_{11}} d\sigma [(\partial_t X_i)^2 - (\partial_\sigma X_i)^2 + \cdots] \quad (4.10) \]
where, \( P_{11} = \frac{N}{R_{11}} \). Using the connection between DLCQ M-theory and BFSS matrix model (Sec. 2.2.1), we can rewrite the action as,
\[ S = \frac{1}{4\pi \beta'} \int dt \int_0^{2\pi \beta' P^+} d\sigma [(\partial_t X_i)^2 - (\partial_\sigma X_i)^2 + \cdots] \quad (4.11) \]
where \( P^+ = \frac{N}{R} \) is the light-cone momentum. Above action is same as the light-cone gauge type IIA Green-Scharzw string with string tension \( \beta' \). If we include \( p \) dependence to be non-zero then the commutator terms are non-zero, and the corresponding fluctuations are not string-like. Full non-perturbative effective action will presumably make these fluctuations massive.

![Figure 4.1: x – q space](image)

### 4.1.1 Two Phases

In this thesis we will calculate the free energy corresponding to two different configuration of D0-branes as done in our paper [24].

**Phase 1:** The background \( X^9 = \tilde{L}^9 p \) gives a configuration where the D0 branes spread out to form a string wound in the compact direction.

**Phase 2:** The background \( X^9 = 0 \) gives a phase where the D0-branes are clustered.

We will consider these two backgrounds to calculate free energy up to one loop level, and compare to find any signature of phase transition. It is important to have a precise definition of the measure in the functional integral. This is described
in the next section and as an example we calculate partition function for $\mathcal{N} = 2$ SUSY harmonic oscillator. For convenience we will drop the tilde sign on the parameters and put it back in the end.

4.2 Defining Measure for $\mathcal{N} = 2$ SUSY in 1D

We define measure such that

$$\int Dx D\psi^* D\psi \exp[-\pi \int_0^\beta dt (\frac{x^2}{2} + \psi^* \psi)] = 1$$

(4.12)

Where $x(t)$ is a bosonic variable, and $\psi(t)$ is its super-partner. The SUSY transformation is given by,

$$\delta x = \epsilon^* \psi + \psi^* \epsilon; \delta \psi^* = -\epsilon^* x; \delta \psi = -\epsilon x.$$  

(4.13)

where $\epsilon^*$ and $\epsilon$ are two infinitesimal anti commuting parameter. From these definition we can define the measure

$$Dx D\psi^* D\psi \equiv dx_0 \prod_{n=1}^\infty (dx_n dx_{-n}) d\psi_0^* d\psi_0 \prod_{m>0} d\psi_m^* d\psi_{-m}^* d\psi_m d\psi_{-m}$$

(4.14)

The Fermionic measure in terms of $\psi_1$ and $\psi_2$, where $\psi = \psi_1 + i\psi_2$ is given by,

$$D\psi^* D\psi \equiv id\psi_{10} d\psi_{20} \prod_{m>0} d\psi_{1m} id\psi_{1m}^* d\psi_{2m} id\psi_{2m}^*$$

(4.15)

where $x(t + \beta) = x(t)$; $x_n$ are Fourier expansion co-efficient for $x, \psi_m$ are Fourier expansion co-efficient for $\psi(t)$, $m$ runs over all integers for periodic boundary condition, but takes only odd values for anti-periodic boundary condition.

$$x(t) = \sum_{n=\infty}^\infty x_n e^{-\frac{2\pi}{\beta} nt}$$

(4.16)
\[ \psi(t) = \sum_{n=-\infty}^{\infty} \psi_n e^{-\frac{2\pi i}{\beta} nt} \quad \text{for periodic boundary condition} \]  
\[ \psi(t) = \sum_{n=-\infty, odd}^{\infty} \psi_n e^{-\frac{2\pi i}{\beta} nt} \quad \text{for anti-periodic boundary condition} \]  
\[ (4.17) \]
\[ (4.18) \]

4.2.1 Zeta Function

We will need the following results [118] for our calculation,

\[ \zeta(s) = \sum_{n=1}^{\infty} n^{-s} \]

\[ \zeta(s)' = -\sum_{n=1}^{\infty} n^{-s} \ln(n) \]

\[ \zeta_{\text{odd}}(s) = \sum_{n=1, n=\text{odd}}^{\infty} n^{-s} \]

\[ = (1 - 2^{-s})\zeta(s) \]

\[ \zeta_{\text{odd}}(s)' = -\sum_{n=1, n=\text{odd}}^{\infty} n^{-s} \ln(n) \]

\[ = 2^{-s} \ln 2 \zeta(s) + (1 - 2^{-s})\zeta(s)' \]  
\[ (4.19) \]

and \( \zeta(0) = -\frac{1}{2} \), \( \zeta(0)' = -\frac{1}{2} \ln(2\pi) \). Which gives \( \zeta_{\text{odd}}(0) = 0 \) and \( \zeta_{\text{odd}}(0)' = -\frac{1}{2} \ln(2) \).

4.2.2 Example: Super-symmetric (\( \mathcal{N} = 2 \)) 1D Harmonic Oscillator

Consider the SUSY Harmonic Oscillator at finite temperature with action,

\[ S = \int_0^\beta dt (\frac{x^2}{2} - \psi^* \dot{\psi} + \frac{x^2}{2} + \psi^* \psi) \]  
\[ (4.20) \]

Where the SUSY transformation is given by,

\[ \delta x = \epsilon^* \dot{\psi} + \psi^* \epsilon; \delta \psi^* = -\epsilon^*(\dot{x} + x); \delta \psi = -\epsilon(-\dot{x} + x). \]  
\[ (4.21) \]
With periodic boundary condition on both $x$ and $\psi$,

$$S = \frac{1}{2}\beta x_0^2 + \beta \sum_{n=1}^{\infty} (1 + \frac{4\pi^2 n^2}{\beta^2}) x_n x_{-n} + \beta \psi_0^* \psi_0 + \sum_{n=1}^{\infty} (\beta + 2\pi in) \psi_n^* \psi_n$$

$$+ \sum_{n=1}^{\infty} (\beta - 2\pi in) \psi_n^* \psi_{-n}$$  \hspace{1cm} (4.22)

So integrating by using the measure defined the partition function is,

$$Z = \int Dx \; D\psi^* D\psi \; e^{-S}$$

$$= \sqrt{\frac{2\pi}{\beta}} \times \prod_{n=1}^{\infty} \left\{ \frac{2\pi}{\beta(1 + \frac{4\pi^2 n^2}{\beta^2})} \right\} \times \beta \times$$

$$\prod_{n=1}^{\infty} (\beta + 2\pi in) \times \prod_{n=1}^{\infty} (\beta - 2\pi in)$$

$$= 1$$  \hspace{1cm} (4.23)

Using $\prod_{n=1}^{\infty} C = C^{\zeta(0)} = C^{-1/2}$, where $C$ is any constant number.

If we keep periodic boundary condition on $x$, but take anti-periodic boundary condition on $\psi$, the super-symmetry breaks and

$$S = \frac{1}{2}\beta x_0^2 + \beta \sum_{n=1}^{\infty} (1 + \frac{4\pi^2 n^2}{\beta^2}) x_n x_{-n} + \sum_{n=1,n=odd}^{\infty} (\beta + \pi in) \psi_n^* \psi_n$$

$$+ \sum_{n=1,n=odd}^{\infty} (\beta - \pi in) \psi_n^* \psi_{-n}$$  \hspace{1cm} (4.24)

So integrating by using the measure defined the partition function is,

$$Z' = \int Dx \; D\psi^* D\psi \; e^{-S}$$

$$= \sqrt{\frac{2\pi}{\beta}} \times \prod_{n=1}^{\infty} \left\{ \frac{2\pi}{\beta(1 + \frac{4\pi^2 n^2}{\beta^2})} \right\} \times$$

$$\prod_{m=1,odd}^{\infty} (\beta + \pi im) \times \prod_{m=1,odd}^{\infty} (\beta - \pi im)$$  \hspace{1cm} (4.25)

Now rearranging the products and using $\zeta$ function to regularize the infinite prod-
ucts, (i.e. using $\prod_{n=1}^{\infty} C = C^{-1/2}$, $\prod_{n=1}^{\infty} n = \sqrt{2\pi}$, $\prod_{n=1,\text{odd}}^{\infty} C = 1$, $\prod_{n=1,\text{odd}}^{\infty} n = \sqrt{2}$) we get,
\[
Z' = \frac{2}{\beta} \prod_{k=0}^{\infty} (1 + \frac{(\beta/2)^2}{\pi^2 (2k+1)^2}) = \coth \beta/2
\]  
(4.26)

Using, $\prod_{k=0}^{\infty} (1 + \frac{4x^2}{\pi^2 (2k+1)^2}) = \cosh x$ and $x \prod_{n=1}^{\infty} (1 + \frac{x^2}{\pi^2 n^2}) = \sinh x$ \cite{118}. Also notice as $\beta \to \infty$, $Z' \to 1$ i.e. super-symmetry is restored in the zero temperature limit.

### 4.3 SUSY Scalar Field theory on $S^1 \times S^1$

We will first calculate action for a SUSY scalar field on $S^1 \times S^1$, which is then related to the action (eqn. (2.24)) we are concerned, in the following section. The Minkowski action is given by,
\[
S_M = \frac{1}{g_s} \int \frac{dt_M}{l_s} \int_0^{2\pi L_5^*} \frac{dx}{2\pi L_9^*} \left\{ (\partial_{t_M} X)^2 - (\partial_x X)^2 + \overline{\psi}(i\gamma^\mu)\partial_\mu \psi \right\}
\]  
(4.27)

$\psi_\alpha$, $\alpha = 1, 2$ are two components (real) of two dimensional Majorana Spinor $\psi$.

$\gamma$ matrices are given by,
\[
\gamma^0 = \gamma_0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma^1 = -\gamma_1 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}
\]  
(4.28)

\[
\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}; \quad g^{00} = -g^{11} = 1
\]  
(4.29)

Fermionic part of the action can be rewritten as,
\[
\overline{\psi}(i\gamma^\mu)\partial_\mu \psi = i\psi_1 (\partial_{t_M} - \partial_x) \psi_1 + i\psi_2 (\partial_{t_M} + \partial_x) \psi_2
\]  
(4.30)

Let us consider $t_M \to -it$, and $t$ compact with periodicity $\beta$. The Euclidean action is given by $S = -i S_M$, we get,
\[
S = \frac{1}{g_s} \int \frac{dt}{l_s} \int_0^{2\pi L_5^*} \frac{dx}{2\pi L_9^*} \left\{ (\partial_t X)^2 + (\partial_x X)^2 + \psi_1 (\partial_t + i\partial_x) \psi_1 + \psi_2 (\partial_t - i\partial_x) \psi_2 \right\}
\]  
(4.31)
Where \( X \) is periodic in both \( t \) and \( x \), \( \psi_\alpha \) is anti-periodic in \( t \) and periodic in \( x \).

\[
X = (2\pi L_9^* \beta)^{1/2} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} X_{nm} e^{-\frac{2\pi}{\beta} nt} e^{-\frac{i}{T_9} mx} \tag{4.32}
\]

\[
\psi_\alpha = (2\pi L_9^* \beta)^{1/4} \sum_{n=-\infty, n=\text{odd}}^{\infty} \sum_{m=-\infty}^{\infty} \psi_{\alpha,nm} e^{-\frac{\pi}{\beta} nt} e^{-\frac{i}{T_9} mx} \tag{4.33}
\]

\( X \) and \( \psi_\alpha \) is real implies \( X_{nm}^* = X_{-n-m} \) and \( \psi_{\alpha,nm}^* = \psi_{\alpha,-n-m} \). \( X_{nm} \) is a dimensionless c-number and \( \psi_{\alpha,nm} \) is a dimensionless Grassmann number.

So the action becomes,

\[
S = \frac{\beta}{2g_s \ell_s} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} (2\pi L_9^* \beta)[(\frac{2\pi n}{\beta})^2 + (\frac{m}{L_9^*})^2] X_{nm} X_{-n-m} \\
+ \frac{\beta}{2g_s \ell_s} \sum_{n=-\infty, n=\text{odd}}^{\infty} \sum_{m=-\infty}^{\infty} \{i \sqrt{(2\pi L_9^* \beta)}[\frac{\pi n}{\beta} + i \frac{m}{L_9^*}] \psi_{1,nm} \psi_{1,-n-m} + i \sqrt{(2\pi L_9^* \beta)}[\frac{\pi n}{\beta} - i \frac{m}{L_9^*}] \psi_{2,nm} \psi_{2,-n-m}\} \tag{4.34}
\]

So we can write \( S = S_B + S_F = S_B + S_{F1} + S_{F2} \). \( S_B \) is the bosonic part. \( S_F \) is the fermionic part of the action, which again receives contribution from two parts \( S_{F1} \) and \( S_{F2} \) corresponding to \( \psi_1 \) and \( \psi_2 \). Each \( \psi_\alpha \) contributes same amount to partition function, \( Z_F = Z_{F1} Z_{F2} = Z_{F1}^2 = Z_{F2}^2 \). The full partition function is given by

\[
Z = Z_B Z_F \tag{4.35}
\]

### 4.3.1 Bosonic Part

We will consider the bosonic part first.

\[
S_B = \frac{\beta}{2g_s \ell_s} (2\pi L_9^* \beta) \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \left[\left(\frac{2\pi n}{\beta}\right)^2 + \left(\frac{m}{L_9^*}\right)^2\right] X_{nm} X_{-n-m} \tag{4.36}
\]
which implies,

\[
S_B = (2\pi L_g^b \beta) \frac{\beta}{g_s I_s} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \left\{ \frac{4\pi^2 n^2}{\beta^2} + \frac{m^2}{L_g^2} \right\} X_{nm} X_{nm}^* \tag{4.37}
\]

\[
= (2\pi L_g^b \beta) \frac{\beta}{g_s I_s} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left\{ \frac{4\pi^2 n^2}{\beta^2} + \frac{m^2}{L_g^2} \right\} [X_{nm} X_{nm}^* + X_{n-m} X_{n-m}^*] \\
+ (2\pi L_g^b \beta) \frac{\beta}{g_s I_s} \sum_{n=1}^{\infty} \frac{4\pi^2 n^2}{\beta^2} X_{n0} X_{n0}^* + (2\pi L_g^b \beta) \frac{\beta}{g_s I_s} \sum_{m=1}^{\infty} \frac{m^2}{L_g^2} X_{0m} X_{0m}^* \tag{4.38}
\]

Now we can calculate partition function easily,

\[
Z = \left( \int_{-\infty}^{\infty} dX_{00} \right) \left( \prod_{n=1}^{\infty} \prod_{m=1}^{\infty} \int_{-\infty}^{\infty} dX_{nm} dX_{nm}^* \right) e^{-\frac{(2\pi L_g^b \beta) \frac{\beta}{g_s I_s} \left( \frac{4\pi^2 n^2}{\beta^2} + \frac{m^2}{L_g^2} \right) X_{nm} X_{nm}^*}} \\
\left( \prod_{n=1}^{\infty} \int_{-\infty}^{\infty} dX_{n0} dX_{n0}^* \right) e^{-\frac{(2\pi L_g^b \beta) \frac{\beta}{g_s I_s} \left( \frac{4\pi^2 n^2}{\beta^2} \right) X_{n0} X_{n0}^*}} \\
\left( \prod_{m=1}^{\infty} \int_{-\infty}^{\infty} dX_{0m} dX_{0m}^* \right) e^{-\frac{(2\pi L_g^b \beta) \frac{\beta}{g_s I_s} \left( \frac{m^2}{L_g^2} \right) X_{0m} X_{0m}^*}} \tag{4.39}
\]

The zero mode integral diverges. So we put a cut-off \( \frac{L_0}{\sqrt{2\pi L_g^b \beta}} \) as the value of zero mode integral.

\[
Z_B = \left( \frac{L_0}{\sqrt{2\pi L_g^b \beta}} \right) \prod_{n=1}^{\infty} \prod_{m=1}^{\infty} \left\{ \frac{2\pi}{(2\pi L_g^b \beta) \frac{\beta}{g_s I_s} \left( \frac{4\pi^2 n^2}{\beta^2} + \frac{m^2}{L_g^2} \right) X_{nm} X_{nm}^*} \right\}^2 \\
\times \prod_{n=1}^{\infty} \left\{ \frac{2\pi}{(2\pi L_g^b \beta) \frac{\beta}{g_s I_s} \frac{4\pi^2 n^2}{\beta^2}} \right\} \times \prod_{m=1}^{\infty} \left\{ \frac{2\pi}{(2\pi L_g^b \beta) \frac{\beta}{g_s I_s} \frac{m^2}{L_g^2}} \right\} \tag{4.40}
\]

Using \( \prod_{n=1}^{\infty} c = c^{\epsilon(0)} = c^{-1/2} \), if we take out the constant factor \( (2\pi L_g^b \beta) \) from the products, it cancels nicely with the factor in zero mode integral. These products
can be rearranged to get,

\[
Z_B = \left\{ L_0 \prod_{n=1}^{\infty} \frac{2\pi}{g_s l_s \left( \frac{4\pi^2 n^2}{\beta^2} + \frac{m^2}{L_9^2} \right)} \right\} \text{free particle with mass} \frac{1}{g_s l_s} \times \\
\left\{ \prod_{m=1}^{\infty} \left[ \frac{2\pi}{\beta g_s l_s \left( \frac{m^2}{L_9^2} \right)} \prod_{n=1}^{\infty} \frac{2\pi}{g_s l_s \left( \frac{4\pi^2 n^2}{\beta^2} + \frac{m^2}{L_9^2} \right)} \right] \right\}^2 
\]

(4.41)

therefore,

\[
Z_B = L_0 \sqrt{\frac{M}{2\pi \beta}} \prod_{m=1}^{\infty} \left\{ \frac{1}{2 \sinh(\beta \omega_m/2)} \right\}^2 
\]

(4.42)

where,

\[
M = \frac{1}{g_s l_s}, \quad \omega_m = \frac{m}{L_9} 
\]

(4.43)

Using,

\[
\eta(ix) = \prod_{k=1}^{\infty} (2 \sinh(\pi kx)) 
\]

(4.44)

where \( \eta(z) \) is Dedekind’s eta function. We get,

\[
Z_B = \frac{L_0}{\sqrt{4\pi^2 g_s l_s L_9^*}} \eta\left( \frac{i\beta}{2\pi L_9^*} \right)^{-2} 
\]

(4.45)

The Partition function in terms of ratio of radii of two \( S^1 \), i.e., \( x = \frac{\beta}{2\pi L_9^*} \),

\[
Z_B = \frac{L_0}{\sqrt{4\pi^2 g_s l_s L_9^*}} \frac{1}{\sqrt{x}} \eta(ix)^{-2} 
\]

(4.46)

Dedekind’s Eta Function has a symmetry given by,

\[
\eta(ix) = \frac{1}{\sqrt{x}} \eta(i/x) 
\]

(4.47)

Which makes the partition function invariant under the transformation \( x \to 1/x \)
For low temperature, \( \frac{\beta}{2\pi L_9^*} \gg 1 \), the free energy takes the form,

\[
F_B(T) = -\frac{1}{\beta} \ln(Z) \simeq -\frac{1}{12L_9^*} - \frac{1}{2} T \ln\left( \frac{L_0^2}{2\pi g_s l_s T} \right) 
\]

(4.48)
which shows $F_B(0) \neq 0$ due to the presence of zero-point energy,

$$F_B(0) = -\frac{1}{12L_9^5} = \sum_{n=1}^{\infty} \frac{n}{L_9^9}$$

(4.49)

using Zeta function regularization. The high temperature expansion, $\frac{\beta}{2\pi L_9} << 1$ is given as,

$$F_B(T) = -\frac{\pi^2 L_9^5 T^2}{3} + \frac{T}{2} \ln(\frac{8\pi^3 g_s l_s L_9^2}{L_0^2} T)$$

(4.50)

### 4.3.2 Fermionic Part

Let us consider the fermionic part of the action,

$$S_F = S_{F1} + S_{F2} = \frac{\beta}{g_s l_s} \sum_{n=-\infty, n=\text{odd}}^{\infty} \sum_{m=-\infty}^{\infty} \left\{ i \sqrt{2\pi L_9^9} \left[ \frac{\pi n}{\beta} + i \frac{m}{L_9^9} \right] \psi_{1,n,m} \psi_{1,-n-m} ight\}$$

$$+ i \sqrt{2\pi L_9^9 \beta} \left[ \frac{\pi n}{\beta} - i \frac{m}{L_9^9} \right] \psi_{2,n,m} \psi_{2,-n-m}$$

(4.51)

By rearranging the sum (we have dropped the index 1 or 2 in $\psi$),

$$S_{F1} = \frac{\beta}{g_s l_s} \sqrt{2\pi L_9^9 \beta} \left\{ \sum_{n,m=1, n=\text{odd}}^{\infty} \left( \frac{\pi n}{\beta} + i \frac{m}{L_9^9} \right) \psi_{nm} \bar{\psi}_{-n-m} \right\}$$

$$+ \sum_{n=1, n=\text{odd}}^{\infty} \left( \frac{\pi n}{\beta} - i \frac{m}{L_9^9} \right) \psi_{nm} \bar{\psi}_{-n-m}$$

$$+ \sum_{n=1, n=\text{odd}}^{\infty} \frac{\pi n}{\beta} \psi_{n0} \bar{\psi}_{-n0}$$

(4.52)

Therefore,

$$Z_{F1} = \left\{ \prod_{n=1, \text{odd}}^{\infty} C \frac{\pi n}{\beta} \right\} \left\{ \prod_{n=1, \text{odd}}^{\infty} \prod_{m=1}^{\infty} C^2 \left( \frac{\pi^2 n^2}{\beta^2} + \frac{m^2}{L_9^2} \right) \right\}$$

(4.53)

where, $C = \frac{\beta}{g_s l_s} \sqrt{2\pi L_9^9 \beta}$

Using, $\prod_{n=1, \text{odd}}^{\infty} C = C^{\text{odd}(0)} = 1$, $\prod_{n=1, \text{odd}}^{\infty} n = e^{-C^{\text{odd}(0)} = \sqrt{2}$ and rearranging
the products we get,

\[
Z_{F_1} = \sqrt{2} \prod_{n=1, \text{odd}}^{\infty} \prod_{m=1}^{\infty} \left(1 + \frac{\left(\frac{\pi^2 L_9^2 n}{\beta}\right)^2}{\pi^2 m^2}\right)
\]

(4.54)

Using, \(\frac{\sinh(x)}{x} = \prod_{k=1}^{\infty} \left(1 + \frac{x^2}{2\pi^2 n^2}\right)\) we get,

\[
Z_{F_1} = \prod_{n=1, \text{odd}}^{\infty} \sinh\left(\frac{n^2 L_9^2 m}{\beta}\right)
\]

\[
= \frac{\prod_{n=1}^{\infty} 2 \sinh\left(\frac{\pi^2 L_9^2 n}{\beta}\right)}{\prod_{n=1}^{\infty} 2 \sinh\left(\frac{2\pi^2 L_9^2 n}{\beta}\right)}
\]

\[
= \frac{\eta(i \frac{\pi L_9}{\beta})}{\eta(i \frac{2\pi L_9}{\beta})}
\]

(4.55)

Where \(x = \frac{\beta}{2\pi L_9^2}\), and using property of Dedekind eta function, we get,

\[
Z_{F_1} = \sqrt{2} \frac{\eta(2ix)}{\eta(ix)}
\]

(4.56)

So we get ,

\[
Z_F = Z_{F_1} Z_{F_2} = Z_{F_1}^2 = 2 \left(\frac{\eta(2ix)}{\eta(ix)}\right)^2
\]

(4.57)

Where \(x = \frac{\beta}{2\pi L_9^2}\).

### 4.3.3 SUSY scalar field partition function

So SUSY partition function for the action (eqn.(4.31)) is given by (eqn.(4.35)),

\[
Z = Z_B Z_F = \frac{2L_0}{\sqrt{4\pi^2 g_s l_s L_9}} \frac{1}{\sqrt{\pi}} \frac{\eta^2(2ix)}{\eta^4(ix)}
\]

(4.58)
using Eqns. (4.46) and (4.57). Compare with (Eqn.(4.46)),

\[
Z_B = \frac{L_0}{\sqrt{4\pi^2 g_s \lambda}} \frac{1}{\sqrt{x}} \eta(ix)^{-2}
\] (4.59)

Now \(Z\) does not have the \(x \to 1/x\) symmetry, which is natural as two directions are not similar due to different boundary condition. At low temperature, i.e. \(x \to \infty\),

\[
Z = 2L_0 \sqrt{\frac{1}{2\pi \beta}}
\]

which is the partition function for a super-symmetric free particle (the zero mode).

4.4 Free energy for two phases of Matrix Model

We use Background Gauge Fixing Method (see Appendix B) to calculate the free energy up to one loop for the action (eqn.(2.24)), which is a 1 + 1 dimensional SYM action. From Appendix B we have,

\[
\ln Z^{(1)} = -\beta F = 5\text{Tr}(\ln \tilde{D}_E^2)_{\text{b Bosonic}} - 4\text{Tr}(\ln \tilde{D}_g^2)_{\text{fermionic}} - \text{Tr}(\ln \tilde{D}_E^2)_{\text{ghost}}
\] (4.60)

where, \((\tilde{D}_E)^2 = (\partial_x)^2 + (\tilde{D}_y)^2\) and \(\tilde{D}_g = \partial_x - i [a_0, \bullet]\). Background of \(A_0\) is zero \(^1\) and background for \(A_0\) is \(a_0\). As described in Section (4.1), for String Phase \(a_0 = \tilde{L}_0 p\) and for Clustered phase \(a_0 = 0\). Therefore,

\[
\tilde{D}_g = \partial_x - i \tilde{L}_0 [p, \bullet] = \partial_x \otimes I + I \otimes \frac{1}{N\tilde{L}_0^*} \partial_\sigma \quad \text{for String Phase}
\]

\[
= \partial_x \quad \text{for Clustered Phase}
\] (4.61)

in the large \(N\) limit. The derivative acts on eigenfunctions functions of form given in (eqn.(4.8)) and can be replaced by \(\partial_\sigma\) as described in Section (4.1). Where \(\sigma\) is a effective periodic coordinate with period \(2\pi N\tilde{L}_0^*\) and \(x\) has a period \(2\pi \tilde{L}_0^*\). Thus \(\tilde{D}_g = \partial_\sigma\) for String Phase and \(\tilde{D}_g = \partial_x\) for Clustered Phase.

\(^1\)At finite temperature zero mode of \(A_0\) is a physical variable (as it can not be gauged away due to periodic nature of Euclidean time) and corresponds to chemical potential. This has been set to zero for our case. More generally one can consider free energy as a function of zero mode of \(A_0\) (as considered in [21] in a different context). Analysis similar to [21], could be done in our case, and keep this for future. Also note that non-zero value of the chemical potential will remove the necessity of IR cut-off and as it introduces a scale and the parameter counting (as discussed in introduction) will match even without the IR cut-off.
The ghost terms (eqn.(4.60)) effectively cancels the two gauge fields, and remaining theory is effectively that of 8 SUSY scalar fields, except the fields are now $U(N)$ matrices in adjoint representation and the derivatives are little complicated than that for scalar fields. For clustered phase at 1-loop the partition function is equivalent to $8N^2$ SUSY scalar fields on $S^1 \times S^1$ with radii $\beta$ and $\tilde{L}_9$. And for string phase it is equivalent to $8N$ (factor of $N$ comes as the fluctuations are considered to be independent of $p$ and there is trace over $p$ direction. The trace over $q$ and integral over $x$ gives the effective integral over $\sigma$) SUSY scalar fields on $S^1 \times S^1$ with radii $\beta$ and $N\tilde{L}_9$.

4.4.1 Phase 1: String Phase

The partition function for phase 1 (string) is given by,

$$
Z_1 = e^{-\beta F_1} = \left\{ \frac{2\tilde{L}_0}{\sqrt{4\pi^2 g_s \tilde{L}_9}} \frac{1}{\sqrt{4\pi^2 N}} \frac{\eta^2(i\frac{2\pi}{N})}{\eta^4(i\frac{\pi}{N})} \right\}^{8N} 
$$

$$
\beta F_1 = -8 N \ln(b) + 8 N \ln(\sqrt{x}) - 16 N \ln(\eta(i\frac{2\pi}{N})) 
+ 32 N \ln(\eta(i\frac{x}{N}))
$$

(4.62)

(4.63)

where $x = \frac{\beta}{2\pi \tilde{L}_9}$, $b = \frac{2\tilde{L}_0}{\sqrt{4\pi^2 g_s \tilde{L}_9}} = \frac{2\tilde{L}_0}{\sqrt{4\pi^2 \tilde{\beta}}}.$

4.4.2 Phase 2: Clustered Phase

The partition function for phase 2 (clustered) is given by,

$$
Z_2 = e^{-\beta F_2} = \left\{ \frac{2\tilde{L}_0}{\sqrt{4\pi^2 g_s \tilde{L}_9}} \frac{1}{\sqrt{4\pi^2 N}} \frac{\eta^2(2ix)}{\eta^4(ix)} \right\}^{8N^2}
$$

$$
\beta F_2 = -8 N^2 \ln(b) + 8 N^2 \ln(\sqrt{x}) - 16 N^2 \ln(\eta(2ix)) 
+ 32 N^2 \ln(\eta(ix))
$$

(4.64)

(4.65)
4.4.2.1 Low temperature expansion

Consider $x >> 1$ and $x/N >> 1$,

$$\beta F_1 \simeq -8N \ln(b) + 8N \ln(\sqrt{x})$$  \hspace{1cm} (4.66)

$$\beta F_2 \simeq -8N^2 \ln(b) + 8N^2 \ln(\sqrt{x})$$  \hspace{1cm} (4.67)

Using,

$$\eta(ix) \simeq e^{-\frac{\pi x}{2}} \text{ for } x >> 1$$  \hspace{1cm} (4.68)

So $\beta F_1 < \beta F_2$, i.e. the string phase will be favored at low temperature if $b/\sqrt{x} \leq 1$.

We can expect a phase transition from the string phase to the clustered phase as the temperature is increased from zero, at $x = b^2$. For the transition temperature to lie in the validity region of the low temperature expansion : $b >> \sqrt{N}$. Let us call this temperature as $T_H$.

$$T_H = \frac{\pi R_{11}^2}{2 L_0^2}$$  \hspace{1cm} (4.69)

In terms of the DLCQ parameters,

$$T_H = \frac{\pi R^-}{2 L_0^2}$$  \hspace{1cm} (4.70)

using the scaling properties discussed in section (2.3). If we now take the limit $L_0 \to \infty$, the transition temperature $T_H \to 0$. So, it is essential to have a finite value of $L_0$ to get phase transition at finite temperature. This is the temperature at which there is a deconfinement transition in the Yang-Mills’ model, which should be same as Hagedorn transition.

4.4.2.2 High temperature expansion

Consider $x << 1$,

$$\beta F_1 \simeq 8N ln\left(\frac{2N}{b}\right) - 8N \ln(\sqrt{x}) - \frac{2N^2 \pi}{x}$$  \hspace{1cm} (4.71)

$$\beta F_2 \simeq 8N^2 \ln\left(\frac{2}{b}\right) - 8N^2 \ln(\sqrt{x}) - \frac{2N^2 \pi}{x}$$  \hspace{1cm} (4.72)
Using,
\[ \eta(ix) \simeq e^{-\left(\frac{1}{2x} + \ln\sqrt{x}\right)} \text{ for } x \ll 1 \]  
(4.73)

We see that, at \( x > \frac{4}{\beta x} \), the clustered phase is favored but at very high temperature we can again have a string phase. This “Gregory-Laflamme” kind of transition will occur at \( x = \frac{4}{\beta} N^{-1/N} \simeq \frac{4}{\beta x} \) (For large \( N \), \( N^{-1/N} \sim 1 \)). This is also consistent with \( b >> \sqrt{N} \). Let us call this temperature as \( T_G \).

\[ T_G = \frac{\widetilde{L}_0^2}{8\pi^3 R^{11} L_0^2} \]  
(4.74)

In terms of the DLCQ parameters,

\[ T_G = \frac{L_0^2}{8\pi^3 R^2 L_0^2} \]  
(4.75)

using the scaling properties discussed in section 2.3. In this case, note \( L_0 \to \infty \) implies \( T_G \to \infty \).

Let us express this in terms of the parameters of Yang-Mills theory: The infrared cutoff on \( A = \frac{X}{L^2} \) is \( \frac{\tilde{L}_0}{L_0} = \frac{1}{L_0} \). Thus we get (up to factors of \( 2\pi \))

\[ T_G = \frac{\tilde{L}_s^2}{R_{11} L_0^2 L_0^2} = \frac{\tilde{L}_s^2}{g_s L_0^2 L_0^2} = \frac{1}{g_Y^{M_{0+1}}} \frac{1}{L_0^2 L_0^2} = \frac{1}{g_Y^{M_{1+1}}} \frac{1}{L_0^2 L_0^2} \]  
(4.76)

### 4.4.3 Numerical results for Phases

Fig.4.2 shows the division of parameter space \((N, x, b)\) into regions where each of the phases are preferred, which is obtained by numerically comparing the free energies obtained in previous sections. The numerical plot re-affirms our analytic result that there exists two phase transition.
Figure 4.2: Solid region represents string phase and empty region is clustered phase.
The BFSS matrix model with tilde parameters is supposed to be a non-perturbative description of M-theory (or IIA String Theory). But the idea of gravity dual [30] of Yang-Mills’ theories allows us to relate the matrix model to super-gravity (with tilde parameters). This issue is discussed in the Chapter (3). One can use this to infer properties of the matrix model.

Two classical solutions of Type IIA super-gravity are: (1) the decoupling limit of black \( D0 \) branes and (2) the BPS \( D0 \) branes. In a recent study [41] phase transition of these two solutions were discussed. It was shown that the IR cut-off plays a crucial role in phase transition. As mentioned earlier this is motivated by the work of [33, 34, 40]. We will redo the analysis of this paper [41] here and try to explain the physical origin of IR cutoff used in [41]. The super-gravity solutions have tilde parameters, but for convenience we will drop the tilde signs and put them back at the end.

### 5.1 Hawking-Page transition in D-branes

Ideally, we should construct the super-gravity solution corresponding to the wrapped membrane. We reserve this for the future. Here we are only interested in understanding the nature of the phase transition and the role of the IR cutoff, so we will just use the solution for \( N \) coincident \( D0 \)-branes used also in [41]. In the decoupling limit, (with \( U = \frac{r}{T} = \text{fixed} \), where \( r \) is radial co-ordinate defined in the transverse space of the brane. \( U \) also sets the energy scale of the dual Yang-Mills theory.) the solution for \( N \) coincident black \( D0 \)-branes in Einstein frame is given
by (eqn. (3.7)),
\[
\begin{align*}
    ds_{Ein}^2 &= \frac{\alpha'}{2\pi g_{YM}^2} \left\{ -\frac{U^{4\beta}}{(g_{YM}^2 d_0^2 N)^{\frac{3}{2}}} \left( 1 - \frac{U_H^7}{U^7} \right) dt^2 \\
    &\quad + \frac{(g_{YM}^2 d_0^2 N)^{\frac{1}{2}}}{U^{\frac{1}{2}}} \left[ \frac{dU^2}{1 - \frac{U_H^7}{U^7}} + U^2 d\Omega_8^2 \right] \right\},
\end{align*}
\]
\begin{equation}
    (5.1)
\end{equation}
\[
    e^\phi = 4\pi g_{YM}^2 \left( \frac{g_{YM}^2 d_0^2 N}{U^7} \right)^{\frac{3}{2}},
\]
\begin{equation}
    (5.2)
\end{equation}
\[
    F_{U_0} = -\frac{\alpha'}{2} \frac{7U^6}{4\pi^2 d_0^2 N g_{YM}^4}.
\]
\begin{equation}
    (5.3)
\end{equation}

where \(d_0 = 2^7 \pi^{9/2} \Gamma(7/2)\) is a constant. Simply setting \(U_H = 0\) gives the solution for \(N\) coincident BPS D0-branes.

The Euclidean action can be obtained by setting \(t = -i\tau\). The Euclidean time \(\tau\) has a period
\[
    \beta = \frac{4\pi g_{YM} \sqrt{d_0 N}}{7U_H^2},
\]
\begin{equation}
    (5.4)
\end{equation}
in order to remove the conical singularity. This is the inverse Hawking temperature of the black D0 brane in the decoupling limit.

Now the on-shell Euclidean action for the two solutions can be calculated and gives,
\[
\begin{align*}
    I_{\text{black}} &= \frac{7^3 V(\Omega_s) \beta}{16^{1/2} G_{10}^\prime} \int_{U_H \text{ or } U_{IR}}^{U_{uv}} U^6 dU \\
    I_{\text{bps}} &= \frac{7^3 V(\Omega_s) \beta^\prime}{16^{1/2} G_{10}^\prime} \int_{U_{IR}}^{U_{uv}} U^6 dU
\end{align*}
\]
\begin{equation}
    (5.5)
\end{equation}
\begin{equation}
    (5.6)
\end{equation}

where \(G_{10}^\prime = \alpha'^{-7} G_{10} = 2^7 \pi^{10} g_{YM}^4\) is finite in the decoupling limit. \(U_{uv}\) is introduced to regularize the action and is taken to \(\infty\) in the end. The temperature of BPS branes \(\beta^\prime\) is arbitrary and can be fixed by demanding the temperature of both the solutions to be same at the UV boundary \(U_{uv}\), which gives \(\beta^\prime = \beta \sqrt{1 - \frac{U_H^7}{U_{uv}^7}}\). \(U_{IR}\) is a IR cut-off which removes the region \(U < U_{IR}\) of the geometry. The integration in the action starts from \(U_{IR}\) for BPS solution and, \(U_{IR}\) or \(U_H\) for the black brane solution depending on \(U_H < U_{IR}\) or \(U_H > U_{IR}\) respectively. If we put \(U_{IR} = 0\) i.e. in absence of the IR cut-off, comparison of the actions
(eqns. (5.5), (5.6)) shows that there is no phase transition, and the black brane phase is always favored. Let us consider the case \( U_H > U_{IR} \),

\[
\Delta I_{bulk} = \lim_{U_{uv} \to \infty} (I_{\text{black}} - I_{\text{bps}}) = \frac{7^2}{16} \frac{V(\Omega_4)\beta}{16\pi G_{10}^9} (-\frac{1}{2} U_H^7 + U_{IR}^7)
\]

(5.7)

Which shows a change in sign as we increase the temperature i.e. \( U_H \) (eqn. (5.4)). The system will undergo a phase transition (“Hawking-Page Phase transition”) from BPS brane to Black brane solution at \( U_{IR}^7 = 2U_H^7 \). Actually, we should also consider Gibbons-Hawking surface term for careful analysis (as done in [41]) which corrects the transition temperature by some numerical factor given by,

\[
\beta_{crit} = \frac{4\pi \tilde{g}_{YM} \sqrt{d_0 N}}{7(\frac{49}{20})^{5/14} \tilde{U}_{IR}^{5/2}}
\]

(5.8)

We see a IR cutoff is essential to realize a phase transition, (as \( \tilde{U}_{IR} \to 0, \beta_{crit} \to \infty \)) so to get confinement-deconfinement phase transition in dual super Yang-Mills theory we have to introduce a IR cutoff.

As mentioned in the introduction, one possible mechanism for the origin of the cutoff for D0 branes can be understood from the analysis of [42]. It was shown that the higher derivative corrections to super-gravity introduce a finite horizon area for extremal D0 brane solution which is otherwise zero. The multigraviton states (with total N units of momentum in the 11th direction) and the single graviton state seem to both be microstates of the same black hole when interaction effects higher order in \( g_s \) are included. Radius of the horizon developed due to higher derivative corrections is \( R \sim \tilde{l}_s^{-1/3} \). So we can get an estimate of IR cutoff by identifying \( R \) with the IR cutoff in our case, \( \tilde{U}_{IR} = \frac{R}{\tilde{l}_s} \sim \tilde{l}_s^{1/3} \sim \tilde{g}_{YM}^{2/3} \), which is finite in the scaling limit. If we plug in this value of \( \tilde{U}_{IR} \) in eqn. (5.8), we get

\[
\beta_{crit} = \frac{4\pi \sqrt{d_0 N}}{7(\frac{49}{20})^{5/14} \tilde{g}_{YM}^{2/3}} \sim \frac{1}{\tilde{U}_{IR}}
\]

(5.9)

In case of D1 brane system, which is just T dual to the system studied above also shows that a IR cutoff is required for phase transition [41]. We were unable to find a analysis like [42] corresponding to wrapped membrane system, which we need to get an estimate of the IR cutoff.

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5.2 Gregory-Laflamme Transition

In our calculation we find a temperature $T_G$, where the $D0$-branes spread out uniformly along the compact space. This configuration is just the one that is favored at very low temperatures. It is not clear whether this perturbative result is reliable. However, a similar phase transition exists in the dual super-gravity theory, known as “black hole-black string” transition or Gregory-Laflamme transition [31, 32, 54, 55, 56, 119, 120]. It is shown in [31], that the near horizon geometry of a charged black string in $R^{8,1} \times S^1$ (winding around the $S^1$) develops a Gregory-Laflamme instability at a temperature $T_{GL} \sim \frac{1}{L^2 g_{YM} \sqrt{N}}$, where $L$ is the radius of $S^1$ and $g_{YM}$ is $1 + 1$ dimensional Yang-Mills' coupling. Below this temperature the system collapses to a black-hole. In the weak coupling limit, the dual $1 + 1$ SYM theory also shows a corresponding phase transition by clustering of eigenvalues of the gauge field in the space-like compact direction below the temperature, $T'_{GL} \sim \frac{1}{L^2 g_{YM}^2 N}$, as shown numerically in [31]. This should be compared to the perturbative result (eqn.( 4.76)) $T_G \sim \frac{1}{g_{YM}^2 (L_0)^2 L_5}$. So the presence of high temperature string phase in our model must correspond to some kind of “Gregory-Laflamme” transition in dual super-gravity. This is (at least superficially) independent of the issue of any classical instability. This is because both solutions may be locally stable, but at finite temperatures it is possible to have a first order phase transition to the global minimum.

In our perturbative result, the high temperature phase is a string rather than a black string i.e. it is the same as the low temperature phase. The question thus arises whether Gregory-Laflamme transitions can happen for extremal objects. We can give a heuristic entropy argument to show Gregory-Laflamme kind of transition is also possible for extremal system. In the original argument [57], it was shown that for extremal branes there is no instability. However, these systems had zero horizon area. Instead, we will here consider a 5 dimensional extremal RN black hole with a large compact direction, and the same solution with the mass smeared uniformly along the compact direction (“RN black ring”). The metric, ADM mass ($M_5$) and entropy ($S_5$) for a 5$d$ extremal RN black hole solution is given by (where
the compact direction is approximated by a non-compact one),

\[ ds_5^2 = -(1 - \frac{r^2}{r_s^2})^2 dt^2 + (1 - \frac{r^2}{r_e^2})^{-2} dr^2 + r^2 d\Omega_3^2 \]  

(5.10)

\[ M_5 = \frac{3\pi}{4G_5} r_e^2 \]  

(5.11)

\[ S_5 = \frac{2\pi^2 r_e^3}{4G_5} = \frac{\pi^2}{2} \left( \frac{4}{3\pi} \right)^{3/2} G_5^{1/2} M_5^{3/2} \]  

(5.12)

where \( G_5 \) is 5d Newton’s constant. Similarly, we can write the metric for 5d extremal RN Black ring, which when dimensionally reduced gives a 4d extremal RN black hole. The metric, ADM mass ( \( M_{(4\times1)} \)) and entropy ( \( S_{(4\times1)} \)) is given by,

\[ ds_{(4\times1)}^2 = -(1 - \frac{R_e}{r})^2 dt^2 + (1 - \frac{R_e}{r})^{-2} dr^2 + dx^2 + r^2 d\Omega_2^2 \]  

(5.13)

\[ M_{(4\times1)} = \frac{R_e}{G_4} \]  

(5.14)

\[ S_{(4\times1)} = \frac{4\pi R_e^2 \times 2\pi L}{4G_5} = \frac{1}{2} G_5 \frac{M^2}{L} \]  

(5.15)

where \( G_4 = \frac{G_5}{2\pi L} \) is 4d Newton’s constant and \( L \) is the radius of the compact direction \( x \). If we consider \( M_5 = M_{(4\times1)} = M \) and compare the entropy,

\[ \frac{S_5}{S_{(4\times1)}} = \frac{16}{9} \frac{L}{r_e} \]  

(5.16)

So, when radius of the compact direction is greater than the radius of the 5d black hole horizon, the black hole solution is entropically favored. As we increase the horizon radius, there may be phase transition when horizon size becomes of the order of the radius of the compact dimension, above which the “string” solution is entropically favored. Our analysis is a simple entropy comparison. As mentioned above, this is independent of the classical stability issue, that was studied in detail in [57]. Therefore, extremal solutions with a finite horizon size may also show a Gregory-Laflamme kind of transition. This needs further study.
In part 1 of the thesis, the finite temperature phase structure of string theory has been studied using the BFSS matrix model which is a 0 + 1 Super-symmetric Yang-Mills (SYM) theory. This is first studied in perturbation theory. This was actually a refinement of an earlier calculation [20] where some approximations were made. The result of this study is that there is a finite and non-zero phase transition temperature $T_H$ below which the preferred configuration is where the $D0$ branes are arranged in the form of a wrapped membrane and above which the $D0$ branes form a localized cluster. It is reasonable to identify this temperature with the “Hagedorn” temperature, which was originally defined for the free string. We have found that $T_H \sim \frac{1}{L_0}$, where $L_0$ is the IR cutoff of the Yang-Mills theory which needs to be introduced to make the calculations well defined.

The 0 + 1 SYM has a dual super-gravity description. Here also it is seen that in the presence of an IR cutoff there is a critical temperature above which the BPS D0 brane is replaced by a black hole.

Simple parameter counting shows that the BFSS matrix model needs one more dimensionful parameter if it is to be compared with string theory, so the IR cutoff $L_0$ can be thought of as one choice for this extra parameter. It makes the comparison well defined by effectively removing the strongly coupled region of the configuration space in SYM as well as in super-gravity. A physical justification for this (beyond parameter counting) comes from the work of [42]. It is shown there that the entropy matching requires even the extremal BPS configuration of $D0$-branes to develop a horizon, due to higher derivative string loop corrections to super-gravity. This is an issue that deserves further study.
Finally, the perturbative result shows a second phase transition at a higher temperature, back to a string like phase. This could be an artifact of perturbation theory. On the other hand, it is very similar to the Gregory-Laflamme instability and there is also some similarity in the expressions obtained in [31] for the critical temperature. This also requires further study.