REFERENCES


108


Progressively Censored Samples from Log-Normal and Logistic Distributions

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1. INTRODUCTION AND SUMMARY:

Cohen [1] has obtained the likelihood estimates of the parameters in case of normal and exponential distributions when the samples are progressively censored. He assumed that the parameters remain the same at each stage of censoring. But there are situations in which it is reasonable to assume that the parameters of the distribution under consideration might change at each stage of censoring, e.g. the remaining items at each stage are checked and the defects are eliminated whenever possible. Srivastava [2] has treated this problem when the items under consideration come from an exponential distribution with different failure rates at different stages. If two stage progressively censored samples are considered, we have the following four types of censoring. Let \( n \) be the total number of the items under consideration. Let \( T_1 \) and \( T_2 \) be the times of censoring (\( T_2 > T_1 \)). Let \( n_i \) be the number of items failed on or before \( T_i \) and \( r_i \) be the number of items eliminated after \( T_i \) (\( i = 1, 2 \)). Then we have the following schemes of experiment when \( n \) and \( r_1 \) are given:

(i) Let \( n_i (i = 1, 2) \) and \( r_2 \) be fixed. Then \( T_i (i = 1, 2) \) are random variables which is termed as Type II censoring by Cohen [1].

(ii) Let \( n_1 \) and \( T_1 \) be fixed. Then \( T_2 \) and \( r_2 \) are random variables. The experiment is stopped at the first stage if \( T_2 \geq T_1 \) and otherwise the experiment is carried to the second stage.

(iii) Let \( T_1 \) and \( n_2 \) be fixed. Then, \( T_2 \) and \( n_1 \) are random variables. If \( n_1 \geq n - r_1 - n_2 \), the experiment is stopped at the first stage, and \( T_2 = T_1 \) and otherwise the experiment is carried to the second stage.

(iv) Let \( T_1 \) and \( T_2 \) be fixed. Then \( n_1 \) and \( n_2 \) are random variables. If \( n_1 \geq n - r_1 \), the experiment is stopped at the first stage, and otherwise the experiment is carried to the second stage. This type of censoring is termed as Type I by Cohen [1].

When there are \( k \) stages of censoring, there are in the above way \( 2^k \) types of censoring and it is difficult to consider all of them at a time. The maximum likelihood estimates for all these types will remain the same, except for variances. Hence, we are considering here the last type of censoring for log-normal and logistic distributions and obtain the maximum likelihood estimates of the parameters involved in each stage of censoring. It is noted that due to different...
parameters at different stages, it leads us to estimate the parameters from truncated censored distributions.

2.1. The Log-normal Distribution

In this section, we shall consider the type I and type II censorings only. Let \( F(x) \) be the distribution of the random variate \( X \), the life of an item. Let \( n \) be the number of items failed at the \( i \)-th stage and let \( x_i^{(1)}, x_i^{(2)}, \ldots, x_i^{(k)} \) be the observations, \( i = 1, 2, \ldots, k \). Let \( r_i \) be the number of items eliminated from the experiment immediately after \( i \)-th stage. For the type I censoring the likelihood is

\[
L \propto \prod_{i=1}^{k} \left\{ \prod_{j=1}^{n_i} f(x_i^{(j)}) \right\} \prod_{i=1}^{k} \left[ 1 - F(T_i) \right]^{r_i}
\]

where \( T_i (i = 1, 2, \ldots, k) \) are the points of censoring and \( f(x) \) is the density function of \( X \). For the type II censoring, \( T_i = \max_{j} (x_{ji}) \). Here also, the likelihood \( L \) is given by (1) except for a constant, or for any type of censoring, it is easy to show that \( L \) is given by (1) except for a constant.

When the life of an item follows the log-normal distribution with different parameters at different stages, then it is easy to verify (see Khatri [3]) that

\[
f(x) = \begin{cases} 
\left( \frac{1}{\sigma_1} \right)^{-1} \phi \left( \frac{\log x - \mu_1}{\sigma_1} \right) & \text{for } 0 < x \leq T_1 \\
\frac{1 - \Phi (\nu_{1,i})}{1 - \Phi (\nu_{k-1,i})} \prod_{j=1}^{n_i} \left\{ 1 - \Phi (\nu_{1,j}) \right\} \left( \frac{1}{\sigma_1} \right)^{-1} \phi \left( \frac{\log x - \mu_1}{\sigma_1} \right) & \text{for } T_{i-1} < x \leq T_i (i = 2, 3, 4, \ldots, k - 1) \\
\frac{1 - \Phi (\nu_{k-1,i})}{1 - \Phi (\nu_{k-1,i})} \prod_{j=2}^{n_i} \left\{ 1 - \Phi (\nu_{k-1,j}) \right\} \left( \frac{1}{\sigma_k} \right)^{-1} \phi \left( \frac{\log x - \mu_k}{\sigma_k} \right) & \text{for } T_{k-1} < x \leq \infty 
\end{cases}
\]

or

\[
F(x) = \begin{cases} 
\Phi \left( \frac{\log x - \mu_1}{\sigma_1} \right) & \text{for } x \leq T_1 \\
\frac{1 - \Phi (\nu_{1,i})}{1 - \Phi (\nu_{k-1,i})} \prod_{j=1}^{n_i} \left\{ 1 - \Phi (\nu_{1,j}) \right\} \Phi \left( \frac{\log x - \mu_k}{\sigma_k} \right) - \Phi (\nu_{k-1,i}) & \text{for } T_{k-1} < x \leq T_k, \text{ for } k = 2, 3, \ldots
\end{cases}
\]

where

\[
y = \log x, \phi(y) = (\sqrt{2\pi})^{-1} \exp \left( -\frac{y^2}{2} \right), \Phi(y) = \int_{-\infty}^{y} \phi(t) dt \text{ and } \nu_{1,i} = (\log T_1 - \mu_1)/\sigma_1.
\]
Then (1) can be rewritten as

\[ L = L_1 \cdot L_2 \cdots L_k \]

where

\[ L_i = (\sigma_i)^{-n_i} \prod_{j=1}^{n_i} \left[ \frac{1}{\sigma_i} \left( \frac{y_{ij} - \mu_i}{\sigma_i} \right) \right] \left( 1 - \Phi(\tau_{ij}) \right)^{n_i-n_i} \]

and

\[ L_i = (\sigma_i)^{-n_i} \prod_{j=1}^{n_i} \left[ \frac{1}{\sigma_i} \left( \frac{y_{ij} - \mu_i}{\sigma_i} \right) \right] \left( 1 - \Phi(\tau_{ij}) \right)^{n_i-n_i} \cdot \left( 1 - \Phi(\tau_{i-1,j}) \right)^{-n_i} \]

We may note from (4) that \( L_i \) is proportional to a likelihood function due to a censored sample from a truncated log-normal distribution, namely

\( \phi([\log x - \mu_i]/\sigma_i)/[1 - \Phi(\tau_{i-1,j})](i = 1, 2, 3, \cdots, k) \).

If \( T_o = 0 \) and hence if \( \mu_i \)'s and \( \sigma_i \)'s are different, then we get the estimates of \( \mu_i \)'s and \( \sigma_i \)'s using the single censored results. Hence the maximum likelihood estimating equations are

\[ \frac{\partial \log L}{\partial \mu_i} = \frac{\partial \log L_i}{\partial \mu_i} = 0 \quad \text{and} \quad \frac{\partial \log L}{\partial \sigma_i} = \frac{\partial \log L_i}{\partial \sigma_i} = 0 \quad \text{for} \quad i = 1, 2, \cdots, k, \]

and we get for \( i = 1, 2, \cdots, k \)

\[ \frac{\partial \log L}{\partial \mu_i} = \frac{n_i (y_i^{(i)} - \mu_i)}{\sigma_i^2} + \left( \frac{n_i^{(i)} - n_i}{\sigma_i} \right) Z_{i,i} \cdot \left( \frac{n_i^{(i)} - n_i}{\sigma_i} \right) Z_{i,i-1,i} = 0 \]

and

\[ \frac{\partial \log L}{\partial \sigma_i} = -\frac{n_i}{\sigma_i} + \frac{n_i}{\sigma_i} \left[ s_i^2 + (y_i^{(i)} - \mu_i)^2 \right] + \left( \frac{n_i^{(i)} - n_i}{\sigma_i} \right) \mu_i Z_{i,i} \]

\[ + \left( \frac{n_i^{(i)} - n_i}{\sigma_i} \right) Z_{i-1,i} \cdot \left( \frac{n_i^{(i)} - n_i}{\sigma_i} \right) Z_{i,i-1,i} = 0 \]

where

\[ g^{(i)} = \sum_{i=1}^{n_i} y_i^{(i)}/n_i, s_i = \sum_{i=1}^{n_i} (y_i^{(i)} - g^{(i)})^2/n_i, Z_{i,i} = \phi(\tau_{i,i})/[1 - \Phi(\tau_{i,i})] \]

and

\[ Z_{i,i} = \phi(\tau_{i,i})/[1 - \Phi(\tau_{i,i})] \]

Note that \( Z_{i,i} = 0 \). \( Z_{i,i} \)'s are the reciprocal of the Mill's ratios. If \( \mu_i \)'s and \( \sigma_i \)'s are equal, then taking sum over \( i = 1, 2, \cdots, k \) in (6) and (7), we get the same result as obtained by Cohen [1]. From (3) and (7) we get

\[ g^{(i)} = \mu_i - \sigma_i \left( \frac{n_i^{(i)} - n_i}{\sigma_i} \right) Z_{i,i} - \left( \frac{n_i^{(i)} - n_i}{\sigma_i} \right) Z_{i,i-1,i} \cdot \left( \frac{n_i^{(i)} - n_i}{\sigma_i} \right) Z_{i,i-1,i} \]
and

\( s_i^2 = s_i^2[1 - n_i^{-1}(\eta_i - \eta_i)Z_{ii} + n_i^{-1}n_{i-1,i}Z_{i-1,i} - n_i^{-1}I \{ (\eta_i - \eta_i)Z_{ii} - n_i^{-1}Z_{i-1,i}\}^2] \)

for \( i = 1, 2, \ldots, k \) and \( Z_{ii} = 0 \). From (8) and (9), we get

\[
\Psi_i = \frac{s_i^2}{\rho_i^2 + (\log T_i - \hat{p}_i^{(r)})^2} = \frac{1 - \xi \eta_i - \gamma_i - \xi^2}{1 + \xi \eta_i - \eta_i} \quad \text{for} \quad i = 1, 2, \ldots, k.
\]

and

\[
\log T_i - \hat{p}_i^{(r)} = \sigma_i \xi_i, \\
n_i^{-1} \{ \log T_i - \log T_{i-1} \} Z_{i-1,i} = \sigma_i \eta_i \quad \text{for} \quad i = 1, 2, \ldots, k
\]

where

\[
\xi_i = \nu_i + (\eta_i - \eta_i)Z_{ii} - n_i^{-1}n_{i-1,i}Z_{i-1,i}, \quad \nu_i = n_i^{-1}(\eta_i - \nu_{i-1})Z_{i-1,i}.
\]

We note that for \( i = 1 \), \( \eta_1 = 0 \), and hence for \( i = 1 \), (10) and (11) are the same as those obtained by Gupta [4] and hence, we get the estimates for \( \mu_i \) and \( \sigma_i \) by using Gupta’s table [4]. For \( i = 2, 3, \ldots, k \) the tables useful in solving (10) and (11) will be published in due course.

### 2.2. Asymptotic Standard Errors of the Estimates of \( \mu_i \) and \( \sigma_i \)

In this section we shall obtain the standard errors of the estimates of \( \mu_i \) and \( \sigma_i \). On differentiating (6) and (7), we obtain

\[
\frac{\partial^2 \log L}{\partial \mu_i^2} = -\frac{n_i}{\sigma_i^2} - \frac{(n_i - n_i)A_i}{\sigma_i^2} + \frac{n_i A_{i-1,i}}{\sigma_i^2}
\]

\[
\frac{\partial^2 \log L}{\partial \sigma_i^2} = -\frac{2n_i (\hat{p}_i^{(r)} - \mu_i)}{\sigma_i^2} - \frac{(n_i - n_i)B_i}{\sigma_i^2} + \frac{n_i B_{i-1,i}}{\sigma_i^2}
\]

\[
\frac{\partial^2 \log L}{\partial \sigma_i^2} = \frac{n_i}{\sigma_i^2} - \frac{3n_i (\hat{p}_i^{(r)} - \mu_i)^2}{\sigma_i^2} - \frac{(n_i - n_i)C_i}{\sigma_i^2} + \frac{n_i C_{i-1,i}}{\sigma_i^2}
\]

where

\[
A_i = Z_{ii}(Z_{ii} - \nu_i), \quad A_{i-1,i} = Z_{i-1,i}(Z_{i-1,i} - \nu_{i-1,i})
\]

\[
B_i = Z_{ii} + \nu_i A_i, \quad B_{i-1,i} = Z_{i-1,i} + \nu_{i-1,i} A_{i-1,i}
\]

\[
C_i = \nu_i(Z_{ii} + B_i), \quad C_{i-1,i} = \nu_{i-1,i}(Z_{i-1,i} + B_{i-1,i})
\]

Now

\[
E(n_i) = E(n_i)[\Phi(\nu_i) - \Phi(\nu_{i-1,i})]/[1 - \Phi(\nu_{i-1,i})]
\]
where \( \mu_{12}, \mu_{31} \) are the probabilities of misclassification of class 1 and 2 points, respectively. The decision rule divides the character space into two regions \( R_1 \) and \( R_2 \) such that an observation vector falling in \( R_i \) is assigned to class \( j \). The probability of misclassifying a sample point drawn from class \( i \) is then defined:

\[
\mu_{ji} = \int_{R_i} f_j(x) \, dx \quad (i \neq j)
\]

Dividing (2.4) by \( (K_1 + K_2) \) yields the expected normalized loss:

\[
E(L) = \frac{E(L)}{K_1 + K_2} = (1 - c)\mu_{12} + c\mu_{31}.
\]

(2.5)

Henceforth “loss” is understood to mean “normalized loss.” As shown by Anderson [1], the likelihood ratio criterion (2.3) minimizes the expected loss (2.5).

3. Proposed Nonparametric Procedure

To estimate \( P(1 \mid x) \) for a point of undetermined source, when the population distribution laws are unknown, insert a hypersphere of radius \( r \) in the sample space, centered at the co-ordinates \( x \) of the element to be classified. If the volume of the hypersphere is \( V \), estimate \( f_1(x) \) and \( f_2(x) \) at the single point \( x \) by

\[
\hat{f}_i(x) = \frac{n_i}{N_i V}
\]

\[
\hat{f}_2(x) = \frac{n_2}{N_2 V}
\]

(3.1)

where \( n_1, n_2 \) are the numbers of known class 1 and 2 points, respectively, that fall inside the sphere. If there are no sample points in the hypersphere, go to the nearest known point and give the unknown element its classification. When \( a \) priori probabilities are not known, but the samples have been drawn at random from a mixed population with proportions \( \pi_1, \pi_2, (\pi_1 + \pi_2 = 1) \), the sample ratios \( N_i/(N_1 + N_i) \) and \( N_2/(N_1 + N_2) \) are reasonable estimates of the prior probabilities. Under these conditions, when the estimates (3.1) along with those for \( \pi_1, \pi_2 \) are introduced into (2.2) and decision rule (2.1) is used, the effect is the same as employing decision rule (9), p. 131, of Anderson [1], but with known functions and constants replaced by appropriate estimates.

The expected loss at stage \( N = N_1 + N_2 \), conditional on the samples, is

\[
E(L(r, N)) = (1 - c)\pi_1\mu_{12}(r, N) + c\pi_2\mu_{31}(r, N),
\]

(3.2)

where \( \mu_{ji}(r, N) \) is the nonparametric analogue of Hills’ \( \alpha (r) \) [7], which he defines as the conditional probability that, given a fixed sample of \( N_1 \) objects from population 1 and of \( N_2 \) objects from population 2, a randomly chosen member of population \( i \) will be misallocated.

Work by Rosenblatt [12] suggests that the radius \( r \) of the hypersphere may have an optimal value for classification purposes. As \( r \) increases, the variance of \( \hat{f}_i(x) \) decreases, but the bias of the estimate increases. More generally, because different values of the hypersphere radius \( r \) lead to different decisions,
The times of censoring are 1150 hours and 1400 hours respectively. After each time of censoring 38 and 20 items are eliminated from inspection.

\[ n = 340, n_1 = 176, n_2 = 95, r_1 = 38, r_2 = 20, T_1 = 1150, T_2 = 1400, \]
\[ y^{(1)} = 6.8230201, s_1^2 = 0.1478021, y^{(2)} = 7.1167998, s_2^2 = 0.0025306, \]
\[ d = \log T_1 - y^{(1)} = 0.2244876, \Psi_1 = 0.7457340, p_1 = n_1/n = 0.5176470. \]

Referring to Gupta's tables [4], the estimates of \( \mu_i \) and \( \sigma_i \) and the estimated variances of \( \hat{\mu}_i \) and \( \hat{\sigma}_i \) are

\[
\begin{align*}
\hat{\mu}_i &= 7.1236870 \\
\hat{\sigma}_i &= 0.4640023 \\
V(\hat{\mu}_i) &= 0.0008100 \\
V(\hat{\sigma}_i) &= 0.0006590.
\end{align*}
\]

Since the censoring has been done at two stages, we will now obtain the estimates of \( \mu_2 \) and \( \sigma_2 \) and the standard errors of these estimates. The iterative method of obtaining the estimates of \( \mu_i \) and \( \sigma_i \) for \( i = 2, 3, \ldots, k \) is given in the appendix.

Now

\[ n^{(2)} = 126, \quad n^{(2)} = 11, \quad X = 1250 \quad \text{and} \quad \lambda = 44. \]

Hence the equations (54) and (55) give

\[ \hat{\mu}_2 = 7.0757518 \quad \text{and} \quad \hat{\sigma}_2 = 1198634. \]

The equations (48) and (49) of the appendix for \( i = 2 \) become

\[ \delta_{21}(-3319.1874) + \delta_{22}(-6551.5309) = -64.4422 \]
\[ \delta_{21}(-6551.5309) + \delta_{22}(16480.8989) = -274.3400. \]

On solving the equations (25) and (26), we get

\[ \delta_{21} = -0.0624143, \quad \text{and} \quad \delta_{22} = 0.0414568. \]

Hence the equation (47) for \( i = 2 \) gives

\[ \hat{\mu}_2 = 7.0133375 \quad \text{and} \quad \hat{\sigma}_2 = 0.1613202. \]
The estimated variance-covariance matrix of $\beta$ and $\delta$ is

\[
\begin{bmatrix}
V(\beta) & \text{cov} (\beta, \delta) \\
\text{cov} (\beta, \delta) & V(\delta)
\end{bmatrix}
= \begin{bmatrix}
1362.9200 & 3125.4573 \\
3125.4573 & 3320.3583
\end{bmatrix}^{-1}
\]

\[
= \begin{bmatrix}
0.0054549 & -0.0020465 \\
-0.0020465 & 0.0008890
\end{bmatrix}.
\]

### 3.1 The Logistic Distribution

In this section, we shall consider the case in which the life of an item has logistic distribution with density

\[
f(x; \alpha, \beta) = \frac{1}{\beta} \exp \left\{ - \left( \frac{x - \alpha}{\beta} \right) \right\}, \quad \alpha \leq x \leq \infty, \quad \beta > 0.
\]

We define

\[
P_i(x) = P(x < x_i) = 1 - Q_i(x) = 1/\left[1 + \exp \left\{ - \left( x - \alpha_i \right)/\beta_i \right\}\right]
\]

where

\[
P_i(x) = \int_{-\infty}^{x} f(y; \alpha_i, \beta_i) \, dy.
\]

In progressively censored samples which change the parameters at each stage of censoring, the likelihood (3) of section 2.1 will have the form with

\[
L = \prod_{i=1}^{m} \left[ P_i(x^{(i)}_i) \text{Pr}(x^{(i)}_i) \right]^{n^{(i)}-n_i}
\]

for $i = 2, 3, \ldots, k$ and $n^{(1)} = n, n^{(i)} = n^{(i-1)} - n_{i-1} - r_{i-1}$ for $i = 2, 3, \ldots, k$.

The maximum likelihood estimating equations are

\[
\frac{\partial \log L}{\partial \alpha_i} = \frac{\partial \log L}{\partial \beta_i} = \frac{1}{\beta_i} \left[ 2 \sum_{i=1}^{m} P_i(x^{(i)}_i) - n_i + (n^{(i)} - n_i)P_i(T_i) - n^{(i)}P_i(T_{i-1}) \right] = 0
\]

and

\[
\frac{\partial \log L}{\partial \beta_i} = \frac{1}{\beta_i} \left[ -n_i + 2 \sum_{i=2}^{m} \left( \frac{x^{(i)}_i - \alpha_i}{\beta_i} \right) P_i(x^{(i)}_i)
- \frac{n_i(x^{(i)}_i - c_i)}{\beta_i} + (n^{(i)} - n_i) \left( \frac{T_i - \alpha_i}{\beta_i} \right) P_i(T_i)
- n^{(i)} \left( \frac{T_{i-1} - \alpha_i}{\beta_i} \right) P_i(T_{i-1}) \right] = 0
\]
We here assume that $a_i = a$ and $p_i = p$, then by dropping the subscripts and superscripts and summing over $i = 1, 2, \ldots, k$, the likelihood estimating equations (34) and (35) can be rewritten as

\begin{align}
\frac{\partial \log L}{\partial \alpha} &= \frac{1}{\beta} \left[ \frac{1}{2} \sum_{i=1}^{n_i} P(x_i) - m + \sum_{i=1}^{k} r_i P(T_i) \right] = 0 \\
\frac{\partial \log L}{\partial \beta} &= \frac{1}{\beta} \left[ -m + 2 \sum_{i=1}^{n_i} \left( \frac{x_i - \alpha}{\beta} \right) P(x_i) - m \left( \frac{\beta - \alpha}{\beta} \right) + \sum_{i=1}^{k} r_i \left( \frac{T_i - \alpha}{\beta} \right) P(T_i) \right] = 0
\end{align}

where $m = \sum_{i=1}^{k} n_i$ is the total number of items which fail. Here also standard iterative methods may be used to solve the equations (35) and (36) to obtain the estimates of $\alpha$ and $\beta$. If $\alpha_0$ and $\beta_0$ are the approximate solutions, new values can be obtained by the method described in the Appendix, using

\begin{align}
\alpha &= \alpha_0 + h \\
\beta &= \beta_0 + k
\end{align}

The number of repetitions of the iterative process for a given degree of accuracy will depend upon the initial approximations. To obtain better first approximation, the logit technique can be used, which is discussed in the next section. In practice, final values of the second partials calculated using equations (38), (39), and (40) in connection with the iterative solution of the estimating equations may be taken to be equal to their expected values, and hence may be used to approximate the variance-covariance matrix of $\alpha$ and $\beta$. 

where

\[ \bar{z}_{it} = \frac{1}{n_i} \sum_{i=1}^{n_i} z_{it} \]
A D A P T I V E  N O N P A R A M E T R I C  C L A S S I F I C A T I O N

of misclassifications that would be observed if the entire population 1 were
to be classified, \( N_1 \) times under the rules of section 3, each time with a different
known class 1 sample point missing, \( P(1 \mid x) \) being estimated from the remaining
\( N_1 - 1 + N_1 \) fixed points of the sample. Thus (4.7) is an unbiased estimate
of the probability of misclassification in the following hypothetical procedure:

Assume

a) A fixed sample of \( N_1, N_2 \) points known to have originated in populations 1
and 2, respectively. These are termed "sample" points or "known" points.

b) Another set of points which are drawn at random from population 1 and
are to be classified. Although the source is given, these points are labeled "un­
known" points for convenience.

When a class 1 unknown point with observation vector \( x \) is to be classified,
select one of the \( N_1 \) class 1 sample points at random, remove it temporarily
from the sample, estimate \( P(1 \mid x) \) for the unknown point by

\[
P_{1,L}(r, N, x) = \frac{\pi r_1 n_1'}{N_1 - 1 + \pi r_2 n_2} ,
\]

and classify the unknown point using decision rule (2.1). Here \( n_1' \) is the number
of class 1 sample points inside the hypersphere of radius \( r \), centered at \( x \), excluding
the randomly selected class 1 known point temporarily removed from the sample;
\( n_2 \) is the number of class 2 sample points inside the hypersphere. Return the
random known point to the sample, but do not add the point just classified.
Repeat the process with the next unknown point of class 1. Denote the prob­
ability of misclassification in this hypothetical procedure by \( \mu_{12,L}(r, N) \).

S-Procedure

On the other hand, expression (4.5) is an unbiased estimate of the probability
of misclassifying type 1 points drawn at random in a hypothetical S-procedure,
the same as the L-procedure in all respects except as follows: Having removed
the random known class 1 point from the sample, center the hypersphere on
the co-ordinates \( x \) of the unknown point and observe \( n_1' \) and \( n_2 \). Refer to an
\( M \)-digit table of random numbers; choose a number, and if this is less than or
equal to \( (n_1'/N_1 - 1)(10)^M \), estimate \( P(1 \mid x) \) for the unknown point \( x \) by

\[
P_{1,s}(r, N, x) = \frac{\pi r_1 n_1''}{N_1 + \pi r_2 n_2} , \quad (n_1'' = n_1' + 1),
\]

and classify the unknown point using decision rule (2.1). (For this purpose,
the random number consisting of \( M \) zeros is assigned numerical value \( 10^M \).)
Otherwise, estimate \( P(1 \mid x) \) by (5.2), with \( n_1'' = n_1' \), label the estimate
\( P_{1,s}(r, N, x) \), and classify the unknown point using decision rule (2.1). Return
the random known class 1 point to the sample (but not the point just classified),
and repeat the procedure with the next unknown class 1 point. Denote the prob­
ability of misclassification in the S-procedure by \( \mu_{12,s}(r, N) \).
approximately lie on the logit regression line. This being a straight line, its
equation may be written as

\[ L = \left( \frac{x - c}{\beta} \right). \]

Thus \( a_0 \) is the values of \( x \) corresponding to the logit value zero, and \( \beta_0 \) is the
reciprocal of the slope of the logit regression line.

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(unpublished).

Appendix

Iterative Method for solving equations (6) and (7):

In order to solve equations (6) and (7) simultaneously for obtaining the
estimates of \( \mu_i \) and \( \sigma_i \), the standard iterative methods may be employed.
Newton's method, for instance, will usually be satisfactory. If \( \mu_i^0 \) and \( \sigma_i^0 \)
are the approximate solutions, let

\[ \mu_i = \mu_i^0 + \delta_i, \quad \sigma_i = \sigma_i^0 + \delta_{i2}, \quad i = 2, 3, \ldots, k \]

where \( \delta_i \) and \( \delta_{i2} \) are the corrections to be determined by the iterative process.
Using Taylor's theorem and neglecting higher powers of \( \delta_i \) and \( \delta_{i2} \) above the
first, we get

\[ \delta_i \frac{\partial^2 \log L}{\partial \mu_i^2} + \delta_{i2} \frac{\partial^2 \log L}{\partial \mu_i \partial \sigma_i} = -\frac{\partial \log L}{\partial \mu_i^0}, \]

\[ \delta_{i2} \frac{\partial^2 \log L}{\partial \sigma_i^2} + \delta_i \frac{\partial^2 \log L}{\partial \sigma_i \partial \sigma_i^0} = -\frac{\partial \log L}{\partial \sigma_i^0}. \]

The corrections \( \delta_i \) and \( \delta_{i2} \) can be determined by solving the equations (48)
and (49) simultaneously. The terms on the right are given by (6) and (7).
The coefficients of \( \delta_i \) and \( \delta_{i2} \) can be obtained from (12), (13), and (14). The
number of repetitions of the iterative process for a given degree of accuracy
will generally depend upon the initial approximations. The use of probit tech-
nique may be made to obtain a better first approximation.

The conditional density function of \( x > T_{i-1} \) is given by

\[ f(x \mid x > T_{i-1}) = \frac{\int_{T_{i-1}}^{x} f(x) \, dx}{\left[ \int_{T_{i-1}}^{x} f(x) \, dx \right]}. \]
Now from (2) and (2'), it is evident that

\[ f(x) = \frac{1 - \Phi(\nu_{i+1})}{1 - \Phi(\nu_{i-1,i})} \prod_{i=2}^{t-1} \left( \frac{1 - \Phi(\nu_{i})}{1 - \Phi(\nu_{i-1,i})} \right) (\nu_{i}x)^{-1} \phi \left( \frac{\log x - \mu_i}{\sigma_i} \right) \]

\[ \text{for } T_{i-1} < x \leq T_i \]

and

\[ \int_{T_{i-1}}^{T_i} f(x) \, dx = \{1 - \Phi(\nu_{i+1})\} \prod_{i=2}^{t-1} \left( \frac{1 - \Phi(\nu_{i})}{1 - \Phi(\nu_{i-1,i})} \right). \]

In view of (50), (51), and (52), it is easy to see that

\[ \Pr \{ x > T_i \mid x > T_{i-1} \} = \frac{1 - \Phi(\nu_{i+1})}{1 - \Phi(\nu_{i-1,i})} \]

and

\[ \Pr \{ x \leq T_i \mid x > T_{i-1} \} = \frac{\Phi(\nu_{i}) - \Phi(\nu_{i-1,i})}{1 - \Phi(\nu_{i-1,i})}. \]

Now \( n^{(i)} \) and \( n^{(i+1)} \) are the number of observations above \( T_{i-1} \) and \( T_i \), respectively. For \( T_{i-1} < X < T_i \), let \( \lambda \) be the number of observations above \( X \). Then from (53) it is easy to see that

\[ \frac{\lambda}{n^{(i)}} = \frac{1 - \Phi(\nu_{i})}{1 - \Phi(\nu_{i-1,i})} \]

and

\[ \frac{n^{(i+1)}}{n^{(i)}} = \frac{1 - \Phi(\nu_{i})}{1 - \Phi(\nu_{i-1,i})}. \]

Where \( z = \log X, \nu_{i} = (Z - \mu_{s})/\sigma_{i} \), and asterisk denotes estimated values. The value of \( \Phi^{*}(\nu_{i-1,i}) \) may be approximately obtained by replacing \( \mu_i \) and \( \sigma_i \) by \( \mu^{(i)} \) and \( \sigma^{(i)} \), respectively. By transforming \( \Phi^{*}(\nu_{i}) \) and \( \Phi^{*}(\nu_{i}) \) into probits by using a standard probit table or using an ordinary table of normal curve areas, we get the initial approximations \( \mu^{(i)} \) and \( \sigma^{(i)} \) for \( i = 2, 3, \cdots, k. \)
A TRIVARIATE EXTENSION OF FREUND'S BIVARIATE DISTRIBUTION

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1. Introduction:

John, E. Freund [1] has proposed a bivariate extension of the exponential distribution with a view to apply the same for the life testing of two component systems, which can work even after one of the components has failed. The purpose of this paper is to present a trivariate extension of the general distribution of three component systems. We have proposed two models of the general distribution and investigated various statistical properties including the likelihood estimates of the parameters involved and their distributions, in case of exponential distribution. An extension of the method developed by Mendenhall, W. and Lehmann, E. H. [2] has been applied to find the negative moments of the positive multinomial distribution.

Freund's arguments in the derivation of the general bivariate distribution can be written in the following way:

Let \( f_0(x^*, y^*) \) be any bivariate density of the distribution of random variables \( X^* \) and \( Y^* \), and let \( f_{1,0}(x^*) = \int_{y^*}^{\infty} f_0(x^*, y^*) \, dy^* \), and \( g_{0,0}(y^*) = \int_{x^*}^{\infty} f_0(x^*, y^*) \, dx^* \). Moreover, let \( f_{2,0}(y^*; x^*) \geq 0 \) for all \( x^* \) and \( y^* \) and \( \int_{x^*}^{\infty} f_{2,0}(y^*; x^*) \, dx^* = 1 \) for all \( x^* \), and let \( g_{1,0}(x^*; y^*) \geq 0 \) for all \( x^* \) and \( y^* \) and \( \int_{y^*}^{\infty} g_{2,0}(x^*; y^*) \, dy^* = 1 \).
for all \(y^{*}\). Then the new bivariate distribution of the random variables \(X\) and \(Y\) in the sense of Freund is given by

\[
\sigma_0(x, y) = \begin{cases} 
 f_1(y) f_2(x; y) & \text{for } -\infty < x < y < \infty \\
 g_1(x) g_2(y; x) & \text{for } -\infty < y < x < \infty
\end{cases}
\]  

(1.1)

Similarly, for the trivariate case, let \(f(x^{*}, y^{*}, z^{*})\), \(x^{*} < y^{*} < z^{*} < \infty\), be the density function of the random variables \(X\), \(Y\), and \(Z\). Let

\[
f_1^{*}(x^{*}) = \int_{x^{*}}^{\infty} \int_{x^{*}}^{\infty} f(x^{*}, y^{*}, z^{*}) \, dy^{*} \, dz^{*},
\]

(1.2)

\[
g_1^{*}(y^{*}) = \int_{y^{*}}^{\infty} \int_{y^{*}}^{\infty} f(x^{*}, y^{*}, z^{*}) \, dz^{*} \, dx^{*},
\]

(1.3)

and

\[
h_1^{*}(z^{*}) = \int_{z^{*}}^{\infty} \int_{z^{*}}^{\infty} f(x^{*}, y^{*}, z^{*}) \, dx^{*} \, dy^{*}.
\]

(1.4)

Also let

\[
f_2^{*}(y^{*}, z^{*}; x^{*}) \geq 0 \text{ and such that } \int_{x^{*}}^{\infty} \int_{x^{*}}^{\infty} f_2^{*}(y^{*}, z^{*}; x^{*}) \, dy^{*} \, dz^{*} = 1 \text{ for all } x^{*},
\]

\[
g_2^{*}(z^{*}; x^{*}; y^{*}) \geq 0 \text{ and such that } \int_{y^{*}}^{\infty} \int_{y^{*}}^{\infty} g_2^{*}(z^{*}; x^{*}; y^{*}) \, dz^{*} \, dx^{*} = 1 \text{ for all } y^{*},
\]

and

\[
h_2^{*}(x^{*}, y^{*}; z^{*}) \geq 0 \text{ and such that } \int_{z^{*}}^{\infty} \int_{z^{*}}^{\infty} h_2^{*}(x^{*}, y^{*}; z^{*}) \, dx^{*} \, dy^{*} = 1 \text{ for all } z^{*}.
\]

Then, the new trivariate density function of the random variables \(X\), \(Y\), and \(Z\) in the sense of Freund can be written as

\[
\sigma_0(x, y, z) = \begin{cases} 
 f_1(y) f_2(x; y) & \text{for } -\infty < x < y < z < \infty \\
 g_1(y) g_2(z; y, x) & \text{for } -\infty < y < z < (x, x) < \infty \\
 h_1(z) h_2(x; y, z) & \text{for } -\infty < z < (x, y) < \infty
\end{cases}
\]  

(1.6)
Further let
\[
\int f(x^*, y^*, z^*) \, dz^* = \begin{cases} 
  f_5(x^*, y^*) & \text{if } x^* > y^* \\
  f_4(x^*, y^*) & \text{if } y^* > x^*
\end{cases} \max(x^*, y^*)
\]
\[
\int f(x^*, y^*, z^*) \, dx^* = \begin{cases} 
  g_5(y^*, z^*) & \text{if } y^* > z^* \\
  g_4(y^*, z^*) & \text{if } z^* > y^*
\end{cases} \max(y^*, z^*)
\]
\[
\int f(x^*, y^*, z^*) \, dy^* = \begin{cases} 
  h_5(z^*, x^*) & \text{if } z^* > x^* \\
  h_4(z^*, x^*) & \text{if } x^* > z^*
\end{cases} \max(z^*, x^*)
\]

Also let
\[
f_4(z^*; \max(x^*, y^*)) \geq 0 \text{ and be such that } \int f_4(z^*; \max(x^*, y^*)) \, dz^* = 1 \text{ for all } (x^*, y^*)
\]
\[
g_4(x^*; \max(y^*, z^*)) \geq 0 \text{ and be such that } \int g_4(x^*; \max(y^*, z^*)) \, dx^* = 1 \text{ for all } (y^*, z^*)
\]
and
\[
h_4(y^*; \max(z^*, x^*)) \geq 0 \text{ and be such that } \int h_4(y^*; \max(z^*, x^*)) \, dy^* = 1 \text{ for all } (z^*, x^*)
\]

Hence the new trivariate density function of the random variables $X$, $Y$ and $Z$ can be written as
\[
\delta(x, y, z) = \begin{cases} 
  f_6(x, y) & \text{for } -\infty < x < y < \infty \\
  f_4(x, y) & \text{for } -\infty < y < x < \infty \\
  g_6(y, z) & \text{for } -\infty < y < z < \infty \\
  g_4(y, z) & \text{for } -\infty < z < y < \infty \\
  h_6(z, x) & \text{for } -\infty < z < x < \infty \\
  h_4(z, x) & \text{for } -\infty < x < z < \infty 
\end{cases}
\]

It is easy to verify that
\[
\int \int \alpha(x, y, z) \, dx \, dy \, dz = 1 \text{ and } \int \int \delta(x, y, z) \, dx \, dy \, dz = 1, \text{ when the integration is taken over the appropriate regions.}
Similarly, the multivariate extension can be written down, but it is not mentioned here.

In particular, suppose that $X, Y$ and $Z$ are random variables representing the life times of three component system, and also suppose that $X^*$ represents the life time of component $A$ if the components $B$ and $C$ are replaced with the components, of the same kind each time they fail (once or more than once). Similarly $Y^*$ and $Z^*$ can be defined. We now assume that $X^*, Y^*$ and $Z^*$ are independent random variables having exponential distributions, namely,

$$f(x^*) = e^{-ax^*}, \quad x^* > 0, \quad a > 0; \quad f(y^*) = e^{-\beta y^*}, \quad y^* > 0, \quad \beta > 0;$$

$$f(z^*) = e^{-\gamma z^*}, \quad z^* > 0, \quad \gamma > 0.$$

It is clear from (1.2), (1.3) and (1.4) that

$$f_1(x^*) = P_r \{ A \text{ fails first at } x^*, \quad B \text{ and } C \text{ working} \} = e^{-(\alpha + \beta + \gamma)x^*}$$

$$g_1(x) = P_r \{ B \text{ fails first at } y^*, \quad C \text{ and } A \text{ working} \} = e^{-(\alpha + \beta + \gamma)y^*}$$

$$h_1(z) = P_r \{ C \text{ fails first at } z^*, \quad A \text{ and } B \text{ working} \} = e^{-(\alpha + \beta + \gamma)z^*} \quad (1.11)$$

Now let the probability density of $Y^*$ and $Z^*$ given that $A$ fails first at $x^*$ be

$$f_2(y^*, z^*; x^*) = e^{-\gamma'(y^*-x^*)-\gamma'(z^*-x^*)}; \quad x^* \leqslant (y^*, z^*) \leqslant \infty, \quad \beta' > 0, \quad \gamma' > 0 \quad (1.12)$$

Let the probability density of $Z^*$ and $X^*$ given that $B$ fails first at $y^*$ be

$$g_2(z^*, x^*; y^*) = e^{-\gamma'(r^*-y^*)-\alpha'(x^*-y^*)}; \quad y^* \leqslant (z^*, x^*) \leqslant \infty, \quad \gamma' > 0, \quad \alpha' > 0, \quad (1.13)$$
and let the probability density of \( X^* \) and \( Y^* \) given that \( C \) fails first at \( z^* \) be

\[
h_2(x^*, y^*, z^*) = a' \beta' e^{-a'(x^*-z^*)} - \beta'(y^*-z^*) ;
\]

\[
z^* < (x^*, y^*) < \infty, \quad a' > 0, \quad \beta' > 0. \quad (1.14)
\]

Hence, using (1.6), the new trivariate density function of the random variables \( X, Y \) and \( Z \) given by

\[
\alpha(x, y, z) = \begin{cases} 
\alpha' \beta' \gamma e^{-(\alpha + \beta + \gamma - \beta')x - \beta' y - \gamma z} ; & 0 < x \leq (y, z) < \infty, \\
\alpha' \beta' \gamma e^{-(\alpha + \beta + \gamma - \alpha')y - \alpha' z - \gamma x} ; & 0 < y \leq (z, x) < \infty, \\
\alpha' \beta' \gamma e^{-(\alpha + \beta + \gamma - \beta')z - \beta' x - \gamma y} ; & 0 < z \leq (x, y) < \infty.
\end{cases} \quad (1.15)
\]

Now the probability that \( A \) fails first at \( x^* \), \( B \) fails first at \( y^* \) and \( C \) working is

\[
\int_{\max(x^*, y^*)}^{\infty} \gamma e^{-\gamma z} \, dz, \text{ if } x^* > y^*
\]

and

\[
f_5(x^*, y^*) = \alpha' \beta' e^{-\gamma x^*} - \beta y^* \quad \text{if } y^* > x^*.
\]

Similarly the probability that \( B \) fails first at \( y^* \), \( C \) fails first at \( z^* \) and \( A \) working is

\[
g_6(y^*, z^*) = \beta y e^{-(\alpha + \beta)y^* - \gamma z^*} \quad \text{if } y^* > z^*,
\]

and

\[
g_4(y^*, z^*) = \beta y e^{-(\alpha + \gamma)z^* - \beta y^*} \quad \text{if } z^* > y^*, \quad (1.17)
\]

and the probability that \( C \) fails first at \( z^* \), \( A \) fails first at \( x^* \) and \( B \) working is

\[
h_3(z^*, x^*) = \gamma ae^{-(\beta + \gamma)z^* - \alpha x^*} \quad \text{if } z^* > x^*,
\]

and

\[
h_4(z^*, x^*) = \gamma ae^{-(\alpha + \beta)x^* - \gamma z^*} \quad \text{if } x^* > z^*. \quad (1.18)
\]
The probability densities of $Z^*$ given that $A$ fails first at $x^*$ and $B$ fails first at $y^*$ are

$$f_3(z^*;x^*) = \gamma e^{-\gamma'(z^*-x^*)}, \quad z^* > x^*, \text{ if } x^* > y^*, \quad (1.19)$$
and

$$f_3(z^*;y^*) = \gamma e^{-\gamma'(z^*-y^*)}, \quad z^* > y^*, \text{ if } y^* > x^*, \quad (1.20)$$
with $\gamma > 0$.

In the same way, the respective probability densities of $X^*$ given that $B$ fails first at $y^*$ and $C$ fails at $z^*$ and of $Y^*$ given that $C$ fails first at $z^*$ and $A$ fails first at $x^*$, are

$$g_3(x^*; y^*) = \alpha e^{-\alpha'(x^*-y^*)}, \quad x^* > y^*, \text{ if } y^* > x^*, \quad (1.21)$$
and

$$g_3(x^*; z^*) = \alpha e^{-\alpha'(x^*-z^*)}, \quad x^* > z^*, \text{ if } x^* > y^*, \quad (1.22)$$
and

$$h_3(y^*; z^*) = \beta e^{-\beta'(y^*-z^*)}, \quad y^* > z^*, \beta > 0 \text{ if } z > x^*, \quad (1.23)$$
$$h_3(y^*; x^*) = \beta e^{-\beta'(y^*-x^*)}, \quad y^* > x^*, \beta > 0 \text{ if } x^* > y^*. \quad (1.24)$$

Hence, using (1.9) the new trivariate density function of the random variables $X$, $Y$ and $Z$ is given by

$$b(x, y, z) = \begin{cases} 
\alpha \beta \gamma e^{-(\alpha + \beta - \gamma')y - \alpha x - \gamma z} & \text{for } 0 \leq y \leq x \leq z \leq \infty \\
\alpha \beta \gamma e^{-(\beta + \gamma - \gamma')y - \alpha x - \gamma z} & \text{for } 0 \leq x \leq y \leq z \leq \infty \\
\beta \alpha e^{-(\alpha + \gamma - \gamma')z - \beta y - \alpha x} & \text{for } 0 \leq z < x \leq \infty \\
\beta \alpha e^{-(\beta + \gamma - \gamma')z - \beta y - \alpha x} & \text{for } 0 \leq x < z \leq \infty \\
\gamma \alpha e^{-(\alpha + \beta - \gamma')x - \gamma z - \beta y} & \text{for } 0 \leq y < z \leq \infty \\
\gamma \alpha e^{-(\alpha + \beta - \gamma')x - \gamma z - \beta y} & \text{for } 0 \leq z < x \leq \infty \\
\end{cases} \quad (1.25)$$

We shall refer to (1.15) and (1.25) as model I and model II respectively. It may be noted that the dependence among $X$, $Y$ and $Z$ is such that, in case of the model I, the failure of the $C$ component changes the parameters of the distributions of the $A$ and $B$ components from $\alpha$ and $\beta$ to $\alpha'$ and $\beta'$; the failure of
the $A$ component changes, the parameters of the distributions of the $B$ and $C$ components from $\beta'$ and $\gamma'$ to $\beta''$ and $\gamma''$; and the failure of the $B$ component changes the parameters of the distributions of the $C$ and $A'$ components from $\gamma$ and $\alpha'$ to $\gamma'$ and $\alpha''$ respectively. In the model II, the failures of $A$ and $B$ components change the parameter of the distribution of $C$ component from $\alpha$ to $\alpha''$ and the failures of $C$ and $A'$ components change the parameter of the distribution of $B$ component from $\beta$ to $\beta''$.

2. Statistical Properties of the Model I:

The moment generating function (m.g.f.) of the distribution in (1.15) is given by

$$
M(t_1, t_2, t_3) = \int \int \int e^{tx_1 + ty_1 + tz} \alpha(x, y, z) \, dx \, dy \, dz
$$

(2.1)

Integrating over the appropriate regions of $x$, $y$ and $z$, the m.g.f. is

$$
M(t_1, t_2, t_3) = \frac{(\alpha + \beta + \gamma)^x}{\left(1 - \frac{t_1 + t_2 + t_3}{\alpha + \beta + \gamma}\right)} \left[ \frac{\beta}{(1 - \frac{t_1}{\beta}) \left(1 - \frac{t_2}{\gamma} \right) \left(1 - \frac{t_3}{\alpha} \right)} \right] + \frac{\gamma}{\left(1 - \frac{t_1}{\alpha'} \right) \left(1 - \frac{t_2}{\beta'} \right) \left(1 - \frac{t_3}{\gamma'} \right) \left(1 - \frac{t_4}{\alpha''} \right)}
$$

(2.2)

Expanding the m.g.f. in powers of $t_1$, $t_2$ and $t_3$, it is easy to see that

$$
E(X) = \frac{\alpha' + \beta + \gamma}{\alpha' (\alpha' + \beta + \gamma)} \gamma, \quad E(Y) = \frac{\beta' + \gamma + \alpha}{\beta' (\alpha' + \beta + \gamma)} \gamma, \quad E(Z) = \frac{\gamma' + \alpha + \beta'}{\gamma' (\alpha' + \beta + \gamma)} \gamma
$$

(2.3)

$$
V(X) = \frac{\alpha'' \beta' + \gamma^2 + 2 \alpha' \beta + 2 \alpha' \gamma + 2 \gamma \alpha}{\alpha'' \beta' (\alpha' + \beta + \gamma)^2}
$$

$$
V(Y) = \frac{\beta'' \gamma' + \alpha^2 + 2 \alpha \beta + 2 \alpha \gamma + 2 \gamma \alpha}{\beta'' \gamma' (\alpha + \beta + \gamma)^2}
$$

(2.4)

$$
V(Z) = \frac{\gamma'' + \alpha^2 + \beta' + 2 \alpha \beta + 2 \beta \gamma + 2 \gamma \alpha}{\gamma'' (\alpha + \beta + \gamma)^2}
$$
A Trivariate Extension of......

\[
\text{cov}(x, y) = \frac{\alpha'\beta' - \alpha\beta}{\alpha'\beta'(\alpha + \beta + \gamma)^2}, \quad \text{cov}(y, z) = \frac{\beta'\gamma' - \beta\gamma}{\beta'\gamma'(\alpha + \beta + \gamma)^2},
\]

\[
\text{cov}(z, x) = \frac{\gamma'\alpha' - \gamma\alpha}{\gamma'\alpha'(\alpha + \beta + \gamma)^2}
\]

(2.5)

The population correlation coefficients are

\[
\rho(x, y) = \frac{\alpha'\beta' - \alpha\beta}{\sqrt{\left\{ (\alpha + \beta + \gamma)^2 - \alpha^2 + \alpha'^2 \right\} \left\{ (\alpha + \beta + \gamma)^2 - \beta^2 + \beta'^2 \right\}}},
\]

\[
\rho(y, z) = \frac{\beta'\gamma' - \beta\gamma}{\sqrt{\left\{ (\alpha + \beta + \gamma)^2 - \beta^2 + \beta'^2 \right\} \left\{ (\alpha + \beta + \gamma)^2 - \gamma^2 + \gamma'^2 \right\}}},
\]

(2.6)

\[
\rho(z, x) = \frac{\gamma'\alpha' - \gamma\alpha}{\sqrt{\left\{ (\alpha + \beta + \gamma)^2 - \gamma^2 + \gamma'^2 \right\} \left\{ (\alpha + \beta + \gamma)^2 - \alpha^2 + \alpha'^2 \right\}}},
\]

(2.7)

It may be noted that each correlation coefficient given in (2.6) lies between \(-\frac{1}{8}\) and 1. The correlation coefficient \(\rho(x, y)\) tends to the limit unity if \(\alpha' \to \infty\) and \(\beta' \to \infty\) i.e., if the components \(A\) and \(B\) have not failed prior to \(C\) then they must fail simultaneously with \(C\). This corresponds to the case where the three-component system cannot function if all the components fail simultaneously. In the same way \(\rho(y, z)\) and \(\rho(z, x)\) both tend to unity when \(\beta' \to \infty\), \(\gamma' \to \infty\), and \(\gamma' \to \infty\), \(\alpha' \to \infty\) respectively. If \(\alpha' \to 0\), \(\beta' \to 0\) and \(\alpha = \beta = \gamma\), \(\rho(x, y)\) tends to the limit \(-\frac{1}{8}\). This corresponds to the case where the third component cannot fail if the first two fail. In the same way \(\rho(y, z)\) and \(\rho(z, x)\) both tend to \(-\frac{1}{8}\) when \(\beta' \to 0\), \(\gamma' \to 0\) and \(\gamma' \to 0\), \(\alpha' \to 0\) and \(\alpha = \beta = \gamma\) in each case.

The marginal densities of \((x, y)\), \((y, z)\) and \((z, x)\) are

\[
\alpha_1(x, y) = \begin{cases} 
\left( \frac{(a\beta' (\alpha + \beta + \gamma - \alpha' - \beta') - \alpha'\beta'\gamma)}{(\alpha + \beta + \gamma - \alpha' - \beta') (\alpha'\beta'\gamma)} e^{-(\alpha + \beta + \gamma - \beta')x - \beta'y} + a\beta' e^{-\alpha'x - \beta'y} \right) & \text{for } x < y, \\
\left( \frac{(\alpha + \beta + \gamma - \alpha' - \beta') (\alpha'\beta'\gamma)}{(\alpha + \beta + \gamma - \alpha' - \beta')} e^{-(\alpha + \beta + \gamma - \beta')y - \beta'x} + a\beta' e^{-\alpha'x - \beta'y} \right) & \text{for } y < x.
\end{cases}
\]

(2.7)
Provided \( \alpha + \beta + \gamma - \alpha' - \beta' \neq 0 \)

\[
\alpha_s(y, z) = \begin{cases} 
\frac{[\beta' \gamma (\alpha + \beta + \gamma - \beta') - \alpha' \beta' y' z] e^{-(\alpha + \beta + \gamma - \beta') y' s} + \alpha' \beta' y' e^{-\beta' y' s}}{(\alpha + \beta + \gamma - \beta')} & \text{for } y < z, \\
\frac{[\beta' \gamma (\alpha + \beta + \gamma - \beta') - \alpha' \beta' y' z] e^{-(\alpha + \beta + \gamma - \beta') y' s} + \alpha' \beta' y' e^{-\beta' y' s}}{(\alpha + \beta + \gamma - \beta')} & \text{for } z < y,
\end{cases}
\]

Provided \( \alpha + \beta + \gamma - \beta' \neq 0 \),

and

\[
\alpha_s(x, s) = \begin{cases} 
\frac{[\alpha' \gamma (\alpha + \beta + \gamma - \alpha' - \gamma') - \alpha' \beta' y' z] e^{-(\alpha + \beta + \gamma - \alpha') x' y' s} + \alpha' \beta' y' e^{-\alpha' x' y' s}}{(\alpha + \beta + \gamma - \alpha')} & \text{for } x < s, \\
\frac{[\alpha' \gamma (\alpha + \beta + \gamma - \alpha' - \gamma') - \alpha' \beta' y' z] e^{-(\alpha + \beta + \gamma - \alpha') x' y' s} + \alpha' \beta' y' e^{-\alpha' x' y' s}}{(\alpha + \beta + \gamma - \alpha')} & \text{for } s < x,
\end{cases}
\]

Provided \( \alpha + \beta + \gamma - \alpha' - \gamma' \neq 0 \).

The marginal densities of \( X, Y \) and \( Z \) are

\[
\alpha^{(1)}(x) = \frac{(\alpha - \alpha') (\alpha + \beta + \gamma) e^{-(\alpha + \beta + \gamma) x} + \alpha' (\beta + \gamma) e^{-\alpha' x}}{(\alpha + \beta + \gamma - \alpha')}; \quad x > 0,
\]

\[
\alpha^{(2)}(y) = \frac{\beta' \gamma (\alpha + \beta + \gamma) e^{-(\alpha + \beta + \gamma) y + \beta' (\gamma + \alpha) e^{-\beta' y}}}{(\alpha + \beta + \gamma - \beta')} \quad ; \quad y > 0
\]

\[
\alpha^{(3)}(z) = \frac{(\gamma - \gamma') (\alpha + \beta + \gamma) e^{-(\alpha + \beta + \gamma) z + \gamma' (\alpha + \beta) e^{-\gamma' z}}}{(\alpha + \beta + \gamma - \gamma')} \quad ; \quad z > 0,
\]

provided, of course,

\( \alpha + \beta + \gamma - \alpha' \neq 0 \), \( \alpha + \beta + \gamma - \beta' \neq 0 \) and \( \alpha + \beta + \gamma - \gamma' \neq 0 \).

When the \( \alpha + \beta + \gamma - \alpha' - \beta' = 0 \), \( \alpha + \beta + \gamma - \beta' - \gamma' = 0 \) and \( \alpha + \beta + \gamma - \gamma' - \alpha' = 0 \) the marginals densities of \((x, y), (y, z)\)
and \((x, z)\) become
\[
\begin{align*}
\alpha_1(x, y) &= \begin{cases} 
\alpha'(\beta + \gamma x) e^{-\alpha'x - \beta'y} & \text{for } y < x, \\
\beta'(\alpha + \alpha'y) e^{-\alpha'x - \beta'y} & \text{for } x < y.
\end{cases} \\
\alpha_2(y, z) &= \begin{cases} 
\beta'(\gamma' + \gamma'az) e^{-\beta'y - \gamma'z} & \text{for } z < y, \\
\gamma'(\beta + \beta'ay) e^{-\beta'y - \gamma'z} & \text{for } y < z.
\end{cases} \\
\alpha_3(y, x) &= \begin{cases} 
\gamma'(\alpha + \alpha'bx) e^{-\gamma'z - \alpha'x} & \text{for } x < z, \\
\alpha'(\gamma + \gamma'bx) e^{-\gamma'z - \alpha'x} & \text{for } z < x.
\end{cases}
\end{align*}
\] (2.13)

The marginal densities of \(X, Y\) and \(Z\) then become
\[
\begin{align*}
\alpha^{(a)}(x) &= \left\{ \alpha = \frac{\alpha' (\beta + \gamma)}{\beta'} \right\} e^{-(\alpha + \beta + \gamma) x} + \frac{\alpha'}{\beta'} (\beta + \gamma) e^{-\alpha'x} + \frac{\beta'}{\beta'} (\gamma + \alpha) e^{-\beta'y} \\
\alpha^{(y)}(y) &= \left\{ \beta = \frac{\beta'(\gamma + \alpha)}{\gamma'} \right\} e^{-(\alpha + \beta + \gamma) y} + \frac{\beta'}{\gamma'} (\gamma + \alpha) e^{-\beta'y} + \frac{\gamma'}{\alpha'} (\alpha + \beta) e^{-\gamma'z} \\
\alpha^{(z)}(z) &= \left\{ \gamma = \frac{\gamma'(\alpha + \beta)}{\alpha'} \right\} e^{-(\alpha + \beta + \gamma) z} + \frac{\gamma'}{\alpha'} (\alpha + \beta) e^{-\gamma'z}.
\end{align*}
\] (2.16)

It is evident from (2.10), (2.11), and (2.16) that when \(\alpha = \alpha', \beta = \beta', \gamma = \gamma', X, Y, \) and \(Z\) become independent with density functions
\[
\begin{align*}
f_1(x) &= \alpha e^{-\alpha'x}, \quad x > 0, \quad f(y) = \beta e^{-\beta'y}, y < 0, \quad f(z) = \gamma e^{-\gamma'z}, z > 0.
\end{align*}
\] (2.19)

3. Estimates of the Parameters of the Model I:

To estimate the parameters \(\alpha, \beta, \gamma, \alpha', \beta'\), and \(\gamma'\) by the method of maximum likelihood, suppose that in a random sample of \(n\) items from a population having trivariate density
(1-d5)), the A component fails $r_1$ times, the B component fails $r_2$ times and the C component fails $r_3$ times such that $r_1 + r_2 + r_3 = n$.

Further write the sum of the life times of the A component which failed first as $\sum_1 X$, the sums of the corresponding B and C components as $\sum_1 Y$ and $\sum_1 Z$; the sum of life times of the B component which failed first as $\sum_1 Y$; the sums of the corresponding C and A components as $\sum_2 Z$ and $\sum_2 X$; the sum of the life times of the C component which failed first as $\sum_1 X$, the sums of the corresponding A and B components as $\sum_2 Y$ and $\sum_2 X$.

Using this notation, the likelihood function of the sample becomes

$$L = (\alpha \beta \gamma')^n (\alpha' \beta' \gamma')^r \exp \left[ - (\alpha + \beta + \gamma - \gamma') \sum_1 X - \beta' \sum_1 Y - \gamma' \sum_1 Z - (\alpha + \beta + \gamma - \alpha') \sum_2 X - \gamma' \sum_2 Y - (\alpha + \beta + \gamma - \alpha') \sum_2 Z \right].$$

(Differentiating Log $L$ partially with respect to (w.r.t.) $\alpha, \beta, \gamma$, $\alpha'$, $\beta'$ and $\gamma'$ and equating these partial derivatives to zero, we get the simultaneous likelihood estimates as follows:

$$\hat{\alpha} = \frac{r_1}{\sum_2 X + \sum_2 Y + \sum_2 Z}, \quad \hat{\beta} = \frac{r_2}{\sum_1 Y + \sum_2 Y + \sum_3 Z}, \quad \hat{\gamma} = \frac{r_3}{\sum_1 X + \sum_2 Y + \sum_3 Z} \quad (3.1)$$

Extending Freund's arguments in case of bivariate distribution to trivariate distribution, it can be shown that the means and variances of $\alpha$, $\beta$ and $\gamma$ are

$$E(\hat{\alpha}) = \frac{n}{n-1} \alpha, \quad V(\hat{\alpha}) = \frac{n \alpha}{(n-1)^2 (n-2)} \quad (3.2)$$

$$E(\hat{\beta}) = \frac{n}{n-1} \beta, \quad V(\hat{\beta}) = \frac{n \beta}{(n-1)^2 (n-2)} \quad (3.3)$$

$$E(\hat{\gamma}) = \frac{n}{n-1} \gamma, \quad V(\hat{\gamma}) = \frac{n \gamma}{(n-1)^2 (n-2)} \quad (3.4)$$
A Trivariate Extension of Higher moments of $\alpha$, $\beta$ and $\gamma$ can be also readily obtained. According to Freund's arguments, the mean and variance of $\hat{\alpha}'$ are

$$E_{r_{n+1}}(\hat{\alpha}') = \frac{r_2 + r_3}{r_2 + r_3 - 1} \alpha' \quad \text{for} \quad r_2 + r_3 > 1,$$

(3.6)

and

$$\nu_{r_{n+1}}(\hat{\alpha}') = \frac{(r_2 + r_3)^2 \alpha'^2}{(r_2 + r_3 - 1)^2 (r_2 + r_3 - 2)} \quad \text{for} \quad r_2 + r_3 > 2.$$  

(3.7)

The subscripts $r_2$ and $r_3$ used in (3.6) and (3.7) indicate that they are held fixed and hence the mean and variance of $\alpha'$ are conditional. Corresponding formulae for the mean and variance of $\hat{\beta}'$ and $\hat{\gamma}'$ can be written down. The means of the reciprocals of $\hat{\alpha}'$, $\hat{\beta}'$ and $\hat{\gamma}'$ are given by

$$E(\hat{\alpha}'^{-1}) = \frac{1}{\alpha'}, \quad E(\hat{\beta}'^{-1}) = \frac{1}{\beta'}, \quad E(\hat{\gamma}'^{-1}) = \frac{1}{\gamma'}.$$  

(3.8)

The asymptotic expressions for the variances of $\hat{\alpha}'^{-1}$, $\hat{\beta}'^{-1}$ and $\hat{\gamma}'^{-1}$ can be obtained by using the methods, developed by Mendenhall and Lehmann [2] and hence we have

$$V(\hat{\alpha}'^{-1}) = \frac{1}{n \alpha'^2} \left(1 + \frac{\alpha}{\beta + \gamma}\right) + O\left(n^{-2}\right),$$

(3.9)

$$V(\hat{\beta}'^{-1}) = \frac{1}{n \beta'^2} \left(1 + \frac{\beta}{\gamma + \alpha}\right) + O\left(n^{-2}\right),$$

$$V(\hat{\gamma}'^{-1}) = \frac{1}{n \gamma'^2} \left(1 + \frac{\gamma}{\alpha + \beta}\right) + O\left(n^{-2}\right).$$

To find the asymptotic values for the means and variances of the reciprocals of $\alpha$, $\beta$ and $\gamma$, one can apply the extension of the methods developed by Mendenhall and Lehmann and hence it can be shown that

$$E(\hat{\alpha}'^{-1}) = \frac{1}{\alpha} + O\left(n^{-1}\right), \quad E(\hat{\beta}'^{-1}) = \frac{1}{\beta} + O\left(n^{-1}\right),$$

$$E(\hat{\gamma}'^{-1}) = \frac{1}{\gamma} + O\left(n^{-1}\right).$$  

(3.10)
4. Statistical Properties of the Model II:

The m.g.f. of the distribution \(1.25\) is

\[
M(t_1, t_2, t_3) = \int \int \int e^{t_1x + t_2y + t_3z} \delta(x, y, z) \, dx \, dy \, dz. \quad (4-1)
\]

Integrating over the appropriate regions of \(x, y\) and \(z\), we get

\[
M(t_1, t_2, t_3) = \frac{1}{(\alpha + \beta + \gamma - t_1 - t_2 - t_3)} \left[ \frac{\beta \gamma}{(1 - \frac{t_1}{\alpha}) (\alpha + \beta - t_1 - t_3)} \right] +
\]

\[
+ \frac{1}{(\alpha + \beta + \gamma - t_1 - t_2 - t_3)} \left[ \frac{\alpha \beta}{(1 - \frac{t_2}{\beta})(\beta + \gamma - t_2 - t_3)} \right] +
\]

\[
+ \frac{1}{(\alpha + \beta + \gamma - t_1 - t_2 - t_3)} \left[ \frac{\beta \gamma}{(1 - \frac{t_3}{\gamma})(\gamma + \alpha - t_3 - t_1)} \right]. \quad (4-2)
\]

Expanding the m.g.f. in power of \(t_1, t_2\) and \(t_3\), we have

\[
E(X) = \frac{1}{(\alpha + \beta + \gamma)} \left[ 1 + \frac{\gamma}{\alpha + \beta} + \frac{\beta \gamma}{\alpha + \beta} \cdot \frac{2\alpha + \beta + \gamma}{(\alpha + \beta)(\gamma + \alpha)} \right],
\]

\[
E(Y) = \frac{1}{(\alpha + \beta + \gamma)} \left[ 1 + \frac{\alpha}{\beta + \gamma} + \frac{\gamma}{\alpha + \beta} + \frac{\alpha \beta}{\beta + \gamma} \cdot \frac{2\beta + \gamma + \alpha}{(\beta + \gamma)(\alpha + \beta)} \right]. \quad (4-3)
\]
A Trivariate Extension of......

$$E(Z) = \frac{1}{(a + \beta + \gamma)^2} \left\{ \frac{1}{\gamma + \alpha + \beta + \gamma} \left[ (\alpha + \beta)^2 \gamma + \frac{\alpha \beta}{\alpha + \gamma}, \frac{\alpha \beta}{(\gamma + \alpha)(\beta + \gamma)} \right] \right\},$$

$$v(x) = \frac{1}{(a + \beta + \gamma)^2} - \frac{1}{(a + \beta + \gamma)^3} \left[ \frac{(\alpha + \beta)^2 \gamma}{\alpha + \gamma}, \frac{\alpha \beta}{\alpha + \gamma} \right] + \frac{2}{(a + \beta + \gamma)},$$

$$v(y) = \frac{1}{(a + \beta + \gamma)^2} - \frac{1}{(a + \beta + \gamma)^3} \left[ \frac{(\beta + \gamma) \gamma}{\beta + \gamma}, \frac{\beta \gamma}{\beta + \gamma} \right] + \frac{2}{(a + \beta + \gamma)},$$

$$v(z) = \frac{1}{(a + \beta + \gamma)^2} - \frac{1}{(a + \beta + \gamma)^3} \left[ \frac{(\gamma + \alpha) \beta}{\gamma + \alpha}, \frac{(\gamma + \beta) \alpha}{\gamma + \beta} \right] + \frac{2}{(a + \beta + \gamma)},$$

$$\text{cov}(X, Y) = \frac{1}{(a + \beta + \gamma)^2} - \frac{1}{(a + \beta + \gamma)^3} \left[ \frac{\gamma}{\gamma + \beta}, \frac{\beta \gamma}{\beta + \gamma} \right],$$

$$\text{cov}(Y, Z) = \frac{1}{(a + \beta + \gamma)^2} - \frac{1}{(a + \beta + \gamma)^3} \left[ \frac{\gamma}{\gamma + \beta}, \frac{\beta \gamma}{\beta + \gamma} \right],$$

$$\text{cov}(Z, X) = \frac{1}{(a + \beta + \gamma)^2} - \frac{1}{(a + \beta + \gamma)^3} \left[ \frac{\gamma}{\gamma + \beta}, \frac{\beta \gamma}{\beta + \gamma} \right].$$
It may be noted that each correlation coefficient lies between $-\frac{1}{2}$ and 1. If $a' \to \infty$, $b' \to \infty$, $\gamma \to \infty$, i.e., if $A$ and $B$ components fail simultaneously with $C$, then the correlation coefficient $\rho(x, y)$ approaches the limit unity. Similarly $\rho(y, z)$ and $\rho(z, x)$ both approach the limit $+1$ when $b' \to \infty$, $\gamma' \to \infty$, $a' \to \infty$ and $\gamma' \to \infty$, $a' \to \infty$, $\gamma \to \infty$ respectively. Physically speaking this corresponds to the case where the three-component system cannot function if all the components fail simultaneously. If $a' \to 0$, $b' \to 0$ and $a = b = \gamma$, $\rho(x, y)$ approaches the limit $-\frac{1}{2}$. In the same way, $\rho(y, z)$, $\rho(z, x)$ both tend to limit $\frac{1}{2}$, when $b' \to 0$, $\gamma' \to 0$, and $\gamma' \to 0$, $a' \to 0$, respectively and $a = b = \gamma$ in each case. Physically speaking, this corresponds to the case where one component cannot fail if the other two fail.

The marginal densities of $(X, Y)$, $(Y, Z)$ and $(Z, X)$ are

$$d_1(x, y) = \begin{cases}
\frac{\alpha(\beta + \gamma)(\beta - \gamma)e^{-(\beta + \gamma)y - \alpha x}}{(\beta + \gamma - \beta')}
- \frac{\alpha(\beta - \gamma)e^{-(\alpha + \beta + \gamma - \beta')x - \beta'y}}{(\beta + \gamma - \beta')}
+ \alpha e^{-(\alpha + \beta - \beta')x - \beta'y}
& \text{for } x < y,
\end{cases} \quad (4.6)
$$

$$d_2(y, z) = \begin{cases}
\frac{\beta(\gamma + \alpha)(\gamma - \alpha)e^{-(\gamma + \alpha)x - \beta'y}}{(\gamma + \alpha - \alpha')}
- \frac{\alpha(\alpha + \beta)e^{-(\alpha + \beta + \gamma - \alpha')y - \alpha'x}}{(\gamma + \alpha - \alpha')}
+ \alpha e^{-(\alpha + \beta - \alpha')y - \alpha'x}
& \text{for } y < x,
\end{cases} \quad (4.7)
$$

Provided $\beta + \gamma - \beta' \neq 0$ and $\gamma + \alpha - \alpha' \neq 0$. 


A Trivariate Extension of 

Provided \( \gamma + \alpha - \gamma' \neq 0, \alpha + \beta - \beta' \neq 0 \), and 

\[
\delta_3(z, x) = \frac{\gamma(\alpha+\beta)(\alpha-a')(\alpha+\beta)x - \gamma a'(a-a')e^{-(\alpha+\gamma-a')x}}{(a+\beta-a')e^{(a+\gamma-a')x}} - \frac{\gamma a'(a-a')e^{-(\alpha+\gamma-a')x}}{(a+\beta-a')e^{(a+\gamma-a')x}}
\]

for \( z < x \), 

\[
\frac{\alpha(\beta+\gamma)(\gamma-\gamma')e^{-(\beta+\gamma)x} - \alpha'(\gamma-\gamma')e^{-(\alpha+\beta+\gamma-x-x')y}}{(\beta+\gamma-\gamma')e^{(\beta+\gamma-x-y')}},
\]

provided \( \alpha + \beta - \alpha' \neq 0 \) and \( \beta + \gamma - \gamma' \neq 0 \). 

(4.8)

The marginal densities of \( X, Y \) and \( Z \) are 

\[
\delta^{(1)}(x) = \frac{(\alpha-a') (\alpha+\beta)e^{-(\alpha+\beta)x}}{(\alpha+\beta-a')} + \frac{(\alpha-a') (\gamma+\alpha)e^{-(\alpha+\gamma)x}}{(\alpha+\gamma-a')e^{(\alpha+\gamma-a')x}}
\]

\[
- \frac{(a-a') (\alpha+\beta+\gamma)e^{-(\alpha+\beta+\gamma)x} - a'\beta(2\alpha-2\alpha'+\beta+\gamma)e^{-(\alpha+\gamma-a')x}}{(\alpha+\beta+\gamma-a')(\alpha+\gamma-a')(\alpha+\beta+\gamma-a')}
\]

Provided \( \alpha + \beta + \gamma - \alpha' \neq 0 \), 

\[
\delta^{(2)}(y) = \frac{(\beta-\beta') (\beta+\gamma)e^{-(\beta+\gamma)y} - (\alpha+\beta)'e^{-(\alpha+\gamma)y}}{(\beta+\gamma-\beta')e^{(\beta+\gamma-\beta')y}} + \frac{(\beta-\beta') (\alpha+\gamma)e^{-(\alpha+\gamma)y}}{(\alpha+\beta+\gamma-\beta')e^{(\alpha+\beta+\gamma-\beta')y}}
\]

Provided \( \alpha + \beta + \gamma - \beta' \neq 0 \), 

\[
\delta^{(3)}(z) = \frac{(\gamma-\gamma')(\gamma+z)e^{-(\gamma+z)x}}{(\gamma+\alpha-\gamma')e^{(\gamma+\alpha-\gamma')x}} + \frac{(\gamma-\gamma')(\beta+\gamma)e^{-(\beta+\gamma)x}}{(\beta+\gamma-\gamma')e^{(\beta+\gamma-\gamma')x}}
\]

\[
- \frac{(\gamma-\gamma')(\alpha+\beta+\gamma)e^{-(\alpha+\beta+\gamma)x} - \gamma'\beta(2\gamma-2\gamma'+\alpha+\beta)e^{-(\gamma+\alpha-\gamma')x}}{(\alpha+\beta+\gamma-\gamma')(\gamma+\alpha-\gamma')(\beta+\gamma-\gamma')},
\]

Provided \( \alpha + \beta + \gamma - \gamma' \neq 0 \). 

If \( \alpha + \beta - \alpha' = 0, \beta + \gamma - \gamma' = 0, \gamma + \alpha - \gamma' = 0, \alpha + \beta - \beta' = 0, \beta + \gamma - \beta' = 0, \gamma + \alpha - \gamma' = 0, \alpha + \beta + \gamma - \alpha' = 0, \alpha + \beta + \gamma - \beta' = 0 \) and \( \alpha + \beta + \gamma - \gamma' = 0 \), the marginal densities in two variables are
The marginal densities of $X$, $Y$ and $Z$ are

$$
\delta_1(x, y) = \left\{ \begin{array}{ll}
\alpha e^{-\beta y - \alpha x} + \gamma \alpha' (y - x) e^{-\alpha x - \beta y} & \text{for } x < y, \\
\alpha e^{-(\alpha + \gamma) x - \beta y} + \beta \gamma' (x - y) e^{-\beta y - \alpha x} & \text{for } y < x,
\end{array} \right. \quad (4.12)
$$

$$
\delta_2(y, z) = \left\{ \begin{array}{ll}
\beta y e^{-\gamma z - \beta y} + \alpha \beta' (z - y) e^{-\beta y - \gamma z} & \text{for } y < z, \\
\beta y e^{-(\beta + \gamma) y - \gamma z} + \gamma \beta' (y - z) e^{-\gamma z - \beta y} & \text{for } z < y,
\end{array} \right. \quad (4.13)
$$

$$
\delta_3(z, x) = \left\{ \begin{array}{ll}
\gamma e^{-\alpha x - \gamma z} + \beta \gamma' (x - z) e^{-\gamma z - \alpha x} & \text{for } x < z, \\
\gamma e^{-(\alpha + \gamma) z - \alpha x} + \alpha \gamma' (z - x) e^{-\alpha x - \gamma z} & \text{for } z < x.
\end{array} \right. \quad (4.14)
$$

The marginal densities of $X$, $Y$ and $Z$ are

$$
\gamma(x) = (\alpha + \gamma \alpha') x e^{-\alpha x}, \quad (4.15)
$$

$$
\gamma(y) = (\beta + \alpha \gamma') y e^{-\beta y}, \quad (4.16)
$$

$$
\gamma(z) = (\gamma + \beta \gamma') z e^{-\gamma z}. \quad (4.17)
$$

when $\alpha = \alpha'$, $\beta = \beta'$ and $\gamma = \gamma'$, it is evident from (4.9)–(4.11) and (4.15)–(4.17) that $X$, $Y$ and $Z$ are independent with densities

$$
\delta^{(1)}(x) = \alpha e^{-\alpha x}, \quad x > 0, \quad \delta^{(2)}(y) = \beta e^{-\beta y}, \quad y > 0, \quad \delta^{(3)}(z) = \gamma e^{-\gamma z}, \quad z > 0. \quad (4.18)
$$

5. Estimates of the Parameters of the Model II

To estimate $\alpha$, $\beta$, $\gamma$, $\alpha'$, $\beta'$ and $\gamma'$ by the method of maximum likelihood in a random sample of $n$ items from a population having trivariate density (1.25), let

- $r_1$ be the items in which $B$ fails first and $A$ fails second,
- $r_2$ be the items in which $A$ fails first and $B$ fails second,
- $r_3$ be the items in which $C$ fails first and $B$ fails second,
- $r_4$ be the items in which $B$ fails first and $C$ fails second,
- $r_5$ be the items in which $A$ fails first and $C$ fails second,
- $r_6$ be the items in which $C$ fails first and $A$ fails second,

such that $r_1 + r_2 + r_3 + r_4 + r_5 + r_6 = n$. 

Further, let the sums of the life times of the B components which failed first and of the A components which failed second be written as \( \Sigma_{1}x \) and \( \Sigma_{1}y \) respectively and the sum of the corresponding C components as \( \Sigma_{1}z \). Similarly \( \Sigma_{i}x \), \( \Sigma_{i}y \) and \( \Sigma_{i}z \) for \( i = 2, 3, 4, 5, 6 \) in each case can be defined. Using this notation, the likelihood of the sample becomes

\[
L = \left( \alpha \beta y \right)^{y_{1} + r_{2}} (\beta r \alpha')^{r_{3} + r_{4}} (y r \beta')^{r_{5} + r_{6}} \exp \left[ -(\alpha + r - r') \right]
\]

\[
\Sigma_{1}x - \beta \Sigma_{1}y - r' \Sigma_{1}z - (\beta + r - r') \Sigma_{1}y - \alpha \Sigma_{1}x - r' \Sigma_{1}z - (\alpha + r - r') \Sigma_{1}y - \alpha' \Sigma_{1}x - r' \Sigma_{1}z - (\alpha + \beta - \alpha') \Sigma_{1}y - \beta' \Sigma_{1}y - \alpha' \Sigma_{1}x - (\alpha + \beta - \beta') \Sigma_{1}y - \beta' \Sigma_{1}y - \alpha' \Sigma_{1}x - (\alpha + \beta - \beta') \Sigma_{1}y - \beta' \Sigma_{1}y - \alpha' \Sigma_{1}x - (\alpha + \beta - \beta') \Sigma_{1}y - \beta' \Sigma_{1}y - \alpha' \Sigma_{1}x - (\alpha + \beta - \beta') \Sigma_{1}y - \beta' \Sigma_{1}y.
\]

(5.1)

The simultaneous likelihood estimates of \( \alpha, \beta, r, \alpha', \beta' \) and \( r' \) are

\[
\hat{a} = \frac{r_{1} + r_{2} + r_{3} + r_{6}}{\Sigma_{1}x + \Sigma_{2}x + \Sigma_{3}y + \Sigma_{4}z + \Sigma_{5}x + \Sigma_{6}y},
\]

(5.2)

\[
\hat{b} = \frac{r_{1} + r_{2} + r_{3} + r_{4}}{\Sigma_{1}y + \Sigma_{2}y + \Sigma_{3}y + \Sigma_{4}z + \Sigma_{5}x + \Sigma_{6}y},
\]

(5.3)

\[
\hat{r} = \frac{r_{3} + r_{4} + r_{5} + r_{6}}{\Sigma_{1}x + \Sigma_{2}y + \Sigma_{3}y + \Sigma_{4}z + \Sigma_{5}x + \Sigma_{6}y},
\]

(5.4)

\[
\hat{a'} = \frac{r_{3} + r_{4}}{\Sigma_{2}x - \Sigma_{3}y - \Sigma_{4}z},
\]

(5.5)

\[
\hat{b'} = \frac{r_{5} + r_{6}}{\Sigma_{1}y - \Sigma_{2}y - \Sigma_{3}y},
\]

(5.6)

\[
\hat{r'} = \frac{r_{1} + r_{2}}{\Sigma_{1}x - \Sigma_{2}z - \Sigma_{3}y},
\]

(5.6)

Following Freund's arguments, the mean and variance of \( \hat{a} \) for fixed \( r_{3} \) and \( r_{4} \) are

\[
E_{r_{3}, r_{4}} (\hat{a}) = \frac{r_{3} + r_{4}}{r_{3} + r_{4} - 1} \alpha for r_{3} + r_{4} > 1,
\]

(5.7)

\[
\text{Var}_{r_{3}, r_{4}} (\hat{a}) = \frac{(r_{3} + r_{4})^{2} (r_{3} + r_{4} - 2) \alpha^{2}}{(r_{3} + r_{4} - 1)^{2} (r_{3} + r_{4} - 2)} for r_{3} + r_{4} > 2.
\]