CHAPTER IV

On Regular Idempotents

It e by any idempotent in a endomorphism ring End \((M_R)\) of \(M\), then we can write

\[
M = eM \oplus (1-e) M.
\]

Conversely if \(M = M_1 \oplus M_2\) be any direct decomposition of \(M\) then there exists idempotent \(e_1\) such that

\[
M_1 = e_1 M \text{ and } M_2 = e_2 M
\]

where \(e_2 = 1-e_1\) is an idempotent orthogonal to \(e_1\)

(that is \(e_1e_2 = 0 = e_2e_1\)).

It shows that properties concerning the possible direct decompositions of a module \(M\) are reflected in its endomorphism ring \(End (M_R)\), more precisely in the set of idempotents in \(End (M_R)\). Thus the study of modules with prescribed behaviour with respect to direct decomposition is part of the question what kind of rings can be realized as endomorphism rings of modules and number of idempotents in it.

In a commutative ring \(R\), whenever we have an idempotent \(e\), the ring \(R\) decomposes into a direct product of the two rings \(R.e\) and \(R. (1-e)\). For many considerations in commutative ring theory, we can often restrict our attention to rings \(R\) which are indecomposable (or
connected), that is R has only the trivial idempotents 0 and 1. For non commutative rings, these remarks remain valid if we replace the word “idempotent” every where by “central idempotent”. For these rings, there may be many non-trivial noncentral idempotents. To understand the structure of these rings, it is often important to study the behaviour of their idempotents.

For any idempotent e in any ring R, we have the following three Benjamin O. Peirce decompositions:

\[
\begin{align*}
(1) & \quad R_R = Re \oplus R (1-e) \\
(2) & \quad R_R = eR \oplus (1-e)R \\
(3) & \quad R = eRe \oplus eR (1-e) \oplus (1-e) Re \oplus (1-e) R (1-e) \\
\text{(as an additive subgroup)}
\end{align*}
\]

The peirce decomposition can be easily illustrated through the complete matrix ring \( R = M_n (S) \), where S is any ring. If \( 1 < r < n \) be any integer and e is the idempotent matrix diagonal \((1, 1, \ldots, 1, 0, \ldots, 0)\) with \( r \) ones, with the complementary idempotent \((1-e) = \text{diagonal}\) \((0, \ldots, 0, 1, 1, \ldots, 1)\) with \( n-r \) ones, then an easy computation shown that.

\[
\begin{align*}
eRe &= \begin{bmatrix} * & 0 \\ 0 & 0 \end{bmatrix}, & eR (1-e) &= \begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix}
\end{align*}
\]
\[(1- e) \text{Re} = \begin{bmatrix} 0 & 0 \\ * & 0 \end{bmatrix}, \quad (1- e) \text{R} (1- e) = \begin{bmatrix} 0 & 0 \\ 0 & * \end{bmatrix}\]

Where \((*)\) denote, respectively blocks of sizes \(r \times r\), \(r \times (n-r)\), \((n-r) \times r\) and \((n-r) \times (n-r)\).

Keeping above view in the mind, in this chapter we study the regular idempotents which is defined as

**Definition 4.1:**

Let \(R\) be any ring and \(e\) be an idempotent in \(R\) is said to be right regular idempotent if every cyclic submodule of \(eR\) is a direct summand of \(eR\).

Similarly we define, an idempotent \(e\) in \(R\) is said to be left regular idempotent if every cyclic submodule of \(Re\) is a direct summand of \(Re\).

**Definition 4.2:**

A non zero left ideal \(I\) in a ring \(R\) is called simple if it is simple as a left \(R\)-module. A simple left ideal is some times called a minimal left ideal since it contains no submodules other than \(0\) and itself.

**Example 4.3:**

Any idempotent of a von Neumann regular ring is a regular idempotent.
Proof:

Let $R$ be any von Neumann regular ring that is for every $r \in R$ there exists $x \in R$ such that $r = rxr$, here $rx$ is an idempotent as $e = rx = rxrx = e^2$. Similarly $xr$ is also idempotent in $R$. Since $Rr = Rrxr \subseteq Rxr \subseteq Rr$, therefore $Rr = Rxr = Re$, that is any principal ideal of $R$ is generated by idempotent and hence it will be direct summand therefore any cyclic left $R$-module will be direct summand of $Re$ whenever $r \in Re \subseteq R$.

Example 4.4 :-

Let $R = \sum_{i=1}^{n} De_{ii} + \sum_{j=2}^{n} De_{ij}$ where $D$ is a division ring and $e_{ij}$ denote the matrix unit whose $(i,j)^{th}$ place is 1 and rest are zero. Thus $R$ is a subring of $n \times n$ upper triangular matrices having all the entries zero except possibly the diagonal and first row.

It is clear that $J = \text{Rad } R = \sum_{j=2}^{n} e_{ij}D$ this implies $(\text{Rad } R)^2 = 0$.

Since the equation $e_{1n}x e_{1n} = e_{1n}$ has no solution for $x$ in $R$, therefore $R$ is not regular ring. Since $J(e_{i}R) = 0, \forall i \geq 2$ where $e_{i} = e_{ii}$, therefore $e_{i}R \cong$ Simple right $R$-module $S_{i}$, for all $i \geq 2$

$e_{i}Re_{i} = De_{ii} \cong D$

$e_{1}R = \sum_{j=1}^{n} e_{1j}D$
\[ e_1 J = J = \bigoplus_{j=2}^n e_{1j} D \cong S_2 \oplus S_3 \oplus \ldots \oplus S_n \]

Hence \( e_{22}, e_{33}, \ldots \ldots \ldots e_{nn} \) are all right regular idempotents and \( e_{11} \) is not a right regular idempotent as the cyclic submodule \( e_1 J \subset e_1 R \) is not a direct summand of \( e_1 R \) which is indecomposable because its endomorphism ring is local. Similarly we can show that \( e_{11} \) is a left regular idempotent and \( e_{22}, \ldots \ldots \ldots, e_{nn} \) are not left regular idempotents.

**Example 4.5:**

Let \( F \) be any infinitely generated free \( R \)-module and \( S=\text{Hom}_R(F,F) \). Choose a basis \( \{x_i\}_{i \in I} \) for \( F \), i.e. \( F=\bigoplus_{i \in I} Rx_i \). Consider the projection mappings \( p_i : F \rightarrow Rx_i \subset F \). Clearly \( p_i \)'s are orthogonal idempotents \( \forall i \in I \) in \( S \) and hence \( P=\bigoplus_{i \in I} p_i S \) is a projective right \( S \)-module.

Corresponding to this projective module \( P \) there exist an idempotent \( e \) in \( \text{Hom}_S(G,G) \) such that \( P= e\text{Hom}_S(G,G) \), where \( G=\bigoplus_{j \in J} S_j \) is a free \( S \)-module. Clearly this \( e \) is an regular idempotent which is not irreducible idempotent. Also \( \text{Hom}_S(G,G) \) in not a regular ring.

**Definition 4.6:**

An idempotent \( e \) (\( \neq 0 \)) is right (resp. left) irreducible if \( eR \) (resp. \( Re \)) is a minimal right (resp. left) ideal of \( R \). [99, Page 323].
Theorem 4.7 :-

Every primitive and left regular idempotent is a left irreducible idempotent.

Proof :-

Let $e$ be a primitive idempotent then $Re$ is indecomposable that is $Re$ has no non-trivial proper direct summand.

Since $e$ is left regular idempotent that is every cyclic submodules of $Re$ is a direct summand and hence $Re$ has no proper submodule this shows that $Re$ is a minimal left ideal that is $e$ is irreducible.

Corollary 4.8:-

If $R$ is a semi perfect ring such that every primitive idempotent is a regular idempotent then $R$ is a semi simple ring.

Proof :-

Since $R$ is semi perfect ring hence we can write

$$R = \bigoplus_{i=1}^{n} e_i R$$

Where $e_i$ (i= 1, 2, 3, ...........n) is a complete set of orthogonal primitive idempotents.

Let $A = \{ e_i R \mid i = 1, 2, 3, ........m, m \leq n \}$ be a complete set of representatives of the indecomposable projective right $R$- modules. Here $e_i$'s are primitive and every primitive are regular then by above
theorem $e_i R$ will be irreducible (Simple) $R$-module. Therefore $R = \oplus_{i=1}^n e_i R$
is a semisimple ring.

**Lemma 4.9:**

Every irreducible idempotent will be regular idempotent

**Proof:**

It is obvious from definitions.

**Lemma 4.10:**

Let $R$ be a ring and $0 \rightarrow K \rightarrow P \rightarrow M \rightarrow 0$ an exact sequence of a
$R$-modules with $P$-projective. Then the following statements are equivalent:

(i) $M$ is flat

(ii) Given any $x \in K$, there exists a homomorphism $g : P \rightarrow K$ such that $xg = x$

(iii) Given any $x_1, x_2, \ldots, x_n$ in $K$, there exists a homomorphism $g : P \rightarrow K$ such that $x_i g = x_i$ for $i = 1, 2, \ldots, n$

**Proof:**

Please see in [172, Lemma 2.2]

**Proposition 4.11:**

Any idempotent $e$ is a left regular if and only if homomorphic
image of left $R$-module generated by $e$ is flat.
Proof:

let e be left regular idempotent then by definition every cyclic submodule of Re is a direct summand. Let

$$0 \rightarrow K \rightarrow Re \rightarrow M \rightarrow 0$$

be any exact sequence of left R-modules and left R-homomorphisms. Since for every $x \in K \subset Re$, Rx is a direct summand of Re. So there exists $g : Re \rightarrow K$ such that $g(x) = x$, therefore by Lemma 4.10, this implies that M is flat.

Conversely:

Suppose homomorphic image of Re is flat, then by above Lemma for any exact sequence of left R-modules and left R-homomorphism

$$0 \rightarrow Rx \rightarrow Re \rightarrow M \rightarrow 0$$

there exists $g : Re \rightarrow Rx \subset Rx$, such that $g(x) = x$, for all $x \in Re$ that is Rx is a direct summand of Re.

Lemma 4.12:

Let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be an exact sequence of left R modules:

(1) If B and C are flat, then A is flat

(2) If A and C are flat then B is flat
Lemma 4.13:

Let $e_1$ and $e_2$ are orthogonal idempotents in a ring $R$, then

$$R e_1 + R e_2 = R e_1 \oplus R e_2 = R (e_1 + e_2)$$

Proof:

Please see in [5, Ex. 7.6, Page 102]

Theorem 4.14:

Let $e_1$ and $e_2$ are orthogonal idempotents. Then $e_1 + e_2$ is a left regular idempotent iff each $e_i$ are left regular ($i=1,2,..$)

Proof:

Suppose $e_1 + e_2$ is a Left regular idempotent, to show $e_i$ ($i=1,2$) are regular. Taking the exact sequence

$$Re_i \longrightarrow M \longrightarrow 0 \quad (i=1,2,....)$$

this can be written as by above Lemma 4.13

$$R(e_1 + e_2) = Re_1 \oplus Re_2 \longrightarrow Re_1 \longrightarrow M \longrightarrow 0$$

Or

$$R(e_1 + e_2) = Re_1 \oplus Re_2 \longrightarrow Re_2 \longrightarrow M \longrightarrow 0$$

In both the cases $M$ being homomorphic image of $R(e_1 + e_2)$ and hence $M$ is flat by assumption that $e_1 + e_2$ is regular this implies $e_1$ and $e_2$ are left regular idempotents
Coversely :-

Let each $e_i$ are regular idempotents then homomorphic images of $Re_i$ (i= 1,2,) are flat by [proposition 4.11]

Let $R(e_1 + e_2) = Re_1 \oplus Re_2 \xrightarrow{f} M \xrightarrow{0}$ be an exact sequence, then $M= f(Re_1) + f(Re_2)$,

We get the exact sequences

\[ 0 \rightarrow f(Re_1) \cap f(Re_2) \rightarrow f(Re_1) + f(Re_2) \rightarrow \frac{f(Re_1)}{f(Re_1) \cap f(Re_2)} \oplus \frac{f(Re_2)}{f(Re_1) \cap f(Re_2)} \rightarrow 0 \]

and

\[ 0 \rightarrow f(Re_1) \cap f(Re_2) \rightarrow f(Re_i) \rightarrow \frac{f(Re_i)}{f(Re_i) \cap f(Re_2)} \rightarrow 0 \ (i = 1,2) \]

Implies that $f(Re_1)$, $f(Re_2)$, $\frac{f(Re_1)}{f(Re_1) \cap f(Re_2)}$, $\frac{f(Re_2)}{f(Re_1) \cap f(Re_2)}$ are flats

Thus \[ \frac{f(Re_1)}{f(Re_1) \cap f(Re_2)} \oplus \frac{f(Re_2)}{f(Re_1) \cap f(Re_2)} \]

is flat, therefore by [Lemma 4.12], then $M= f(Re_1) + f(Re_2)$ is flat that is $(e_1 + e_2)$ is a left regular idempotent.

**Corollary 4.15** :

If $e$ is left regular idempotent. Then $J(Re) = 0$.

**Proof**:

Jacobson radical of $Re$ is the sum of small submodules. Every cyclic submodule of $Re$ is direct summand and direct summand cannot be a small submodule hence $J(Re)=0$