CHAPTER 0

INTRODUCTION

Injective modules first appeared in the context of abelian groups (Z-modules). In 1935 Zippin observed that an abelian group is divisible if and only if it is a direct summand of any larger group containing it as a subgroup, and that divisible abelian groups can be completely described.

In 1940 Baer, R. [12] had first investigated the general notion for modules. It is interesting in retrospect that the theory of these modules was investigated long before the dual notion of projective modules was considered. The "injective" and "projective" terminology originated with Cartan and Eilenberg in 1956 [19].

Proposition [52, 4.1], [Baer’s Criterion] :

Let M be a right module over a ring R. Then M is injective if and only if for every right ideal I of R and every \( f \in \text{Hom}_R (I, M) \), there exists \( a \in M \) such that \( f(r) = ar \) for all \( r \in I \).

Baer worked with what he called "complete" modules over a ring R, namely modules M such that every homomorphism from a one-sided ideal of R to M extends to a homomorphism from R to M. In 1940, Baer [12; Theorem 3], proved that every module is a submodule of a complete module and that a module is complete if and only if it is
a direct summand of every module that contains it [12, Theorem 1]. In
Eckmann and Schopf [29, 2] had introduced the method we have
given for embedding modules into injectives.

In 1953 Eckmann and Schopf (29, 4), had first developed the
concept of injective envelope (hull). The name "injective envelope"
appeared in a paper of Matlis [103, p. 512], and the name "injective
hull" in a paper of Rosenberg and Zelinsky [139, p. 373].

In 1959 Papp [136, Theorem 1] and 1962 Bass [14, Theorem 1.1]
had proved independently that a ring R is right noetherian if and only
if every direct sum of injective right R-modules is injective.

Some basic result concerning to injective and projective modules
which are repeatedly used are as follows.

(1) A direct sum of left R-modules is projective if and only if
each summand is projective.

(2) A direct product of left R-modules is injective if and only if
each factor is injective.

(3) In [52, Theorem 4.4] Every module is a submodule of an
injective module.

(4) Every module is a quotient of a projective module.

Further these two concepts are interconnected as follows by
characterising left hereditary ring.
Theorem [19]:

For a ring $R$ the following statements are equivalent:

1. $R$ is left hereditary.
2. Each submodule of a projective left $R$-module is projective.
3. Each quotient of an injective left $R$-module is injective.

In 1964, J.P. Jans, gave the following results

Theorem [86]:

Let $M$ be a left $R$-module. Then following statements are equivalent:

1. $P$ is projective.
2. Every exact sequence $\frac{R}{M} \rightarrow P \rightarrow 0$ splits.
3. $P$ is direct summands of a free left $R$-module $F$.

The concept of $M$-injective and $M$-projective modules have been introduced by Sandomierski in his Ph.D. thesis in 1964, and independently by E. de Robert in 1969.

In 1953 Eidenberg-Cartan defined the simple and semisimple modules and characterised semi simple rings in terms of injective and projective modules. It is clear by the following theorem that over semisimple rings projective and injective modules are precisely the same.
Theorem [19]:

For each ring $R$ (with unit element $1 \neq 0$) the following properties are equivalent:

1. $R$ is semi-simple as a left $R$-module.
2. Every left ideal of $R$ is a direct summand of $R$.
3. Every left ideal of $R$ is injective.
4. All left modules over $R$ are semi-simple.
5. All exact sequences

$$0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$$

of left $R$-modules split.

6. All left $R$-modules are projective.
7. All left $R$-modules are injective.

In [8] Azumaya and Varadarajan proved that the direct sum of any family of $M$-injective modules is $M$-injective if and only if every cyclic submodule of $M$ is noetherian.

The concept of coflat modules is first given by R.F. Damiano [26] in 1979. He has shown that these modules are naturally dual to flat modules. It is proved that the $\psi$-Baer criterion provides a
characterisation of coflat modules dual to the characterisation of flat modules.

Eilenberg characterised semi-simple rings as rings over which every module is projective or equivalently, every module is injective. An analogous characterisation holds for rings in which every module is flat or for rings over which every module is coflat.

**Theorem [26]:**

For a ring R, the following are equivalent:

1. R is von Neumann regular.
2. Every left (right) R-module is flat.
3. Every left (right) R-module is coflat.

In 1973 C. Webb [174] has shown that if $M_R$ is a R-module and $N_i$ is a family of left R-modules then it is always true that the groups $M \bigotimes_R (\bigoplus_i N_i)$ and $\bigoplus_i (M \otimes_R N_i)$ are isomorphic. However, it is not hard to see that the groups $M \bigotimes_R \prod_i N_i$ and $\prod_i (M \otimes_R N_i)$ are not necessarily isomorphic. C. Webb gave the following results:

1. If $M$ is finitely generated and projective then $\varphi$
2. $F$ is surjective (for all families $(N_i)$) if and only if finitely generated).
3. Suppose $I$ is a set and $M_i$ is a right R-module for every $i \in I$. Let $M \cong \bigoplus M_i$. Then $\varphi$ is injective for $M$ if and only if $\varphi$ is injective for every $M_i$. If $M'$ is pure in $M$ and $\varphi$ is injective for $M$ the $\varphi$ is injective for $M'$. 
This thesis is mainly concerned with the study of coflat module (generalization of injective module) and various idempotents such as regular idempotent and local idempotent in the endomorphism ring of on R-modules. We also study the natural homomorphism

$$
\phi_M : U \otimes_R \text{Hom}_R (M, N) \rightarrow \text{Hom}_U (U \otimes_R M, U \otimes_R N)
$$

define by

$$
\phi_M (a \otimes f) ((b \otimes x)) = ab \otimes f(x)
$$

for all \( a, b \in U, \quad x \in M, \quad f \in \text{Hom}_R (M, N) \).

A brief abstract of material included in the present thesis is given below:

Chapter 'I' is consists of definitions and basic theorems which are used in successive chapters. Mainly we have given the definitions of sub-module, quotient module, cyclic module, annihilator of R-module, module homomorphism, indecomposable module, endomorphism ring, exact sequence, commutative diagram, direct sum and direct product, free and finitely generated module, finitely presented module, group of homomorphism, tensor product, direct summand, injective module, projective module, injective hull (injective envelope), projective cover, M-injective module, M-projective module, flat module, simple module, simple ring, ideal, maximal ideal, jacobson radical, idempotent, regular module, regular ring, nil, nilpotent, T-nilpotent, perfect ring, semi-perfect ring, noetherian
module, noetherian ring, artinian module, division ring, local ring, semi-local ring, hereditary module.

Chapter 'II' is equipped with following out comes :

**Theorem 1:**

Any right R-module M is coflat if and only if :

1. \( l_M (A \cap B) = l_M (A) + l_M (B) \)
   
   for \( A \) and \( B \) are finitely generated right ideals of \( R \).

2. \( l_M r_R (a) \subseteq Ma, \) for all \( a \in R \)

**Theorem 2:**

Following conditions are equivalent for a right coflat ring \( R \)

1. Every finitely generated right ideal is projective.

2. Every quotient of right coflat R-module is right coflat.

3. Every finitely generated right ideal is projective relative to right coflat R-module.

**Theorem 3:**

Following conditions are equivalent :

1. Every finitely generated right ideal of \( R \) is projective.

2. Every quotient of right coflat module is right coflat.

3. Every quotient of an injective module is right coflat.

Chapter 'III' is concerned with the natural homomorphism

\[ \varphi_M : U \otimes \text{Hom}_R (M, N) \rightarrow \text{Hom}_U (U \otimes M, U \otimes N) \]
define by

\[ \phi_M (a \otimes f) \{(b \otimes x)\} = ab \otimes f(x) \]

for all \( a, b \in U, \quad x \in M, \quad f \in \text{Hom}_R(M, N) \).

Various properties of \( \phi_M \) are given. The result of this chapter are as follows.

**Theorem 1:**

If \( M \) is finitely generated and free \( R \)-module, then \( \phi_M \) is an isomorphism.

**Theorem 2:**

If \( M = \bigoplus_{i \in I} M_i \) be any \( R \)-module. Then \( \phi_M \) is monomorphism if and only if \( \phi_{M_j} \) is monomorphism for all \( j \).

**Corollary 3:**

For any free module \( M = \bigoplus_{i \in I} R_i \), \( \phi_M \) is always monomorphism.

**Corollary 4:**

\( M \) be any projective module, then \( \phi_M \) is a monomorphism.

**Theorem 5:**

For any \( R \)-algebra \( U \), following conditions are equivalent:

1. \( \phi_M : U \otimes_R \text{Hom}_R(M, N) \to \text{Hom}_U (U \otimes_R M, U \otimes_R N) \) is an isomorphism for any free module \( M \).
(2) \( g : U \otimes \prod_{i \in I} N_i \to \prod_{i \in I} (U \otimes N_i) \) is an isomorphism, where \( N_i \cong N \ \forall \ i \in I \).

(3) \( U \) is finitely presented.

(4) \( U \) is a module-finite \( R \)-algebra (i.e. \( U \) is finitely generated \( R \)-module).

**Corollary 6:**

For any \( R \)-algebra \( U \),

\[
g : U \otimes \prod_{i \in I} N_i \to \prod_{i \in I} (U \otimes N_i)
\]

is always monomorphism where \( N_i \cong N \).

**Theorem 7:**

Let \( U_R \) be any finitely presented module then followings are equivalent:

(1) \( U \) is \( M \)-flat.

(2) \( U \) is strongly \( M \)-flat.

**Theorem 8:**

Let \( U \) be any commutative and finitely generated \( R \)-algebra, then the natural homomorphism \( \varphi_U \) is an isomorphism if and only if \( R \) is a noetherian ring.
Theorem 9:
If $P_i$ be any finitely generated and projective module then $\varphi_p$ is an isomorphism.

In chapter "IV" we define an regular idempotent with example. Following results of this chapter are as follows.

Theorem 1:
Every primitive and left regular idempotent is a left irreducible idempotent.

Corollary 2:
If $R$ is a semi perfect ring such that every primitive idempotent is a regular idempotent then $R$ is a semi simple ring.

Theorem 3:
Any idempotent $e$ is a left regular if and only if homomorphic image of left $R$-module generated by $e$ is flat

Theorem 4:
Let $e_1$ and $e_2$ are orthogonal idempotents. Then $e_1+e_2$ is a left regular idempotent if and only if each $e_i$ are left regular ($i=1,2,..$)

Corollary 5:
If $e$ is left regular idempotent. Then $J(Re) = 0.$
Chapter "V" is concerned with the local idempotents. The results of this chapter are as follows.

**Theorem 1:**

Let $e$ be any non-trivial idempotent in a ring $R$. Then following statements are equivalent:

1. $e$ is local idempotent
2. $eJ$ is maximal submodule of $eR$ which contains every proper submodule of $eR$.
3. Every homomorphic image of $eR$ is indecomposable
4. $eR$ is indecomposable and every homomorphic image of $eR$ has a projective cover.
5. $eR$ is a projective cover of each of its nonzero homomorphic images
6. $eR = P(R/L)$ for some maximal right ideal $L$ of $R$
7. $J(eR)$ is small in $eR$ and $eR/J(eR) = eR/eJ$ is a simple $R$-module.

**Theorem 2:**

Let $R$ by any semi simple artinian ring. Then for a primitive idempotent $e$ following conditions are equivalent :

1. $e$ is an irreducible idempotent.
2. $e$ is local idempotent.
3. $e$ is regular idempotent.