CHAPTER 1

INTRODUCTION

1.1 HISTORICAL OUTLINE

Research in the field of solid mechanics is not only for basic understanding of mechanical phenomenon but also for advancement of engineering methodology in most of the areas throughout mechanical and structural technology. Advancement of this subject is centralized for the assurance of safety, reliability and economy in the design of different types of devices, which are used in nuclear and gas turbine, aerospace and surface transportation technology, earthquake resistant designs, offshore structures, orthopedic devices, material processing and manufacturing technologies.

The mathematical theory of elasticity is used to determine the state of strain or displacement, in a solid body which is subjected to the action of equilibrating system of forces, with the endeavors to obtain results which are practically important in architecture, engineering and all other useful arts in which the material of construction is solid.

The nature of the resistance of solids to rupture was first considered by famous mathematician Galileo (1638). Although he assumed solids as inelastic and not under the influence of any law that connect the displacements with forces or of any physical hypothesis which is capable of yielding such laws. But his enquiries gave the direction which was subsequently followed by many investigators of this field. In the history of the theory that was started by Galileo, there are two great landmarks: first is the discovery of Hooke’s law in 1660 and the second is formulation of the general equations by Navier in 1821. Hooke’s law gives the necessary experimental foundation for the theory. When the general equations are obtained, all questions related to small strain of elastic bodies were reduced to a matter of mathematical formulation.

In 1678 Hooke gave the famous law of proportionality of stress and strain which is based on his name, in the words “Ut tensio sic vis; that is, power of any spring is in the same proportion with the tension thereof”. With the help of “spring” Hooke explained, any “spring
body” and by “tension” he explained “extension” or more generally “strain”. He discovered this law in 1660, but it was not published until 1676 and then published only under the form of the anagram, ceiiinosssttuu. This law is the basis of the mathematical theory of elasticity and afterward the generalization of this has been made. During the earlier period in the history of science (1638-1820), while the various investigations on special problems were made, there was a cause of work that led to various generalizations. Earlier in the eighteenth century the Newtonian concept of material bodies, displaced the Cartesian conception of a plenum pervaded by “vortices”. Newton regarded his “molecules” as of finite sizes and definite shapes but his successors gradually simplified them into material points.

Navier was the first who investigated the general equations of equilibrium and equations of vibration of elastic solids. He set out these equations from the Newtonian concept of the constitution of the bodies with the assumptions that the elastic reactions are the result of variations in the intermolecular forces which arise from changes in the molecular configuration. He regarded the molecules as material points and assumed that the force between two molecules, whose distance is slightly increased, is proportional to the product of the increment of the distance and some function of the initial distance.

Consideration of plastic state of matter is very important in many branches of science and engineering. The scientific study of the plastic state of metals began in 1864. In this year Tresca published a number of experiments on punching and extrusion, which led him to state that a plastic yielding of material starts when the maximum shear stress attained a critical value. Tresca’s yield criterion was applied by Saint-Venant to determine the stresses in a partially plastic cylinder under torsion or bending (1870) and in a completely plastic tube under internal pressure (1872). Levy (1871), adopted Saint-Venant’s concept of an ideal plastic material, and proposed three-dimensional relationships between stress and rate of plastic strain. In 1900, Guest investigated the yielding in hollow tubes under combined axial tension with internal pressure and obtained results which are in full agreement with the maximum shear stress criterion (1913). These results were further interpreted by Hencky after some years and he stated that yielding occurs when the elastic shear-strain energy reaches some critical value. In 1923, Nadai investigated theoretically and experimentally that the plastic zone is a twisted prismatic bar of arbitrary contour. Afterwards this subject has been studied intensively.
The criterion of new engineering constructions and devices which operate under either elevated or low temperature conditions are the result of various developments in this area. The influence of high temperature conditions on material properties has been considered in the design of steam turbines, automobile parts, oil refineries and other chemical equipments. In recent years high speed aircraft, gas turbines, missiles, rockets and nuclear reactors are among the developments in which high temperature condition exists. In these applications, the influence of high temperature on the material properties is an important consideration for designers. The extent of this influence of temperature depends upon many factors including the material, loading conditions and state of stress. Creep is defined as time-dependent deformation produced in the solids subjected to stresses. In case of many materials such as metals and alloys, elevated temperature is applied to produce creep. The first recorded experiment relating to creep appeared in 1830’s in suspension bridges, measuring instruments and steam engines. After a long time, Andrade (1910) gave the concepts of primary, secondary and tertiary creep in case of uniaxial creep tests with constant load or stress. In 1929, Norton discovered the exponential law, which applies to many metals, i.e.

$$\dot{\varepsilon} = \frac{d\varepsilon}{dt} = k\sigma^n,$$

(1.1)

where $\dot{\varepsilon}$ is strain rate, $\sigma$ is stress and $k$ and $n$ are constants.

Around 1930’s, Bailey had shown that creep of structural metals takes place under constant volume and this deformation is not influenced by superimposed hydrostatic pressure. With these facts of creep deformations and assumption of isotropy, Odquist [83] deduced constitutive relationships for secondary creep under triaxial stresses, which are same as von-Mises’ equation of time-dependent plasticity. A lot of progress has been made theoretically as well as experimentally in the field of creep deformations to define certain aspects but still no complete theory is available to explain this complex creep phenomenon.

1.2 ELASTIC-PLASTIC AND CREEP PHENOMENON

The “theory of elasticity” deals with the systematic study of stresses, strains and displacements in an elastic body under the influence of external forces. All structural materials possess the property of elasticity up to certain extent, that is, if external forces producing deformation in a structure that do not exceed a certain limit, the deformation disappears when
the forces are removed. In various engineering disciplines the main purpose of studying elasticity is to analyze the stresses and displacements of structural and machine elements in the elastic range and to check the strength, stiffness and stability of materials.

“Theory of plasticity” is the name given to the mathematical study of stresses and strains in plastically deformed solids. This theory follows the well-established setup formed by the “theory of elasticity”. The relation between elastic and plastic properties of metals to crystal structures and cohesive forces belongs to the subject known as ‘metal physics’. The theory of plasticity takes certain experimental observations of the macroscopic behavior of a plastic solid in uniform states of combined stresses at its starting point. The task of this theory includes two parts. In the first part the explicit relationship between stress and strain are constructed which closely agree with the observations and in second, mathematical techniques are developed for calculating non-uniform distributions of stresses and strains in bodies which are permanently deformed. Unlike elastic solids in which the state of strain depends only on the final state of stress, the deformation in plastic solid is determined by the complete history of applied load. The problem of plasticity is therefore an essential increment in nature.

The condition in which material deforms continuously under applied load for a prolonged period of time, usually at elevated temperature is called ‘creep’. Such type of constant load test performed on any material is called a ‘creep test’. The conventional stress (load divided by initial cross-section) is called a ‘creep-stress’. The gradual strain is called ‘creep-strain’, and the graph between strain and time is called ‘creep-curve’ and the slope of the curve is called the ‘creep-rate’. Creep strains may be elastic, plastic or a combination of both and it might occur too slowly to detect or too rapidly to follow. Even in such cases, it is evident that creep strain is always time-dependent. In many materials including metals and alloys, elevated temperatures is applied to produce creep. Change in temperature causes thermal expansion and has ill effects upon the material behavior. Creep occurs at normal temperatures in various non-metallic materials such as plastics, wood and concrete.

The theory of elasticity and plasticity describe the mechanics of deformations in most engineering solids. The theory of elasticity and plasticity applied to metals and alloys are based on experimental studies of the relation between stress and strain in a polycrystalline aggregate under simple loading conditions. In order to understand the limitations of these theories, the engineers, are mainly interested in design and manufacturing of the structure of
metals. The validity of any mathematical model of deformations and stress-distribution depends on one’s ability to access the mechanical properties and characteristic of the material to be used and to apply this information in the actual calculations. Due to this reason, testing of material properties is a matter of utmost importance for engineers. No design can be successful unless it is firmly based on good understanding of the potential of the material and its consequent ability of satisfactory response to a given system of loading. It is important to notice that any technique of mechanical testing can provide information only about the average material properties and cannot give any explanation of behavior of material in a particular situation. We can only expect that in the absence of any unknown internal defects, i.e., metallic and non-metallic inclusions, alloy will behave in the way suggested by the results of test.

The ‘tensile test’ is basic method for bringing out the behavior of structural materials. The stress-strain behavior is completely defined in figure 1.1. In first part of the test, it is observed that material obeys Hooke’s law, i.e. the material is in fully elastic state and stress is proportional to strain, giving the straight-line graph. Some point $A$ is eventually reached, where the linear nature of the graph ceases is termed as a ‘limit of proportionality’. For a short period beyond this point the material may still be in elastic state i.e. deformations are completely recovered when load is removed but Hooke’s law does not apply. The limiting point $B$ for this condition is termed as ‘elastic limit’. In most of the practical problems it is assumed that points $A$ and $B$ are coincident. Beyond this limit of elasticity, plastic deformations occur and strains are not totally recoverable. There will thus be some permanent deformations in the material after the removal of load. After the points $C$ (upper limit point) and $D$ (lower yield point), rapid increase in strain occurs without corresponding increase in load or stress. The graph thus becomes much shallower and covers a much greater portion of the strain axis than the elastic range of the material. The capacity of a material to allow these large plastic deformations is called ductility. Beyond the yield point some extra load is required to take the strain to point $E$ on the graph. Between the points $D$ and $E$ the material is said to be in the elastic-plastic state. Some of the sections remain elastic and hence contributes in recovery of the original dimensions when load is removed. Beyond $E$ the cross-sectional area of the bar reduces rapidly over a relatively small length of the bar and the bar is said to be
in necking position. This necking takes place when the load reduces, and fracture of the bar finally occurs at the point \( F \).

\[
\begin{align*}
&\text{Elastic} \\
&\text{Partially plastic}
\end{align*}
\]

**Figure 1.1:** Typical tensile test curve for mild-steel (Stress-strain curve).

In experimental curve (figure 1.1) the strain is usually defined as 
\[
e = \left( \frac{l - l_0}{l_0} \right)
\]
where \( l \) is the length of specimen after deformation which was originally of length \( l_0 \). The experimental results are plotted using the measures 
\[
e = \log \left( \frac{l}{l_0} \right)
\]
For \( e \leq 1 \) the two measures virtually coincide, for example in the initial elastic region.

One of the characteristic features of metal’s stress-strain curve is that the slope in the plastic region is much smaller than in the elastic region. The stress-strain curve is the characteristic of material and depends not only on the chemical constitution but also on heat-treatment and methods of fracture. The strain at fracture is measure of ductility, i.e., if the strain at fracture is ‘large’, the material is said to be ductile and if it is ‘small’, the material is said to be brittle. The terms ‘large’ and ‘small’ are usually related to the strain at yield. The behavior of stress-strain i.e. the initial elasticity followed by plasticity to fracture does not depend upon the time. As soon as the stress changes, strain changes accordingly. Elastic and plastic behaviors are thus said to be instantaneous. As strain rate of the material increases, the strain-hardening effect also increases and therefore the level of yield stress rises. The typical
effect on the stress-strain curve is shown in figure 1.2 when tensile test is performed at different strain rates from very slow to very fast. This shows that an increase in strain rate causes an increase in elastic modulus, yield stress, fracture stress and decrease in ductility. Usually an increase in the temperature of test decreases both the yield and elastic modulus. So we can say that the temperature also influences the fracture ductility. It is observed that at high temperature plastic strains, under the effect of a small stresses, grows with time. Creep is the gradual increase of the plastic strains in a material with time at constant load. Particularly at high temperature some materials are susceptible to creep phenomenon and even under the constant load strains can increase continually until fracture. This type of fracture is particularly occurs in turbine blades, nuclear reactors, furnaces, rocket motors, etc. The general form of the strain and time graph or creep curve is shown in figure 1.3 for two typical operating conditions. In each of these cases curve exhibits four main features:

1. **An initial strain**, due to initial load. In most of the cases this is an **elastic strain**.
2. **A primary creep** region, during which the creep rate (slope of the graph) vanishes.
3. **A secondary creep** region, when the creep rate is sensibly constant.
4. **A tertiary creep** region, during which the creep rate accelerates to final fracture.

![Figure 1.2: Effect of strain-rate on tensile test.](image1)

![Figure 1.3: Creep curve](image2)

In general, strain-time curve is of the shape as mentioned in figure 1.3 where four stages of creep curve can be distinguished. At $t = 0$, the curve $OA$ shows an instantaneous
response $\varepsilon_0$, which depends on the magnitude of stress and it can be elastic or elastic-plastic. In portion $AB$ creep strain rate is comparatively high but decreases with time due to strain hardening. This portion of curve is called primary creep or transient creep or logarithmic creep and can occur at both low and high temperatures. In portion $BC$ the graph is linear which shows that creep strain rate or maximum creep rate is sensibly constant and this portion is known as secondary creep. At some point, another transition occurs and creep rate again accelerates to final stage. This final stage, which is called tertiary creep, represents a region where creep rate continues to increase and finally results in fracture. In case of lower stresses and temperature the final stage of creep is not observable during the usual time covered by creep tests. The curve also shows that for all materials, rupture condition eventually occurs.

We usually associate creep with high temperatures, but the requirement of temperature actually depends upon the material. Potentially, creep can takes place at any temperature above absolute zero. In various non-metallic materials such as concrete, wood and natural high polymers, creep occurs at normal temperature. In typical structural materials like Magnesium (Mg), Lead (Pb) and its alloys, creep occurs at room temperature or below room temperature whereas in other materials such as metals, creep occurs at high temperature. This creep phenomenon of metals is known for several years. But recently some progress is seen in formulation of realistic picture of the mechanism for creep. It was the complexity of this process which is responsible for slow progress in developing a satisfactory understanding of the theory of creep. A lot of progress has been made theoretically as well as experimentally to define certain major aspects of creep, yet no complete theory has been developed to explain this complex creep phenomenon.

When deformation of metals occurs in a particular direction, as in rolling, drawing, stretching or forging, the mechanical properties may be anisotropic on a macroscopic scale. Single crystals are usually highly anisotropic materials with respect to plastic deformation. The fabrication procedures used to produce metals in this section give rise to anisotropic mechanical properties. The Hill’s theory [51], is not suitable for the mechanical anisotropy of any of the anisotropic metals except that of materials exhibiting planar isotropy. It does not allow the presence of a Bauschinger effect and can be applied to anisotropic materials of restricted orthotropic symmetry. However, due to recent aerospace and commercial
applications, such as helicopter, rotor blade, compressors, fly wheel and automobile structures etc., anisotropic non-homogeneous materials are effectively used and it enhances the scope of further research in this area.

A number of review papers, books and proceedings of symposia have been published with macroscopic elastic, plastic and creep behaviour of anisotropic materials, see for example Altenbach and Skrzypek [2], Boyle and Spence [12], Bridgman [13], Chakrabarty [14], Finnie and Heller [24], Goodier and Hoff [26], Green [27], Han and Reddy [47], Hetnarski and Ignaczak [50], Hill [51], Johnson and Meller [58], Kraus [65], Love [69], Lubahn and Felger [70], Marin [71], Marsden and Hughes [72], Nabarro and Villiers [76], Nadai [77], Odquist [83], Parkus [91], Penny and Marriott [93], Sadd [98], Sokolinikoff [133], Swainger [134] amd Timoshenko and Goodier [138], .

1.3 FINITE DEFORMATION

Lots of countable work has been done on the theory of finite deformation. This theory of finite deformation [133] has been applied to solve various problems of solid mechanics which cannot be dealt with the classical theory of infinitesimal deformation. It has qualitatively predicted a yield point and combined it into one of the two theories of elastic failure, i.e., principal maximum stress hypothesis and maximum shear stress theory or maximum principal stress difference theory. It has also proved that yield stress in compression can be several times of the tension, which is in agreement with Bauschinger effect. The applications of this theory gives axial stresses in cylinders subjected to torsional loads, which are neglected in the classical theory.

To define finite deformation (the deformations in which displacement and its derivatives are not linear) in a continuous medium, two methods are generally used, namely, the Lagrangian method and the Eulerian method. The components of strain in these two methods are described by the co-ordinates of a particle either in the strained state or in the unstrained state as independent variables. Many researchers have adopted the Lagrangian method due to its mathematical convenience whereas Seth [100-113] and many other authors [8-11, 28-45, 87-88, 115-128] emphasized on the use of Eulerian method. Seth [100] has quoted in this connection: “Like body stress equations, these (strain components) should be referred to actual position of a point P of the material in the strained condition, and not to the
position of point considered before strain. Apparently, Filon and Coker were the first to notice
this point and emphasize on its importance.

Eulerian and the Lagrangian viewpoints coincide with each other for infinitesimal
deformations, so there is no need for the distinction between these. If the co-ordinates of a
particle lying on curve $C_0$ (before deformation) be taken as $(a_1, a_2, a_3)$ and the co-ordinates
of the same particle after deformation (now lying on the curve $C$) are taken as $(x_1, x_2, x_3)$ then
the elements $dS_0$ and $dS$ of the arc of the curves $C_0$ and $C$ respectively are given by

$$dS_0^2 = da_1^2 + da_2^2 + da_3^2 = da_i da_i, \quad (i = 1, 2, 3) \tag{1.2}$$
$$dS^2 = dx_1^2 + dx_2^2 + dx_3^2 = dx_i dx_i, \quad (i = 1, 2, 3) \tag{1.3}$$

In Eulerian description $x_i$ are independent variables and equation of transformation is given
by $a_i = a_i(x_1, x_2, x_3)$. We can also write it as

$$da_i = a_{i,j} dx_j. \quad (i, j = 1, 2, 3) \tag{1.4}$$

By using equation (1.4) in equation (1.2) and (1.3), we have

$$dS_0^2 = da_i da_i = a_{i,j} a_{i,k} dx_j dx_k, \quad (i, j, k = 1, 2, 3) \tag{1.5}$$
$$dS^2 = dx_i dx_i = \delta_{jk} dx_j dx_k. \tag{1.6}$$

The necessary and sufficient condition for the transformation $a_i = a_i(x_1, x_2, x_3)$ to be one of
rigid body motion is that $dS^2$ and $dS_0^2$ should be equal for all curves $C_0$. Hence we take the
difference $dS^2 - dS_0^2$ as a measure of strain and write

$$dS^2 - dS_0^2 = 2 e_{jk}^A dx_j dx_k. \tag{1.7}$$

The strain tensor $e_{jk}^A$ for infinitesimal strains was introduced by Cauchy and for finite strains
by Almansi and Hamel. This strain tensor is known as Almansi strain tensor. From the
expression (1.5) and (1.6), we have

$$dS^2 - dS_0^2 = (\delta_{jk} - a_{i,j} a_{i,k}) dx_j dx_k. \tag{1.8}$$

On comparing equations (1.7) and (1.8), we obtain

$$2 e_{jk}^A = \delta_{jk} - a_{i,j} a_{i,k}. \tag{1.9}$$

On writing the strains $e_{jk}^A$ in terms of the displacement components $u_i = x_i - a_i$, we get
\[ 2e^A_{jk} = u_{j,k} + u_{k,j} - u_{i,j}u_{i,k}, \quad (1.10) \]

where \( e^A_{jk} \) are called the Eulerian strain components.

If Lagrangian co-ordinates are used then \( a_i \) are taken as independent variables and equations of transformation are of the form \( x_i = x_i(a_1, a_2, a_3) \). We can also write
\[
dx_i = x_{i,j}da_j. \quad (i, j = 1,2,3) \quad (1.11)
\]

Thus the elements \( dS_0 \) and \( dS \) of the arc of the curve \( C_0 \) and \( C \) are given by the equation
\[
dS_0^2 = da_ida_i = \delta_{jk}da_jda_k, \quad (1.12)
\]
\[
dS^2 = dx_idx_i = x_{i,j}x_{i,k}da_jda_k. \quad (1.13)
\]

The Lagrangian components of strain \( \varepsilon_{jk} \) are defined as
\[
dS^2 - dS_0^2 = 2\varepsilon_{jk}da_jda_k. \quad (1.14)
\]

Also from equations (1.12) and (1.13), it has been obtained
\[
dS^2 - dS_0^2 = (x_{i,j}x_{i,k} - \delta_{jk})da_jda_k. \quad (1.15)
\]

Comparing equations (1.14) and (1.15), it has been obtained
\[
2\varepsilon_{jk} = x_{i,j}x_{i,k} - \delta_{jk}. \quad (1.16)
\]

Expressing \( \varepsilon_{jk} \) in terms of displacement components \( u_i = x_i - a_i \), we get
\[
2\varepsilon_{jk} = u_{j,k} + u_{k,j} + u_{i,j}u_{i,k}. \quad (1.17)
\]

In equation (1.10) differentiation is carried out with respect to the variable \( x_i \) while equation (1.17) is differentiated with respect to the variable \( a_i \), the typical expressions for \( e^A_{jk} \) and \( \varepsilon_{jk} \) in an unabridged form can be written as
\[
e^A_{xx} = \frac{\partial u}{\partial x} - \frac{1}{2}\left[ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial x} \right)^2 \right], \quad e^A_{aa} = \frac{\partial u}{\partial a} + \frac{1}{2}\left[ \left( \frac{\partial u}{\partial a} \right)^2 + \left( \frac{\partial v}{\partial a} \right)^2 \right],
\]
\[
2e^A_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - \left[ \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \frac{\partial v}{\partial y} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right], \quad \text{and}
\]
\[
2e^A_{ab} = \frac{\partial u}{\partial b} + \frac{\partial v}{\partial a} + \left[ \frac{\partial u}{\partial a} \frac{\partial u}{\partial b} + \frac{\partial v}{\partial a} \frac{\partial v}{\partial b} + \frac{\partial w}{\partial a} \frac{\partial w}{\partial b} \right],
\]
where $e_{xx}^A$ and $e_{aa}^A$ represent the extension of vectors which are originally parallel to the co-ordinate axes, while $e_{ab}^A$, $e_{xy}^A$ represent shear strain or change of angle between vectors originally at the right angles.

In case of small deformations i.e. when the displacements and its derivatives are small, the two representations, Eulerian and Lagrangian, become same and therefore it is incorporeal which system is employed. In this case, if we neglect the non-linear terms occurring in the partial derivatives in equation (1.10) and (1.17) then both these equations reduce to

$$e_{jk}^A = \frac{1}{2} [u_{j,k} + u_{k,j}],$$(j, k = 1, 2, 3)

which exists in the case of infinitesimal transformations.

In cylindrical polar co-ordinates, the components of strain in (1.10) are given by

$$e_{rr}^A = \frac{\partial u}{\partial r} - \frac{1}{2} \left[ \left( \frac{\partial u}{\partial r} \right)^2 + \left( \frac{\partial v}{\partial r} \right)^2 + \left( \frac{\partial w}{\partial r} \right)^2 - v^2 \right],$$

$$e_{\theta\theta}^A = \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{u}{r} \frac{1}{2r^2} \left[ \left( \frac{\partial u}{\partial \theta} \right)^2 + r^2 \left( \frac{\partial v}{\partial \theta} \right)^2 + \left( \frac{\partial w}{\partial \theta} \right)^2 - vr^2 \frac{\partial u}{\partial \theta} + u \frac{\partial v}{\partial \theta} - v \frac{\partial u}{\partial \theta} + ur^2 \frac{\partial v}{\partial \theta} + u^2 + r^2 v^2 \right],$$

$$e_{zz}^A = \frac{\partial w}{\partial z} - \frac{1}{2} \left[ \left( \frac{\partial u}{\partial z} \right)^2 + r \left( \frac{\partial v}{\partial z} \right)^2 + \left( \frac{\partial w}{\partial z} \right)^2 \right],$$

$$e_{r\theta}^A = \frac{1}{2} \left[ \frac{1}{r} \left( \frac{\partial u}{\partial \theta} \right) + \left( \frac{\partial v}{\partial r} \right) - \frac{u}{r} \right] - \frac{1}{2r} \left[ \frac{\partial u}{\partial \theta} \frac{\partial u}{\partial \theta} + r \frac{\partial (rv)}{\partial \theta} \frac{\partial u}{\partial \theta} + \frac{\partial w}{\partial \theta} \frac{\partial w}{\partial \theta} - r^2 v \frac{\partial u}{\partial \theta} + u \frac{\partial (rv)}{\partial \theta} - rv \frac{\partial v}{\partial \theta} \right],$$

$$e_{\theta z}^A = \frac{1}{2} \left[ \frac{1}{r} \left( \frac{\partial v}{\partial z} \right) + \frac{1}{r} \left( \frac{\partial w}{\partial \theta} \right) - \frac{1}{2r} \left[ \frac{\partial u}{\partial \theta} \frac{\partial u}{\partial \theta} + r \frac{\partial v}{\partial \theta} \frac{\partial u}{\partial \theta} + \frac{\partial w}{\partial \theta} \frac{\partial w}{\partial \theta} - v \frac{\partial u}{\partial \theta} + r^2 \frac{\partial v}{\partial \theta} \right] \right],$$

$$e_{zr}^A = \frac{1}{2} \left[ \frac{\partial w}{\partial r} + \frac{\partial u}{\partial z} \right] - \frac{1}{2} \left[ \frac{\partial u}{\partial z} \frac{\partial u}{\partial r} + \frac{\partial v}{\partial z} \frac{\partial v}{\partial r} + \frac{\partial w}{\partial z} \frac{\partial w}{\partial r} + r v \frac{\partial v}{\partial z} \right],$$

(1.18)

where $u, v, w$ and $e_{rr}^A$, $e_{\theta\theta}^A$, $e_{zz}^A$, $e_{r\theta}^A$, $e_{\theta z}^A$, $e_{zr}^A$ are the components of the displacement $u_i$ and strain tensor $e_{ij}^A$ respectively.

In problems of cylinders and disks, the distribution of stresses is symmetrical about the axis of symmetry and hence the stresses and strains are independent of $\theta$ direction. The components of displacement in cylindrical polar co-ordinates are given by [100-101]

$$u = r(1 - \beta), \quad v = 0 \quad \text{and} \quad w = dz,$$  (1.19)
where $\beta$ is a function of $r = \sqrt{x^2 + y^2}$ only and $d$ is a constant.

Substituting equation (1.19) in (1.18), the finite components of strain are obtained as follows

$$e_{rr}^A = \frac{1}{2} \left[ - (r\beta' + \beta)^2 \right], \quad e_{\theta\theta}^A = \frac{1}{2} \left[ - \beta^2 \right], \quad e_{zz}^A = \frac{1}{2} \left[ - (1 - d)^2 \right], \quad e_{x\theta}^A = e_{z\theta}^A = e_{r\theta}^A = 0,$$

where $\beta' = \frac{d\beta}{dr}$.

1.4 TRANSITION

The theory of deformation for plasticity does not satisfy the condition of continuity of the relationship between stress and deformation in transition from one state to another. In the analysis of such theories of plasticity one may come across the similar circumstances in which some of the consequences of these theories appear to be contradictory or physically unacceptable. According to classical theory of plasticity, the elastic region and plastic region are two different regions, separated by a yield surface which depends on the symmetry and other physical considerations. In other words perfect elasticity and ideal plasticity are two extreme conditions of the material and it is not possible to draw a sharp line between them. This linear theory has given rise to number of important results, which have been incorporated by experiments more exactly, then one can expect. However, the following assumptions are main drawbacks of this theory:

(i) Even when the material at some point has yielded, the material at a neighboring point still remains in elastic state.

(ii) Existence of an assumed yield surface which separates the elastic and plastic regions.

(iii) For any given material there exists a function of the three principal stresses which always has a value when yielding begins irrespective of the stress state.

(iv) Same functional relationship applies to all materials although the numerical value of the function is different for different materials.

Transition from elastic state to plastic state is obtained in current literature by using semi-empirical yield conditions like Tresca or Von-Mises’. The stresses are obtained from the solution of elastic state and substituted in the yield condition to obtain the transition surface. When plastic state tends to set in, the stress-strain relationship undergoes a change. This change is reflected in governing equations. The classical theory does not take this
phenomenon into account. Tresca pointed out that there exists a mid-zone between elastic region and plastic region which was also supported by Todhunter and Pearson. Sir Lawerence expressed that in the transition state whole of the material participates and not a selected region or a line as assumed in classical theories. This was also supported by Bragg. A recent numerical study on flow and deformation theories of plasticity was taken into account to see up to what extent a continuous approximation including the idea of transition in terms of the stress-strain law, would give satisfactory convergent solution. The results obtained showed excellent approximations and convergence of solution of elastic-plastic transition.

The increasing demand of high speed technology in transportation, communication and energy conservation has motivated researchers to consider non-linearity in deformations. But some researchers still find difficulty in getting rid of a century old habit of analytical and experimental approach in continuum mechanics, which was totally depends upon linearization. If a state $A$ changes into state $B$ through a transition state $T$, $A$ and $B$ may be almost linear but $T$ is essentially non-linear. Since these non-linear problems are difficult to investigate, therefore researchers have to take the artifice of replacing them by singular, non-differentiable or discontinuous surfaces. This discontinuous treatment requires the use of ad-hoc and semi-empirical laws, which may or may not exist. Linearization of non-linear problems by perturbation, boundary layers and other techniques may not be able to provide satisfactory explanation of some important characteristics of non-linearity. The existence, uniqueness and stability of linear fields are well established. Nature does not always coincide with our abstract concepts of linearity, smoothness, symmetry, identity, homomorphism and isotropy. It is true that physical phenomena tend to behave linearly in course of time, which may be millions of years but the demands of modern technology want to compress years into fraction of a second. This transition, which frequently occurs in nature, should be taken into account. In fact all linear-disciplinary fields, which are so important in modern research, leads to important transition problems. But both the macro and micro analysts have given very little attention to them due to involvement of non-linearity in their analytic treatment.

As transition fields are asymptotic in nature so there arise some singularities or criticalities in the differential system describing physical phenomena. If the singularities are not recognized in one plane, they can be reconized in some other plane through continuos mapping. Transition field may also be treated as asymptotic subspace resulting from
intersection of two different spaces. Also according to theoretical approach, all continuous
deformations form a symmetric group. At transition, the nature of this group changes. For
example, an elastic body which belongs to the orthogonal group transforms to uni-molecular
in plastic state. Lastly, from the macroscopic point of view, one can imagine that at transition
the macroscopic structure of an element breaks down, with the result of this, the
corresponding transformation matrix becomes singular.
In general, the material from elastic state under some external load can lead to the following:

(i) Plastic state
(ii) Creep state
(iii) First to plastic then to creep or vice-versa.

When the material under some experiment goes to plastic state or creep state from
elastic state then transition takes place. All these cases of transition are expected to occur
when some functions of elasticity of the medium takes critical values. These functions are
called transition functions. These may be either principal stresses or difference of principal
stresses or stress invariants or any suitable combination of these. These critical (asymptotic)
values are determined at the transition points of the differential equation governing the
medium. If number of transition states occurs at the same point, the transition function will
have different limiting values and the point is known as multiple points, each branch of which
will then correspond to different state.

Transition from one state to other is a natural phenomenon and there is hardly any
branch of science or technology in which we do not come across the transition phenomena.
Elasticity-plasticity, visco-elastic, creep, fatigue, relaxation are well known examples of
transition. In classical theory, some yield conditions are required for elastic-plastic
deformations and for creep, not only the yield condition but also a number of creep strain laws
are required. All these assumptions leave little dissatisfaction. More scientific advancement
attempts are required to reduce the need of these assumptions. This is the motivation behind
the treatment of transition problems in solid mechanics.
1.4.1 IDENTIFICATION OF THE TRANSITION STATE

When a material yields at any point, it is reasonable to expect that the material will also yield at its neighboring points, rather than remain in the elastic state which is completely against the plastic state of the nearby particles. The plastic yielding of a material is the result of breaking down of its internal or macroscopic structure. This yielding will be complete or partial depending on the existing physical conditions of the material. This further leads to the identification of two material states i.e. a transition state and plastic state.

There are various ways to explain how transition may occur from one state into another state:

(i) At transition, the differential system defining the elastic state should attain some criticality or singularity.

(ii) If we consider the plastic state as an image of the elastic state under the transformation \( x^k = x^k(X^k) \), where \((X, Y, Z), (x, y, z)\) being the co-ordinates of a point in the undeformed and deformed state, respectively. Then at transition, due to relationship between elastic and plastic state the Jacobian of the transformation should be zero or infinite. This means when transition occurs, one-to-one correspondence does not exist between elastic and plastic state.

1.4.2 GENERAL TREATMENT OF TRANSITION THEORY

Consider the transformation \( x^k = x^k(X^k) \), which maps the metric space \( A \) into metric space \( B \). If we identify \( B \) as the plastic state and \( A \) as the elastic state, then the isomorphism of the transformation is presumably destroyed, since this process of deformations is irreversible and the material may change from elastic to the plastic, creep or fatigue state. Hence, the Jacobian of transformation will therefore correspond to the transition state from \( A \) to \( B \).

When a continuum changes from a state \( A \) to another state \( B \), as has been explained before, the invariants of the stress and strain tensors satisfies some constraint. This constraint should be obtainable from the condition, namely the vanishing of Jacobian of transformation, since the latter corresponds to the transition state.

If \( u, v \) and \( w \) are the displacements along the rectangular cartesian coordinate axes, then

\[
X = x - u, \quad Y = y - v, \quad Z = z - w,
\]
where \((X, Y, Z), (x, y, z)\) are the co-ordinates of a point in the undeformed and deformed state, respectively. Hence the Jacobian

\[
J = \frac{\partial(X, Y, Z)}{\partial(x, y, z)} = \begin{vmatrix}
\frac{\partial X}{\partial x} & \frac{\partial Y}{\partial y} & \frac{\partial Z}{\partial z} \\
\frac{\partial X}{\partial x} & \frac{\partial Y}{\partial y} & \frac{\partial Z}{\partial z} \\
\frac{\partial X}{\partial x} & \frac{\partial Y}{\partial y} & \frac{\partial Z}{\partial z}
\end{vmatrix} = \begin{vmatrix}
1 - \frac{\partial u}{\partial x} & -\frac{\partial v}{\partial x} & -\frac{\partial w}{\partial x} \\
\frac{\partial u}{\partial y} & 1 - \frac{\partial v}{\partial y} & -\frac{\partial w}{\partial y} \\
\frac{\partial u}{\partial z} & -\frac{\partial v}{\partial z} & 1 - \frac{\partial w}{\partial z}
\end{vmatrix}
\]

(1.21)

is referred to the deformed state and

\[
J = \frac{\partial(x, y, z)}{\partial(X, Y, Z)} = \begin{vmatrix}
\frac{\partial x}{\partial X} & \frac{\partial y}{\partial X} & \frac{\partial z}{\partial X} \\
\frac{\partial x}{\partial Y} & \frac{\partial y}{\partial Y} & \frac{\partial z}{\partial Y} \\
\frac{\partial x}{\partial Z} & \frac{\partial y}{\partial Z} & \frac{\partial z}{\partial Z}
\end{vmatrix} = \begin{vmatrix}
1 + \frac{\partial u}{\partial X} & \frac{\partial v}{\partial X} & \frac{\partial w}{\partial X} \\
\frac{\partial u}{\partial Y} & 1 + \frac{\partial v}{\partial Y} & \frac{\partial w}{\partial Y} \\
\frac{\partial u}{\partial Z} & \frac{\partial v}{\partial Z} & 1 + \frac{\partial w}{\partial Z}
\end{vmatrix}
\]

(1.22)

to the undeformed state.

From (1.21), we have

\[
J^2 = \begin{vmatrix}
1 - \frac{\partial u}{\partial x} & -\frac{\partial v}{\partial x} & -\frac{\partial w}{\partial x} \\
\frac{\partial u}{\partial y} & 1 - \frac{\partial v}{\partial y} & -\frac{\partial w}{\partial y} \\
\frac{\partial u}{\partial z} & -\frac{\partial v}{\partial z} & 1 - \frac{\partial w}{\partial z}
\end{vmatrix} = \begin{vmatrix}
1 - 2e_{xx} & -2e_{xy} & -2e_{xz} \\
-2e_{yx} & 1 - 2e_{yy} & -2e_{yz} \\
-2e_{zx} & -2e_{zy} & 1 - 2e_{zz}
\end{vmatrix} = \left(\begin{array}{c}
\frac{\partial u}{\partial x}^2 + \frac{\partial v}{\partial x}^2 + \frac{\partial w}{\partial x}^2 \\
\frac{\partial u}{\partial y}^2 + \frac{\partial v}{\partial y}^2 + \frac{\partial w}{\partial y}^2 \\
\frac{\partial u}{\partial z}^2 + \frac{\partial v}{\partial z}^2 + \frac{\partial w}{\partial z}^2
\end{array}\right)^2
\]

(1.23)

where \(1 - 2e_{xx} = -2\frac{\partial u}{\partial x} + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\),

and \(1 - 2e_{xy} = \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}\right) + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\right) + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y}\), etc.

From (1.23), if \(J = 0\), which is a transition condition corresponding to asymptotically large extensions, we have

\[
1 - 2J_1^2 + 4J_2^2 - 8J_3^3 = 0,
\]

(1.24)

where

\[
J_1 = e_{xx} + e_{yy} + e_{zz}, \quad J_2 = e_{xx}e_{yy} + e_{yy}e_{zz} + e_{zz}e_{xx} - e_{xy}e_{yx} - e_{xz}e_{zx} - e_{yz}e_{zy},
\]

\[
J_3 = e_{xx}e_{yy}e_{zz} - e_{xx}e_{yz}e_{zy} + e_{xy}e_{zx}e_{yz} - e_{xy}e_{zx}e_{yz} + e_{xz}e_{yx}e_{zy} - e_{xz}e_{yx}e_{zy}.
\]
In short, we can also write it as
\[ J'_1 = \frac{1}{1!} \delta^k e^l_k, \quad J'_2 = \frac{1}{2!} \delta^{km}_{ln} e^l_k e^n_m, \quad J'_3 = \frac{1}{3!} \delta^{km}_{lnq} e^l_k e^n_m e^q_p. \]

The symbol \( \delta^{km}_{ln} \) and \( \delta^{km}_{lnq} \) are generalized Kronecker deltas.

Referring to the principal axes of the strain ellipsoid in the deformed state, we have from equation (1.24)
\[ 8J_3 - 4J_2 + 2J_1 = 1, \quad (1.25) \]
where
\[ J_1 = e_{11} + e_{22} + e_{33}, \quad J_2 = e_{11}e_{22} + e_{22}e_{33} + e_{33}e_{11}, \quad J_3 = e_{11}e_{22}e_{33}, \quad (1.26) \]
and \( e_{11}, e_{22}, e_{33} \) are the principal strains.

Relations which are similar to equation (1.25) should hold at transition for materials such as isotropic, anisotropic, homogeneous or heterogeneous and should come out from the transition condition independent of the constitutive equations and the momentum equations. The strain (stress) invariants are, in general, independent of each other however a functional relation exists between them only at the transition state because of the constraint.

Now by constitutive equations for isotropic material \( \tau_{ij} = \lambda \delta_{ij} e_{aa} + 2 \mu e_{ij} \), we obtain
\[ e_{11} = \frac{1}{E} \left[ \tau_{11} - \sigma (\tau_{22} + \tau_{33}) \right], \quad e_{22} = \frac{1}{E} \left[ \tau_{22} - \sigma (\tau_{11} + \tau_{33}) \right], \quad e_{33} = \frac{1}{E} \left[ \tau_{33} - \sigma (\tau_{22} + \tau_{11}) \right], \]
where \( e_{ii} \) are principal strains.

Hence
\[ J_1 = \left( 1 - \frac{2\sigma}{E} \right) I_1, \quad J_2 = \frac{1}{E^2} \left[ (1 + \sigma)^2 I_2 - \sigma (2 - \sigma)I_1^3 \right] \]
and
\[ J_3 = \frac{1}{E^3} \left[ (1 + \sigma)^3 I_3 - \sigma (1 + \sigma)^2 I_1 I_2 - \sigma^2 I_1^3 \right], \]
where the \( I_k \)'s are stress invariants, \( J_k \)'s are strain invariants, \( E \) is Young’s modulus of elasticity and \( \sigma \) is Poisson’s ratio.

Now from (1.25), we get
\[ \frac{8}{E^3} \left[ (1 + \sigma)^3 I_3 - \sigma (1 + \sigma)^2 I_1 I_2 + \sigma^2 I_1^3 \right] - 4 \frac{1}{E^2} \left[ (1 + \sigma)^2 I_2 - \sigma (2 + \sigma)I_1^2 \right] + 2 \frac{1}{E} [(1 - 2\sigma) I_1] = 1. \]

(1.27)
The relation (1.27) should hold at transition. Now equation (1.27) in a simpler form can be written using the following notation:

\[ I_2 = (\tau_{11} - \tau_{22})^2 + (\tau_{22} - \tau_{33})^2 + (\tau_{33} - \tau_{11})^2 = 2I_1^2 - 6I_2. \]

Taking \( T_\mu = \frac{\tau_\mu}{E}, \quad K_1 = \frac{I_1}{E}, \quad K_2 = \frac{I_2}{E^2}, \quad K_3 = \frac{I_3}{E^3} \) and \( K'_2 = \frac{I'_2}{E^2} \), equation (1.27) becomes

\[
8 \left[ (1 + \sigma)^3 K_3 + \frac{1}{6} \sigma (1 + \sigma)^2 K_1 K'_2 - \frac{1}{3} \sigma (\sigma^2 - \sigma + 1) K'_1 \right] \\
+ \frac{4}{3} \left[ \frac{1}{2} (1 + \sigma)^2 K'_2 - (1 - 2\sigma)K'_1 \right] + 2[1 - 2\sigma)K_1] = 1.
\]

(1.28)

The invariant relation (1.28) between the stress invariants should hold good at transition state. The condition involves elastic effect. For the fully plastic state i.e. when material is incompressible then \( \sigma \rightarrow 1/2 \), from equation (1.28), we obtain

\[
3K'_2 + 2 \left( 27K_3 + \frac{3}{2}K_1K'_2 - K'_1 \right)^3 = 2.
\]

(1.29)

Equation (1.29) is taken as the most general yield condition for all types of media irrespective of their properties. Rewriting the equation (1.29), we get

\[
3 \left[ (T_{11} - T_{22})^2 + (T_{22} - T_{33})^2 + (T_{33} - T_{11})^2 \right] \\
+ 2 \left[ (2T_{11} - T_{22} - T_{33}) (2T_{22} - T_{11} - T_{33}) (2T_{33} - T_{22} - T_{11}) \right] = 2.
\]

(1.30)

This is the general form of the yield condition which has been obtained independent of the equations of equilibrium. Equation (1.30) can also be written as

\[
L_1^2 + L_2^2 + L_3^3 + 2L_1L_2L_3 = 2,
\]

(1.31)

or

\[
(T_{11}^d)^2 + (T_{22}^d)^2 + (T_{33}^d)^2 + 6T_{11}^d T_{22}^d T_{33}^d = \frac{2}{9}.
\]

(1.32)

where \( L_1 = (2T_{11} - T_{22} - T_{33}) \), etc. and \( T_{\mu}^d \) are the deviatoric stress tensors in non-dimensional form. It is interesting to note that equation (1.30) reduces to Hencky-Von-Mises’ or Tresca’s yield condition in some special cases.

**PLANE STRAIN**

In the case of plane strain, \( e_{33} = 0 \) so that \( J_3 = 0 \). From equation (1.25) we get
\[ 4J_2 - 2J_1 + 1 = 0. \]

and in the limiting incompressible case \((\sigma \rightarrow 1/2)\), equation (1.29) gives

\[ K_2' = \frac{2}{3}, \]

or equivalent to

\[ (\tau_{11} - \tau_{22})^2 + (\tau_{22} - \tau_{33})^2 + (\tau_{33} - \tau_{11})^2 = \frac{8}{3} y^2, \]

which is exactly same as Von-Mises’ yield criterion, i.e.

\[ (\tau_{11} - \tau_{22})^2 + (\tau_{22} - \tau_{33})^2 + (\tau_{33} - \tau_{11})^2 = 2y^2, \]

where \(y\) is the yield stress in tension.

We have noticed here that the constant \(2y^2\) is replaced by \(\frac{8}{3} y^2\). This is to be expected as (1.33) is a particular case of (1.29).

**PLANE STRESS**

In plane stress case, suppose \(T_{33} = 0\). Then the equation (1.30) of yield condition reduces to

\[ (T_{11} + T_{22} + 1)(2T_{11} - T_{22} - 1)(2T_{22} - T_{11} - 1) = 0. \]

This yields the following three conditions

\[ T_{11} + T_{22} + 1 = 0, \quad T_{aa} = -1, \quad \tau_{aa} = -2y, \]

\[ 2T_{11} - T_{22} - 1 = 0, \quad T_{11}^{d} = \frac{1}{3}, \quad \tau_{11}^{d} = \frac{2}{3}y, \]

\[ 2T_{22} - T_{11} - 1 = 0, \quad T_{22}^{d} = \frac{1}{3}, \quad \tau_{22}^{d} = \frac{2}{3}y. \]

The last two conditions are exactly same as Tresca’s yield condition. In a number of cases the yield takes place through a deviatoric principal stress taking on a maximum value. The yield condition (1.29), which is an invariant relation, can be assumed as general yield condition for all types of materials, irrespective of their properties. While Von-Mises’ or Tresca’s yield condition does not consider the distinction between the yield stress in tension and yield stress in compression, as in equation (1.29) and hence includes Baushinger’s effect.
1.5 GENERALIZED STRAIN MEASURE

The response of all materials to the external loading, in general, is non-linear in character. The division of deformation into different states is the result of linearization of engineering problems. In case of large deformations such as plastic flow, creep and fatigue, the current treatment requires a number of ad-hoc and semi-empirical laws. The use of these semi-empirical laws has made the problem complicated without evolving any simple concept governing them. One source of complications is the use of classical measures of deformation produced in a medium even when we know that non-linearity is a characteristic of such deformed media. Abstract measure theory has been highly developed even though it has not been suitably used in the problems of non-linear mechanics. In classical mechanics, ordinary measures are found sufficient and therefore there is no need to extend them. The equation of equilibrium and the concept of stresses are well defined, but the measures of deformation are flexible. A continuous approach requires the introduction of non-linear measures. Deformation fields associated with irreversible phenomenon such as elastic-plastic deformations, creep, relaxation, fatigue and fracture, etc. are non-linear in character as explained by extensive experimental studies. The classical measures of deformation are not sufficient to deal with transitions and hence the corresponding constitutive equations of the medium are complicated. On the basis of above analysis it is found that we should have to construct generalized measure of deformation to overcome the difficulty of using classical measures in continuum mechanics.

Rabotnov has pointed out the ambiguities of the experimental data which involves in the choice of suitable constitutive equations for different states of creep described in the figure 1.3. There are two parameters which characterize each of these states, one is for the measure and other for the irreversibility. Therefore, it is expected that a generalized measure concept in which these two parameters are experimentally determined may give better results in creep deformation. The strain-rate described in four stages of creep-elastic, transient, secondary and rupture is different in each case (figure 1.3). As the deformations are non-linear so there arises the need of generalization of strain rate measure so that they can be used in all the stages of creep.

Seth [108] has defined the generalized principal strain measure as
\[ e_{ii} = \int_0^1 \left( 1 - 2e_{ii}^A \right)^{\frac{n-1}{2}} \, de_{ii}^A = \frac{1}{n} \left[ 1 - \left( 1 - 2e_{ii}^A \right)^\frac{n}{2} \right], \]

where ‘\( n \)’ is the measure and \( e_{ii}^A \) is principal Almansi finite strain components. In Cartesian coordinates we can write down the generalized measure in terms of any other measures.

For uni-axial case, it is given by Seth [108] as
\[ e = \frac{1}{n} \left[ 1 - \left( \frac{l_0}{l} \right)^n \right], \]

where \( l_0 \) and \( l \) are lengths of the rod in initial and strained states respectively. For \( n = 0, 1, 2, -1, -2 \), generalized strain measure gives the Hencky, Swainger, Almansi, Cauchy and Green measures respectively. Seth [102-104] has shown that creep strain laws such as Norton’s, Kachanov, Odquist, Andrade’s laws etc. used in current literature can also be derived from the generalized measure.

Using equation (1.20) in equation (1.36), we have obtained generalized strain measures as
\[ e_{rr} = \frac{1}{n} \left[ 1 - \left( r + \beta' \right)^n \right], \quad e_{\theta\theta} = \frac{1}{n} \left[ 1 - \beta^n \right], \quad e_{zz} = \frac{1}{2} \left[ 1 - (1 - d)^n \right], \quad e_{r\theta} = e_{\theta r} = e_{zz} = 0, \]

where \( r \beta' = \beta P \), \( \beta' = \frac{d \beta}{dr} \) and \( n \) is the measure.

The generalized strain measure not only gives many well known strain measures for different values of \( n \), but it is also used to find the creep stresses when combined with the transition points of the governing differential equations. Seth has also shown that the transition point analysis does not require any assumptions of incompressibility, creep strain law and yield conditions like in classical theory. He successfully applied transition theory [100, 104, 112] to a large number of problems [100-113]. The asymptotic solution of the governing differential equations at the transition point gives the same results which are also obtained by assuming yield criteria when they exist. The most important contribution of these generalized measures is that they eliminate the use of semi-empirical laws and jump conditions. If such laws exist, they come out with the analytic treatment as a particular case. Also the important feature of non-linear measure is that, they explain transition without assuming any conditions to match the two solutions at transition.
In the present study, an attempt has been made to analyze some problems of technical importance on the basis of transition theory and generalized strain measure.

1.6 CONSTITUTIVE EQUATIONS

Equations describing the individual material and its response to applied loads are known as constitutive equations. The macroscopic behavior of materials is described by these constitutive equations. Materials in the solid state behave in such a complex way that when entire range of possible temperatures and deformations is considered, it is not feasible to write down one equation or set of equations to describe accurately a real material over its entire range of behavior. Instead of formulating separate equations to describe the various kinds of ideal material’s response, the equations should be designed to approximate physical observations of a real material’s response over a suitably restricted range. The classical equations were introduced separately to meet specific needs and to make the description of many physical situations as simple as possible. Some of the ideas are illustrated below which are involved in formulating simple equations for such ideal materials.

ELASTIC STATE

A material is said to be in perfect elastic state when it recovers its original shape completely after removal of load causing the deformation. Also, there is one to one relationship between the state of stress and state of strain at a given temperature and material obeys Hooke’s law, which states that stresses are proportional to strains. Under triaxial loading, classical theory of elasticity assumes that generalized Hooke’s law expresses each stress component as a linear combination of strains, i.e.

\[ T_{ij} = C_{ijkl} e_{kl}, \quad (i, j, k, l = 1,2,3) \]  

(1.38)

where \( T_{ij} \) and \( e_{ij} \) are the stress and strain tensor respectively.

These nine equations contain eighty one material constants \( C_{ijkl} \), but not all the constants are independent. Due to the symmetry of \( T_{ij} \) and \( e_{ij} \) these eighty one material constants reduces to only thirty six material constants. The number of independent constants is further reduced due to their symmetric properties. If we assume the existence of strain energy density \( U \) when the system is isothermal, the number of elastic constants is reduced to twenty
one. For monoclinic materials (materials having only one plane of elastic symmetry), there are only thirteen independent elastic constants. In case of orthotropic materials (materials which possess three orthogonal planes of elastic symmetry) the number of elastic constants is further reduced to nine. For transversely isotropic materials (material properties are same in one direction and different in other), there are only five independent elastic constants. Finally, there are only two independent elastic constants known as Lame’s parameters for isotropic elastic materials (materials in which elastic properties are independent of orientation of axes).

A material in which the elastic properties depend on the orientation of the sample is called anisotropic. If the anisotropic elasticity of the material is to be fully described, a total of twenty one independent parameters are needed in $6 \times 6$ compliance matrix $[C]$ as follows:

$$
[C] = 
\begin{bmatrix}
C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\
C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\
C_{33} & C_{34} & C_{35} & C_{36} \\
Symmetry & C_{44} & C_{45} & C_{46} \\
& C_{55} & C_{56} \\
& & C_{66}
\end{bmatrix}
$$

For transversely isotropic materials, the constitutive equations in matrix form are given as follows:

$$
\begin{bmatrix}
T_1 \\
T_2 \\
T_3 \\
T_4 \\
T_5 \\
T_6
\end{bmatrix} = 
\begin{bmatrix}
C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\
C_{21} & C_{22} & C_{23} & 0 & 0 & 0 \\
C_{31} & C_{32} & C_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & C_{44} & 0 & 0 \\
0 & 0 & 0 & C_{55} & 0 & C_{66} \\
0 & 0 & 0 & 0 & 0 & C_{66}
\end{bmatrix}
\begin{bmatrix}
e_1 \\
e_2 \\
e_3 \\
e_4 \\
e_5 \\
e_6
\end{bmatrix}
$$

or in cylindrical polar co-ordinates, these can be written as

$$
T_{rr} = C_{11} e_{rr} + C_{12} e_{\theta \theta} + C_{13} e_{zz}, \quad T_{\theta \theta} = C_{21} e_{rr} + C_{22} e_{\theta \theta} + C_{23} e_{zz}, \quad T_{zz} = C_{31} e_{rr} + C_{32} e_{\theta \theta} + C_{33} e_{zz}, \\
T_{\theta \phi} = 2C_{44} e_{\theta \phi}, \quad T_{x \phi} = 2C_{55} e_{x \phi}, \quad T_{r \phi} = 2C_{66} e_{r \phi}, 
$$

where

$$
C_{12} = (C_{11} - 2C_{66}) = C_{21}, \quad C_{13} = C_{31} = C_{23} = C_{32}, \quad C_{55} = C_{44}, \quad C_{11} = C_{22}, C_{33}.
$$

(1.39)
Since 1950, the theory of elasticity for anisotropic bodies has been continuously developed and enriched with new investigations. Thus, the general theory has been made on a rigorous scientific basis and a number of laws have been established on this theory. The theory that was first worked out by Saint-Venant and Bekhterev has been revived. The development and construction of many new anisotropic materials is of great importance. These new materials possess a number of advantages over those previously known materials (i.e. glass-fiber reinforced plastics). Thus, over many years this branch of science has made great progress, both in theoretical and purely practical way in constructing new anisotropic materials.

Since the class of materials which obey the generalized Hooke’s law is quite wide, and therefore it is necessary to give a classification which reflects their distinguishing features. On the basis of elastic properties all material bodies may be divided into homogeneous, non-homogeneous and also into isotropic, anisotropic material body. A homogeneous body, with regards to the elastic properties means one whose elastic properties are the same at different points. Non-homogeneous materials are those which are having different elastic properties at different points. If the elastic characteristics of the material, for example the elastic moduli, vary from point to point in a continuous manner, then non-homogeneity is said to be continuous. However, if the elastic characteristic undergoes some discontinuities in passing from one point to another point then non-homogeneity is said to be discontinuous.

The modern structures are not only made of materials which are usually considered as homogeneous and isotropic in design, but also of anisotropic materials, which are having different elastic properties in different directions. A most common example of such materials is natural wood. The elastic modulus of wood in tension parallel to the grain is considerably greater than the corresponding modulus of wood in tension perpendicular to the grain, and therefore its elastic constants depend on the direction in relation to the wood fibers. Synthetic materials such as delta wood, aircraft plywood, fabric laminate are anisotropic materials which are used in aircraft construction. Anisotropy of concrete has been investigated by many authors. In order to design anisotropic members under elastic strains, it is necessary to know how stresses and strains in anisotropic bodies are theoretically determined, i.e. how to solve problems of the theory of elasticity for anisotropic bodies. In the case of an anisotropic homogeneous body the numbers of independent elastic constants are twenty one. To solve
problems of stress and strain in an anisotropic body, it is necessary to proceed from the equations of the theory of elasticity with the assumptions of different elastic properties in different directions.

At present the theory of elasticity for isotropic bodies is well developed but is not so developed for anisotropic bodies. For isotropic materials which are having same elastic properties in all directions, we have two independent elastic constants i.e.

\[ C_{11} = C_{22} = C_{33}, \quad C_{12} = C_{21} = C_{13} = C_{31} = C_{23} = C_{32} = C_{11} - 2C_{66}. \]  

(1.40)

In terms of Lame’s constant \( \lambda \) and \( \mu \), these constants are written as

\[ C_{12} = \lambda, \quad C_{66} = \frac{1}{2} (C_{11} - C_{12}) = \mu \quad \text{and} \quad C_{11} = \lambda + 2\mu. \]

(1.41)

By substituting equations (1.40) and (1.41) in equation (1.39), the result may be written in the form as

\[ T_y = \lambda \delta_y I_1 + 2\mu e_y, \]

(1.42)

where \( \delta_y \) is Kronecker delta and \( I_1 = e^i_i \) is the first strain invariant.

The coefficients of the constitutive equations discussed above specify the relationship between stress and strain for the materials which depends on the temperature. But we generally assume that the dependence is very small such that the coefficient may be treated as constants during the deformation. Even though the variation of the elastic constants is neglected with temperature, it is necessary to take thermal expansion of the material, which produces dimensional changes as large as those produced by the applied forces. If these dimensional changes are prevented by surrounded material then thermal stresses are induced along with stresses related to the strains according to the elastic constitutive equations. The thermo elastic constitutive equations for transversely isotropic material are given by

\[ T_{rr} = C_{11}e_{rr} + (C_{11} - 2C_{66})e_{\theta \theta} + C_{13}e_{zz} - \beta_1 T, \quad T_{\theta \theta} = (C_{11} - 2C_{66})e_{rr} + C_{11}e_{\theta \theta} + C_{13}e_{zz} - \beta_2 T, \]
\[ T_{zz} = C_{13}e_{rr} + C_{13}e_{\theta \theta} + C_{33}e_{zz} - \beta_3 T, \quad T_{rz} = T_{r\theta} = T_{\theta z} = 0, \]

(1.43)

where \( \beta_1 = C_{11} \alpha_1 + 2C_{12} \alpha_2, \quad \beta_2 = C_{12} \alpha_1 + (C_{22} + C_{33}) \alpha_2 \) and \( \alpha_i \) are the thermal expansion coefficients.

The constitutive equations for thermo elastic isotropic material are given by

\[ T_y = \lambda \delta_y I_1 + 2\mu e_y - \xi T \delta_y, \]

(1.44)

where \( \xi = \alpha (3\lambda + 2\mu) \), \( \alpha \) is thermal expansion coefficient and \( T \) is the temperature.
For the special case of steady state heat flow, we have
\[ \nabla^2 T = T_{,tt} = 0. \quad (1.45) \]

**PLASTIC STATE**

As metals obey Hooke’s law only up to certain range of strain i.e., up to the elastic limit. The behavior of metals beyond their elastic limit is rather complicated than discussed in section 1.2. For the analysis of continuum stress and strain distributions, a constitutive theory of plasticity should satisfy the yield condition under combined stresses. This is because of the reason that uniaxial condition \(|T| = Y\) is not appropriate when there is more than one stress component. For an ideal plastic solid that obeys Von-Mises’ yield criterion and flow rule, the following simplifications are considered:

(i) The plastic strain assumes incompressibility and the plastic strain deviation tensor is same as the plastic strain tensor.

(ii) The material is in elastic state and obeys Hooke’s law as long as the second invariant \(J_2 = \frac{1}{2} T_{ij} T_{ij}\) of the stress deviation tensor is less than a constant \(K^2\). In other words, when no change in plastic strain can occur \(e_{ij}^p = 0\) i.e. \(J_2 < K^2\). \quad (1.46)

(iii) Yielding takes place (or elastic limit is reached) only when \(J_2 = K^2\). \quad (1.47)

When the yield condition \(J_2 - K^2 = 0\) prevails, then the rate of change of plastic strain is proportional to the stress deviation, \(e_{ij}^p = \frac{1}{\mu} J_{ij}, \mu > 0\), \quad (1.48)

where \(\mu\) is a positive proportionality factor, whose dimensions are equal to the coefficients of viscosity of a fluid. Any stress-state corresponding to \(J_2 > K^2\) cannot be realized in the material.

The above set of laws has two essential parts (i) the yielding criterion and (ii) stress-strain relations between elastic-plastic states. In these specifications, the yielding depends on the second invariant of stress deviation tensor. Such yield criterion was first given by Von-Mises’. The constant \(K\) can be identified with the yield stress in case of simple shear while for a work hardening material, \(K\) will be allowed to change with strain history.

Sometimes Tresca’s yield condition is used instead of Von-Mises’ yield condition. Tresca’s criterion states that maximum shear stress must have the constant value \(K\) during
plastic flow. To express Tresca’s idea analytically, it is simpler to use the principal stresses \((T_1, T_2, T_3)\). If it is known that \(T_1 \geq T_2 \geq T_3\) then Tresca’s yielding condition is

\[
f \equiv T_1 - T_3 - 2K = 0.
\]

(1.49)

However \(f\) is not analytic in this form. It violates the rule in such a manner that principal axes should not affect the yield function. To obey this rule, we observe that Tresca’s condition states that during plastic flow one of the differences of stresses \(|T_1 - T_2|, |T_2 - T_3|, |T_3 - T_1|\) has the value \(2K\). Hence we may write

\[
f \equiv \left[ (T_1 - T_2)^2 - 4K^2 \right] \text{ or } \left[ (T_2 - T_3)^2 - 4K^2 \right] \text{ or } \left[ (T_3 - T_1)^2 - 4K^2 \right] = 0.
\]

(1.50)

This equation is symmetric with respect to the principal stresses.

In the above treatments, the ideal theories of elasticity and plasticity are treated separately and then combined together through a semi-empirical law, called the yield condition. What actually happens is that when a medium starts to yield, a constraint is placed on the invariant of the strain tensor of the field such that they satisfy functional relation of the form

\[
f(I_1, I_2, I_3) = 0,
\]

(1.51)

where \(I_1, I_2\) and \(I_3\) are the first, second and third strain invariants.

The form of \(f\) should be determined from the condition that the modulus of transformation takes on a singular value like zero or infinity and not by any ad-hoc conditions. In current treatment, it is argued that \(I_1\) vanish due to incompressibility of the material and \(I_3\) is very small, thus equation (1.51) can be reduce to the form

\[
I_2 = \text{a constant},
\]

(1.52)

which is known as the Huber-Von-Mises’ yield condition. If two of the principal stresses are equal or one is the arithmetic mean of the other two, condition (1.52) reduces to the Tresca’s yield condition. But it is clear that equation (1.52) cannot account for the Bauschinger effect, for which \(I_1\) must appear in the yield condition. Seth [113] has expressed that when the material is in the fully plastic state then the yield stress \(Y\) in simple tension is

\[
Y = \frac{E}{n},
\]

(1.53)
where $E$ is the response coefficient in the transition range. It is also concluded that the yield stress in compression is twice than in tension and the general form of yield condition also contains the Bauschinger effect.

The stress-strain relationships for an elastic-plastic solid were first proposed by Prandtl (1924) for the case of plane strain deformation. The general form of the equations was given by Reuss (1930). Reuss assumed that increment in plastic strain, denoted by a superscript ‘$p$’ in the following equations, is at any instant proportional to the instantaneous stress deviation $J_{ij}$ and shear stresses. Thus

$$\frac{de_{ij}^p}{J_{ij}} = \frac{de_{11}^p}{J_{11}} = \frac{de_{22}^p}{J_{22}} = \frac{de_{33}^p}{J_{33}} = \frac{de_{12}^p}{J_{12}} = \frac{de_{23}^p}{J_{23}} = \frac{de_{31}^p}{J_{31}} = d\lambda \quad \text{or} \quad de_{ij}^p = J_{ij}d\lambda, \quad (i, j = 1,2,3) \quad (1.54)$$

where $d\lambda$ is an instantaneous positive constant of proportionality which may vary throughout a straining program.

The equations state that a small increment of plastic strain depends upon the current deviatoric stress, and not on increment. The equations are the statement about the ratio of the plastic strain increments in $x, y, z$ directions. These equations do not give direct information about their absolute magnitude. The total increment in strain is equal to the sum of the elastic strain increment (denoted by a superscript ‘$e$’) and the plastic strain increment. Thus

$$de_{ij} = de_{ij}^p + de_{ij}^e = J_{ij}d\lambda + \left\{ \frac{dJ_{ij}}{2G} + \frac{(1 - 2\nu)}{E} \delta_{ij}d\lambda \right\}, \quad (1.55)$$

where $\nu$ is the Poisson’s ratio.

Since the plastic strain causes no change in plastic volume, then condition of incompressibility can be written in terms of the principal or normal strains as

$$de_{11}^p + de_{22}^p + de_{33}^p = 0 \quad \text{or} \quad de_{ii}^p = 0. \quad (1.56)$$

Equation (1.54) then gives

$$\frac{de_{11}^p - de_{22}^p}{T_{11} - T_{22}} = \frac{de_{22}^p - de_{33}^p}{T_{22} - T_{33}} = \frac{de_{33}^p - de_{11}^p}{T_{33} - T_{11}} = d\lambda. \quad (1.57)$$

Equation (1.57) states that Mohr circles of stress and plastic strain increment are similar. Equation (1.54) can be rewritten in terms of normal stresses and resulting equations are

$$de_{11}^p = \frac{2}{3}d\lambda \left[ T_{11} - \frac{1}{2}(T_{22} - T_{33}) \right]. \quad (1.58)$$

Equation (1.55) thus consists of three equations of the type
\[ de_{11}'' = \frac{2}{3} d\lambda \left[ T_{11} - \frac{1}{2} (T_{22} - T_{33}) \right] + \frac{dT_{11} - \nu (dT_{22} + dT_{33})}{E}, \]  

(1.59)

and three equations of the type

\[ de_{23} = T_{23} d\lambda + \frac{dT_{23}}{2G}. \]  

(1.60)

Finally from equation (1.55), it is observed that the volumetric and deviatoric strain increments are separated in the expression of total strain increment. Including the Von-Mises’ yield criterion, the Prandtl-Reuss equations are also written as

\[ de'_{ij} = J_{ij} d\lambda + \frac{dJ_{ij}}{2G}, \quad de_{ij} = \frac{(1 - 2\nu)}{E} dT_{ii}, \quad J_{ij} J_{ij} = 2K^2. \]  

(1.61)

These equations for elastic-plastic solid are generally difficult to handle in case of real problems and, as a result of this very few solutions are available.

In problems of large deformations, the elastic strains may often be neglected altogether. The material is then considered as perfectly plastic solid. When the stresses are below the yield point of the material, no straining takes place, and the total increments in strain are identical. Stress-strain relations for such type of materials are proposed by Levy and Von-Mises’. In presenting the relationship between stress and strain, we have not followed the historical development of the field. At present time, it is more logical to consider the Levy-Mises’ equations as a special form of the Prandtl-Reuss equations. However, it was first proposed by Saint-Venant (1870) that principal axes of strain increment coincided with the axes of principal stress. The general relationship between strain increment and the reduced stresses was first introduced by Levy (1871) and independently by Von-Mises’ (1913). These equations are now known as Levy-Mises’ equations and are written as

\[ \frac{de_{11}}{J_{11}} = \frac{de_{22}}{J_{22}} = \frac{de_{33}}{J_{33}} = \frac{de_{12}}{J_{12}} = \frac{de_{23}}{J_{23}} = \frac{de_{31}}{J_{31}} = d\lambda. \]  

(1.62)

The superscript ‘p’ of equation (1.54) may be dropped, since the total strain increments are identical. Further, the Mohr circle of stress and strain increment is identical. In terms of total stresses, the Levy-Mises’ relation has three equations of the type

\[ de_{11} = \frac{2}{3} d\lambda [T_{11} - (T_{22} + T_{33})], \]  

(1.63)

and three are of the type
Since the elastic strains are not taken into account, the Levy-Mises’ relations obviously cannot be used to obtain information about “Elastic Spring-back” or residual stresses. Prandtl-Reuss equations are useful in such cases.

**CREEP STATE**

Problems of creep are more complex as compared to problems of plasticity. Laboratory tests for creep under complex stress conditions have technical difficulties and these experiments must be performed very carefully for the reliability of results. Therefore, the available experimental data does not give reliable basis for creep theory which is capable of describing the behavior of materials under complex stress conditions. Moreover, the test is performed on plane stresses and we have no information about performance of creep with stresses on the three axes.

Like plasticity theory, the theory of creep under complex stress is based on certain speculative considerations, which are partially confirmed by experiments. There are many ways by which this theory can be extended to varying stressed states. In real situations the nature of stressed state usually varies little with time, and therefore, the different theories lead to different results. For steady state of creep, Odquist [83] has formulated the constitutive equations by considering the rate of strain energy function $\dot{W}$ with Von-Mises’ yield criterion. It relates the rate of steady state of creep to the second invariant of the stress deviator tensor in the following form

$$
\dot{\varepsilon}_{ij} = \frac{\partial \dot{W}}{\partial J_{ij}} = \frac{3}{2} \left( \frac{\sigma_c}{\sigma} \right)^{n-1} \frac{J_{ij}}{\sigma_c},
$$

(1.65)

where $\dot{\varepsilon}_{ij}$, $J_{ij}$, $\sigma_c$ are the strain rate tensor, stress deviator tensor and effective stress respectively, $\sigma_c$ and $n$ are material constants.

Stress-strain relations in above mentioned form are mostly used to analyze the creep problems based on the following hypothesis:

(i) The material is incompressible.

(ii) Creep strain rate is independent of superimposed hydrostatic pressure.

(iii) Existence of flow potential with Von-Mises’ yield condition.
(iv) Norton’s law holds as the special case, i.e. for uniaxial case.

An alternate approach to the problem of multi-axial stationary creep is made possible by Wahl [140] who has considered maximum shear stress (Tresca) as a stress invariant together with the associated flow rule for the body relations.

It has already been mentioned in section 1.4 that transition state exists when a material goes from elastic to plastic state or to creep state. In classical treatment different types of constitutive equations are used for each of these states, which are based on some hypothesis and simplify the problem to some extent. At first the deformations are assumed to be small for the applicability of infinitesimal strain theory and then the constitutive equations of the material are simplified by assuming the condition of incompressibility of material. By using Seth’s transition theory, it is shown that same constitutive equations are used for different states of deformation, though the elastic constants have different meanings in each state. Equations of equilibrium for a body in cylindrical polar co-ordinates, in the radial direction are given by

\[
\frac{\partial (T_{rr})}{\partial r} + \frac{1}{r} \left( \frac{\partial T_{r\theta}}{\partial \theta} \right) + \left( \frac{\partial T_{rz}}{\partial z} \right) + \frac{1}{r} (T_{rr} - T_{\theta\theta}) + F_r = 0,
\]

\[
\frac{\partial (T_{r\theta})}{\partial r} + \frac{1}{r} \left( \frac{\partial T_{\theta\theta}}{\partial \theta} \right) + \left( \frac{\partial T_{\theta\theta}}{\partial z} \right) + \frac{2}{r} T_{r\theta} + F_\theta = 0,
\]

\[
\frac{\partial (T_{rz})}{\partial r} + \frac{1}{r} \left( \frac{\partial T_{r\theta}}{\partial \theta} \right) + \left( \frac{\partial T_{r\theta}}{\partial z} \right) + \frac{2}{r} T_{rz} + F_z = 0.
\]

(1.66)

where \( F_r, F_\theta, F_z \) are the body forces along \( r, \theta, z \) direction.