ON AUGMENTING RIGHT TRUNCATED EXPONENTIAL AND POWER FUNCTION STRENGTH WITH NORMAL STRESS

7.1. Introduction

If Y denotes the strength of a component and X the stress imposed on it, the reliability of the component is defined as

\[ R = P(Y > X). \]  

(7.1.1)

Such reliability of a component is known as its strength reliability. The determination of stress-strength reliability was first considered by Birnbaum (1956). Lloyed and Lipow (1962) described an application where X is the maximum chamber pressure generated by the ignition of a solid propellant in a rocket engine. Estimation of R under independent stress-strength model was considered by Doenton (1973), Beg and Singh (1979), Nandi and Aich (1994), Basu (1981), Reiser and Guttman (1986), Rehman, Ullah and Singh (2000), Srivastava (2005) and many others.

Alam and Roohi ((2002), (2003)) have studied the problem of stress-strength reliability in a different perspective. To achieve desired level of reliability of a component, they found the required parametric values of the assumed distributions of strength of a component. They assumed exponential stress and exponential and power function strength. It is assumed that the desired strength of equipment should be limited to a finite range. This is due to the strength of a component having n sub-components, all the sub-components are not likely to have an infinite strength, depending on the strength of the weakest components. Here, right truncated exponential and power function strength with normal stress is considered to achieve a desired level of \( P(Y > X) \) by obtaining required values of parameter of strength distributions.
7.2. Derivation of the Results under Power Function Stress

Let X and Y have the probability density function (pdf) given by

\[ f(x) = \frac{1}{\sqrt{2\pi}} \sigma e^{-\frac{1}{2} \left( \frac{x-\mu}{\sigma} \right)^2}, \quad -\infty < x, \mu < \infty, \sigma > 0. \]  

(7.2.1)

and

\[ g(y) = \left( \frac{a}{\theta} \right)^{a-1} y^{a-1}, \quad 0 < y < \theta, a > 0. \]  

(7.2.2)

Since the upper limit of Y is \( \theta \), Y cannot exceed X if X exceeds \( \theta \). Hence it is necessary to consider \( P(X > \theta) \) first. It will indicate the change of alternative failure of the strength Y against the stress X.

We may regard such an eventuality as a 'disaster'.

Here,

\[ P(X > \theta) = 1 - \Phi \left( \frac{\theta - \mu}{\sigma} \right) \]

\[ = 1 - \Phi(k) \]

\[ = \alpha \quad (\text{say}), \]

(7.2.3)

where \( k = \frac{\theta - \mu}{\sigma} \).

Of course this probability should be very small which implies that for fixed \( \theta \) and known \( \mu \) and \( \sigma \), \( k \) can be determined, which has to be a large positive real number. Clearly, chances of disaster recede as \( \theta \) increases with respect to the standard deviation (\( \sigma \)) of the stress. Alternatively we may find the required values of \( k \) for a fixed tolerance level \( \alpha \).

Table 7.1 gives the values of \( k \) for selected tolerance level \( \alpha \).

**Table 7.1: Values of \( k \) for selected tolerance level \( \alpha \)**

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>0.00005</th>
<th>0.0005</th>
<th>0.001</th>
<th>0.02</th>
<th>0.05</th>
<th>0.1</th>
<th>0.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k )</td>
<td>3.891</td>
<td>3.291</td>
<td>3.09</td>
<td>2.576</td>
<td>1.645</td>
<td>1.282</td>
<td>0</td>
</tr>
</tbody>
</table>

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Obviously the ultimate strength capability $\theta$ must increase if we wish to have a small tolerance level. Once the value of $k$ is fixed as per requirement, $\theta$ is fixed as $\theta = \mu + k_1 \sigma$. Now the reliability $R$ can be written as

$$R = P(Y > X)$$

$$= \int_{-\infty}^{\infty} \int_{0}^{1} x f(x) g(vx) dv dx$$

(7.2.4)

However, in this particular case, we have

$$R = \int_{-\infty}^{\infty} \int_{0}^{1} x f(x) g(vx) dv dx$$

$$= \frac{\mu + k_1 \sigma}{x} \int_{-\infty}^{\mu + k_1 \sigma} a \frac{x^{a-1} v^{a-1}}{(\mu + k_1 \sigma)^a} dv dx$$

$$= \Phi(k_1) - \frac{1}{(\mu + k_1 \sigma)^a} \frac{\mu + k_1 \sigma}{\sqrt{2\pi \sigma}} e^{-\frac{1}{2} \left( \frac{x - \mu}{\sigma} \right)^2}$$

$$= \Phi(k_1) - \frac{1}{(\mu + k_1 \sigma)^a} \mu_a(k_1),$$

where $k_1 = \mu + k_1 \sigma$

and

$$\mu_a'(k_1) = a^{th} \text{ raw moment of } N(\mu, \sigma^2) \text{ distribution, right truncated at } k_1.$$  

(7.2.5)

To evaluate $\mu_a'(k_1)$ we use the following recurrence relation of right truncated normal distribution.
a \mu'_{a-1}(k_1) = \frac{1}{\sigma^2} \mu'_{a+1}(k_1) + \left( \frac{k_1 - \mu}{\sigma} \right) \mu'_a(k_1), \quad a=2,3,...

with

\mu'_1(k_1) = \mu - \sigma \frac{\phi(b)}{\Phi(b)}, \quad b = \frac{k_1 - \mu}{\sigma} = k

and

\mu_2(k_1) = \text{variance of right truncated normal distribution, truncated at } k_1

= \sigma^2 \left[ 1 - b \frac{\phi(b)}{\Phi(b)} - \left( \frac{\phi(b)}{\Phi(b)} \right)^2 \right],

where \phi(b) = \frac{1}{\sqrt{2\pi}} e^{-b^2/2},

\Phi(b) = \int_{-\infty}^{b} \phi(x)dx.

Without loss of generality we can assume \( \mu = 0 \) and \( \sigma = 1 \) so that \( k_1 \) becomes \( k \) and (7.2.5) reduces to

\[ R = \Phi(k) - \frac{1}{k^a} \mu'_a(k) \tag{7.2.6} \]

Using equation (7.2.6) we obtain strength reliability for selected values of \( a \) and \( k \) as shown in Table 7.2.

**Table 7.2 Values of R for different choice of \( a \) and \( k \)**

<table>
<thead>
<tr>
<th>( a )</th>
<th>1.2</th>
<th>1.5</th>
<th>1.8</th>
<th>2.1</th>
<th>2.4</th>
<th>2.7</th>
<th>3.0</th>
<th>3.3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td>2</td>
<td>1.373</td>
<td>0.581</td>
<td>0.701</td>
<td>0.777</td>
<td>0.827</td>
<td>0.863</td>
<td>0.889</td>
<td>0.908</td>
</tr>
</tbody>
</table>
7.3. Derivation of the Results under Right Truncated Exponential Strength

Here we assume the probability density function of the right truncated exponential distribution of strength as

\[ g(y) = \frac{y}{\beta} e^{-\frac{y}{\beta}}, \quad 0 < y < \theta, \beta > 0 \]

And the distribution of stress (X) follows normal distribution defined in (7.2.1).

With the similar arguments as put up in Section 7.2, we find R from equation (7.2.4)

\[ R = P(Y > X) \]

\[ = \int_\theta^{\infty} x f(x) \left\{ \frac{\mu+k\sigma}{x} \frac{e^{-v/\beta}}{\beta(1-e^{-\mu+k\sigma}/\beta)} \right\} dx \]

\[ = \frac{1}{1-e^{-\mu+k\sigma}/\beta} \left[ e^{-\mu/\beta} \int_{-\infty}^{\sigma(\mu/\beta)-w^2/2} dw \right. \]

\[ - e^{-\mu+k\sigma} \mu/\beta \left. \int_{-\infty}^{\infty} e^{-w^2/2} dw \right] \]

\[ = \frac{1}{1-e^{-\mu+k\sigma}/\beta} \left[ e^{-\mu/\beta} \left\{ \frac{\sigma^2}{2\beta^2} - e^{-(\mu+k\sigma)/\beta} \int_{-\infty}^{\infty} e^{-w^2/2} dw \right\} \right. \]

\[ - e^{-(\mu+k\sigma)/\beta} \Phi(k) \]

(7.3.1)
Here the probability of disaster, \( \alpha = P(X > 0) \) has to be small which implies \( k \) must be positive. As shown by Gradshteyn and Ryzhik (2006) for \( k > 0 \), we have

\[
\int_{u}^{\infty} e^{-u-u^2/4\beta}du = \sqrt{\pi\beta} e^{\beta^2} \left[ 1 - \Phi \left( \sqrt{\beta + \frac{u}{2\beta}} \right) \right]; \beta > 0, u > 0.
\]

Thus for \( \mu = 0, \sigma = 1 \), the values of \( R \) from (7.3.1) can be computed for different choice of \( \beta \) and \( k \). For selected values of \( \beta \) and \( k \), the values of \( R \) given in Table 7.3.

**Table 7.3 Values of \( R \) for different choice of \( \beta \) and \( k \)**

<table>
<thead>
<tr>
<th>( \beta )</th>
<th>1.2</th>
<th>1.5</th>
<th>1.8</th>
<th>2.1</th>
<th>2.4</th>
<th>2.7</th>
<th>3.0</th>
<th>3.3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.7 5</td>
<td>0.845</td>
<td>0.849</td>
<td>0.863</td>
<td>0.873</td>
<td>0.879</td>
<td>0.883</td>
<td>0.885</td>
<td>0.886</td>
</tr>
<tr>
<td>0.8 7</td>
<td>0.692</td>
<td>0.719</td>
<td>0.740</td>
<td>0.754</td>
<td>0.764</td>
<td>0.770</td>
<td>0.774</td>
<td>0.776</td>
</tr>
<tr>
<td>0.9 9</td>
<td>0.456</td>
<td>0.506</td>
<td>0.541</td>
<td>0.566</td>
<td>0.582</td>
<td>0.593</td>
<td>0.601</td>
<td>0.606</td>
</tr>
<tr>
<td>0.9 5</td>
<td>0.355</td>
<td>0.416</td>
<td>0.459</td>
<td>0.488</td>
<td>0.509</td>
<td>0.522</td>
<td>0.532</td>
<td>0.538</td>
</tr>
<tr>
<td>1.0 8</td>
<td>0.264</td>
<td>0.334</td>
<td>0.384</td>
<td>0.419</td>
<td>0.443</td>
<td>0.460</td>
<td>0.471</td>
<td>0.479</td>
</tr>
<tr>
<td>1.1 0</td>
<td>0.097</td>
<td>0.189</td>
<td>0.254</td>
<td>0.299</td>
<td>0.331</td>
<td>0.353</td>
<td>0.369</td>
<td>0.381</td>
</tr>
<tr>
<td>1.1 3</td>
<td>0.021</td>
<td>0.124</td>
<td>0.196</td>
<td>0.246</td>
<td>0.282</td>
<td>0.308</td>
<td>0.326</td>
<td>0.339</td>
</tr>
<tr>
<td>1.1 8</td>
<td>0.021</td>
<td>0.124</td>
<td>0.196</td>
<td>0.246</td>
<td>0.282</td>
<td>0.308</td>
<td>0.326</td>
<td>0.339</td>
</tr>
</tbody>
</table>
7.4. Discussion

While manufacturing an item with its strength following power function or right truncated exponential distribution, it is likely that the possible values of $\theta$ may have an upper limit say $\theta_0$. For example the capacity of accelerating an engine must be subject to a maximum possible speed. For fixed tolerance level $\alpha$, the desired value of $\theta$ say $\theta_a$ can be obtained from (7.2.3) and if $\theta_a \leq \theta_0$ we may obtain the required value of the parameter $\alpha$ and $\beta$ from Tables 7.2 and 7.3 respectively so that the desired strength reliability can be achieved. However, if $\theta_a > \theta_0$, we will have to either adjust $\alpha$ or look for an alternative item.