CHAPTER V

ON THE $F_A$-SUMMABILITY OF THE SEQUENCE

OF FOURIER COEFFICIENT

5.1. Given an infinite matrix $A = (a_{n,k})$ of real or complex numbers and let $\{S_k\}$ be any sequence of complex numbers. The sequence $\{t_n\}$ define by,

$$(5.1.1) \quad t_n = \sum_{k=0}^{\infty} a_{n,k} S_k ;$$

is called an $A$-transform of $\{S_k\}$ whenever the series converges for $n = 0,1,2, \ldots$. The sequence $\{S_k\}$ is said to be $A$-summable to $S$ if $\{t_n\}$ converges to $S$.

A bounded sequence $\{S_k\}$ is said to be $F_A$-summable to the limit $S$ if,

$$A_{n,p} = \sum_{k=0}^{\infty} a_{n,k} S_{k+p} ;$$

tends to $S$, as $k \to \infty$, uniformly in $p = 0,1,2, \ldots$.

If,

$$a_{n,k} = \frac{1}{n}, \quad k = 0,1,2,3, \ldots, \quad (n-1)$$

$$= 0, \quad k \geq n,$$

1) Lorentz, G.G. (1)
$F_A$-summability is the same as what is generally known as almost convergence of the sequence $\{S_p\}$\(^2\). Moreover if we take $p=0$, the above definition reduce to that of $A$-summability.

A sequence $\{S_p\}$ is called almost $A$-summable if $A$-transform of $\{S_p\}$ is almost convergent\(^3\).

If we superimpose the method $F_A$ on the Cesàro-mean of order one, we obtain another method of summation viz. $F_A-(C,1)$. Further, if we put,

$$p = o,$$

and

$$a_{n,k} = \frac{1}{n}; \quad (k \leq n)$$

$$= 0; \quad (k > n),$$

we get the well-known method of summation by arithmetic means or $(C,1)$summability.

5.2. Let $f(t)$ be a periodic function with period $2\pi$ and integrable in the sense of Lebesgue over $(0,2\pi)$.

2) Lorentz, G. G. (1)

3) King, J. P. (1)
Let,

\[ \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt), \]

be the Fourier series of \( f(t) \), then the conjugate series of \( (5.2.1) \) is

\[ \sum_{n=1}^{\infty} (b_n \cos nt - a_n \sin nt) = \sum_{l=1}^{\infty} B_n(t). \]

We write,

\[ \Psi(t) = \{ f(x+t) - f(x-t) - L \}, \]

\[ G_k(t) = \left( \frac{\sin kt}{kt^2} - \frac{\cos kt}{t} \right), \]

\[ G_{k+p}(t) = \frac{\sin (k+p)t}{(k+p) t^2} - \frac{\cos (k+p)t}{t}, \]

\[ H_{k+p}(t) = \sum_{m=1}^{k+p} g_m(t). \]

5.3. Fejér\(^4\) has shown that if \( \lambda = \{ f(x+) - f(x-) \} \)

exists and is finite, then the sequence \( \{ nB_n(x) \} \) is summable

\( C, r \), \( r > 1 \), to the value \( \frac{\lambda}{2} \), and if \( \Psi(t) \) is of bounded

variation, the theorem holds true for \( r > 0 \). This theorem was

extended for fractional integral by Chow\(^5\), Siddiqi\(^6\) for the

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4) Fejér, L. (1)
5) Chow, H.C. (1)
6) Siddiqi, J.A. (3)
first time extended the result of Fejér to the matrix summability. Further he\textsuperscript{7}) has given a necessary and sufficient condition on the matrix-$A$ for the validity of his theorem and derived certain consequences for the Fourier coefficient of continuous functions of bounded variation.

Mazhar and Siddiqi\textsuperscript{3}) obtained a necessary and sufficient conditions in order that the sequence \( \{ nB_n(x) \} \) be $F_A$-summable or in order that it be $F_B$-summable to $\frac{j}{\pi}$.

Further, they\textsuperscript{9}) have obtained necessary and sufficient conditions in order that the sequence \( \{ nB_n(x) \} \) be almost $A$-summable to $\frac{j}{\pi}$.

In fact they proved the following theorem,

**Theorem** \textsuperscript{10}) : If,

\[
A = (a_{n,k}) \text{ is regular, then for } f(x) \in B.V.(0,2\pi)
\]

and for every $x \in (0,2\pi)$ the sequence \( \{ nB_n(x) \} \) is $F_A$-summable to the limit $\frac{j}{\pi}$.

If and only if,

\[
\lim_{n \to \infty} \sum_{k=0}^{\infty} a_{n,k} \cos (k+p) t = 0,
\]

uniformly in $p$ in every interval $0 < \delta \leq t \leq \pi$.

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7) Siddiqi, J.A. (2)
8) Mazhar, S.M. and Siddiqi, A.H. (1)
9) Mazhar, S.M. and Siddiqi, A.H. (2)
10) Mazhar, S.M. and Siddiqi, A.H. (1)
It may be remarked that theorem M includes as a particular case for \( p = 0 \) and \( a_{n,k} = 0, \ k > n \) the following theorem of Siddiqi\(^{11}\).

**Theorem B.** If,

\[
A = (a_{n,k}) \text{ be regular, then for every } f f(x) \in \mathcal{B.V.}
\]

\((0,2\pi) \) and for every \( x \in (0,2\pi) \);

\[
\lim_{n \to \infty} \sum_{k=0}^{n} a_{n,k} k B_k(x) = \frac{j}{\pi},\]

if and only if,

\[
\lim_{n \to \infty} \sum_{k=0}^{n} a_{n,k} \cos kt = 0,
\]

in every \( 0 < \delta < t < \pi \).

For the first time Sharma\(^{12}\) has studied the product summability \( A^4(C,1) \) of the sequence \( \{ nB_n(x) \} \) and proved:

**Theorem C\(^{13}\):** If,

\[
(5.3.1) \quad \psi(t) = 0 \quad \left( \log \frac{1}{t} \right)^{-1}, t \to 0,
\]

and for some \( r \) with \( 0 < r < 1 \),

\[
\lim_{n \to \infty} \sum_{k=0}^{n} k^r | \triangle \lambda_{n,k} | = o(1),
\]

\[\hline\]

11) Siddiqi, J.A. (2)

12) Sharma, P.L. (1)

13) Sharma, P.L. (2)
then the sequence \( \{ nB_n(x) \} \) is summable \( A^* (C, 1) \) to the sum

\[
\frac{1}{\pi} \Delta \lambda_{n, k} = (a_{n, k} - a_{n, k+1}).
\]

It may be remarked that the regular matrices which satisfy the condition (5.3.1) with \( r = 0 \) are called strongly regular.\(^{14}\)

Sharma\(^{15}\) has further generalized theorem C to include a theorem of Varshney\(^{16}\) on Harmonic summability by proving:

Theorem D. If,

\[
(5.3.2) \quad \int_0^t | \Psi(t) | \, dt = o \left( \frac{t}{\log \frac{1}{t}} \right), \quad t \rightarrow 0,
\]

and for some \( r \) with \( 0 < r < 1 \),

\[
(5.3.3) \quad \lim_{n \rightarrow \infty} \sum_{k=0}^{n} k^r | \Delta^2 a_{n, k} | = o(1),
\]

then the sequence \( \{ nB_n(x) \} \) is summable \( A^* (C, 1) \) to the sum \( \frac{1}{\pi} \),

where

\[
\Delta^2 a_{n, k} = \Delta (a_{n, k} - a_{n, k+1}).
\]

\(^{14}\) Lorentz, G.G. (1)  
\(^{15}\) Sharma, P.L. (1)  
\(^{16}\) Varshney, O.P. (1)
Kashroo\textsuperscript{17) } generalise Theorem C and D by proving

Theorem K. If,

\begin{equation}
\int_0^t |\psi(t)| \, dt = o\left(\frac{t}{\log \frac{1}{t}}\right), \text{ as } t \to 0,
\end{equation}

and for some \( r \) with \( 0 < r < 1 \),

\begin{equation}
|a_n^m - a_{n+1}^m| = o(1)
\end{equation}

then the sequence \( \{ n B_n(x) \} \) is summable \( A-(C,1) \) to the sum \( \frac{\ell}{\pi} \).

Here, we shall prove

Theorem II. If,

\begin{equation}
\int_0^t |\psi(t)| \, dt = o\left(\frac{t}{\log \frac{1}{t}}\right), \text{ as } t \to 0,
\end{equation}

and for some \( r \) with \( 0 < r < 1 \),

\begin{equation}
\sum_{n} (k+p)^r |a_n^k - a_{n+k+1}^k| = o(1),
\end{equation}

uniformly with respect to \( p \), then the sequence \( \{ n B_n(x) \} \) is summable \( F_A-\cdot(C,1) \) to the sum \( \frac{\ell}{\pi} \).

\textsuperscript{17) Kashroo (Thesis) (1)
5.4. Proof of the theorem

If we denote the \((C,1)\) transform of the sequence \(\{nB_n(x)\}\) by \(a_n\), we have after Mohanty and Nanda\(^{18}\).

\begin{equation}
\alpha_n - \frac{l}{\pi} = \frac{1}{\pi} \int_0^5 \psi(t) \left[ \frac{\sin nt}{nt^2} - \frac{\cos nt}{t} \right] dt + o(1),
\end{equation}

by Riemann–Lebesgue theorem, where \(\delta\) is constant and greater than zero.

On account of the regularity of the \(A\)-method of summation, we need only to prove that

\begin{equation}
I = \frac{1}{\pi} \sum \lim n \to \infty a_n, k \int_0^5 \psi(t) \varphi_{k+p}(t) dt
= o(1),
\end{equation}

as \(n \to \infty\) uniformly with \(p = 0, 1, 2, \ldots\),

where

\begin{equation}
\varphi_{k+p} = \left( \frac{\sin (k+p)t}{(k+p)t^2} - \frac{\cos (k+p)t}{t} \right).
\end{equation}

We require the following estimates which can be easily obtained by expanding sine and cosine in power of \((k+p)\) and \(t\):

\(^{18}\) Mohanty, R. and Nanda, M. (1)
\begin{equation}
\varrho_{(k+p)}(t) = O\left( (k+p)^2 t \right)
= O(k+p)
\end{equation}

and

\begin{equation}
\varrho_{(k+p)}(t) = O(t^{-1}).
\end{equation}

It is also known that 19)

\begin{equation}
\sum_{\mu=1}^{(n+p)} \frac{\sin \mu t}{\mu} = o(1).
\end{equation}

Now,

\begin{equation}
|I| = \left| \frac{1}{\pi} \sum a_{n,k} \times \right.
\left. \left\{ \int_0^{(k+p)^{-1}} + \int_{(k+p)^{-1}}^{(k+p)^{-1}} + \int_{(k+p)^{-1}} \right\} \times \right.
\left. \varrho_{(k+p)}(t) \psi(t) \, dt \right|
\end{equation}

\begin{equation}
= \left| \frac{1}{\pi} \sum a_{n,k} (P + Q + R) \right|.
\end{equation}

Using (5.4.4) we get

\begin{equation}
|P| = O(n) \int_0^{(k+p)^{-1}} \left| \psi(t) \right| \, dt
= o(1),
\end{equation}

by the hypothesis (5.3.6)

Also, with (5.3.6) and (5.4.5) we write

\[ |\Omega| = O(1) \left( \int_{(k+p)^{-1}} \frac{|\Psi(t)|}{t} \, dt \right) \]

\[ = O(1) \left\{ \left[ \frac{\Psi(t)}{t} \right]^{(k+p)^{-x}} \right\}^{(k+p)^{-1}} \]

\[ + \int_{(k+p)^{-1}} \frac{\Psi(t)}{t^2} \, dt \}

\[ = o(1) + o(1) \left( \int_{(k+p)^{-1}} \frac{dt}{t \log \frac{1}{t}} \right) \]

\[ = o(1). \]

Thus the first two terms in (5.4.7) can be made as small as we please by choosing \( n \) sufficiently large, uniformly with respect to \( p \).

With the help of (5.4.3) and (5.4.5), we write

\[ G(k+p)(t) = \sum_{\nu=1}^{(k+p)} g_{\nu}(t) \]

\[ = \frac{1}{t^2} \sum_{\nu=1}^{(k+p)} \frac{\sin \nu t}{\nu} - \frac{1}{t} \sum_{\nu=1}^{(k+p)} \cos \nu t \]
\[ = o(\frac{1}{t^2}) - \frac{1}{t} \cdot o(\frac{1}{t}) \]
\[ = o\left(\frac{1}{t^2}\right). \]

It is easy to see that \( \Sigma |a_{n,k}| < \infty \) and

\[ \Sigma (k+p)^T |a_{n,k} - a_{n,k+1}| < \infty, \]

imply that \((k+p)^T a_{n,k} \to 0\), hence using Abel–transformation, we write

\[ \left| \frac{1}{\pi} \sum_{l=1}^{n} a_{n,k} \right| \]
\[ = \left| \frac{1}{\pi} \sum_{l=1}^{n} a_{n,k} \int_{(k+p)^T}^{5} \Psi(t) \left[ G(k+p)(t) - G(k+p-1)(t) \right] dt \right| \]
\[ \leq \left[ \frac{1}{\pi} \sum_{l=1}^{n-1} (a_{n,k} - a_{n,k+1}) \int_{(k+p)^T}^{5} \Psi(t)G(k+p)(t) dt \right] \]
\[ + \left| \frac{1}{\pi} \sum_{l=2}^{n} a_{n,k} \int_{(k+p)^T}^{(k+p-1)^T} \Psi(t) G(k+p)(t) dt \right| \]
\[ + o(1) \]
\[ = R_1 + R_2, \text{ say}. \]
It follows that

\[ R_1 = O(1) \left\{ \sum_{l}^{n-1} |a_{n,k} - a_{n,k+1}| \int_{(k+p)-r}^{\delta} \frac{|\zeta(t)|}{t^2} \, dt \right\} \]

\[ = O(1) \left\{ \sum_{l}^{n-1} |a_{n,k} - a_{n,k+1}| \left\{ \frac{\zeta(t)}{t^2} \right\}^{\delta} \right\} \]

\[ + \int_{(k+p)-r}^{\delta} \frac{2\zeta(t)}{t^3} \, dt \right\} \]

\[ = O(1) \left\{ \sum_{l}^{n-1} |a_{n,k} - a_{n,k+1}| \left\{ \frac{1}{t \log \frac{1}{t}} \right\}^{\delta} \right\} \]

\[ + \int_{(k+p)-r}^{\delta} \frac{dt}{t^2 \log \frac{1}{t}} \right\} \]

\[ = O(1) \left\{ \sum_{l}^{n-1} |a_{n,k} - a_{n,k+1}| \right\} \]

\[ + o(1) \left\{ \sum_{l}^{n-1} \frac{(k+p)^r}{r \log (k+p)} \right\} a_{n,k} - a_{n,k+1} \right\} \]

\[ + o(1) \left\{ \frac{1}{\log \delta} \sum_{l}^{n-1} \left\{ (k+p)^r - \frac{1}{\delta} \right\} \right\} X \]

\[ |a_{n,k} - a_{n,k+1}| \right\} \]

\[ = o(1) , \text{ by } (5.3.7). \]

Moreover,
\[ R_2 = o(1) \left\{ \sum_{n} a_{n,k} \mid \frac{(k+p-1)^{-r}}{r \log (k+p-1)} \left\{ \frac{1}{r \log (k+p-1)} \right\} \right\} \]

\[ = o(1) \left\{ \sum_{n} a_{n,k} \mid \left\{ \frac{1}{r \log (k+p-1)} \right\} \left\{ (k+p)^r - (k+p-1)^r \right\} \right\} \]

\[ = o(1), \text{ for } 0 < r < 1, \]
since $(a_n, k)$ is regular row infinite matrix and

$$\left\{ \frac{(k+p)^r}{r \log (k+p)} - \frac{(k+p-1)^r}{r \log (k+p-1)} \right\} \text{ and}$$

$$\left\{ (k+p)^r - (k+p-1)^r \right\}, \text{ is a null sequence.}$$

This completes the proof of the theorem.